GRAPHS WITH VALENCY \(k\), EDGE CONNECTIVITY \(k\), CHROMATIC INDEX \(k+1\) AND ARBITRARY GIRTH* 

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Abstract. We prove that if \(k \geq 3\) and there exists a regular graph with valency \(k\), edge connectivity \(k\) and chromatic index \(k + 1\), then there exists such a graph of any girth \(g \geq 4\).  

\(G = (X, E)\) is called a graph if \(X\) is a finite set and \(E \subseteq \{(x_1, x_2) | x_1, x_2 \in X \text{ and } x_1 \neq x_2\}\). We call \(X\) the set of vertices and \(E\) the set of edges of \(G\). For each \(x \in X\), \(d(x)\) is the number of edges containing \(x\). We call \(d(x)\) the valency of the vertex \(x\). If \(d(x)\) is the same for all \(x \in X\), then \(G\) is called regular. \(G\) has edge connectivity \(k\) if \(k\) is the smallest number of edges whose removal disconnects \(G\). \(G\) has chromatic index \(\lambda\) if its edges can be partitioned into \(\lambda\) disjoint classes (colors), but no fewer, so that two edges of the same class (color) have no common vertices. \(G\) has girth \(g\) if no cycle has length shorter than \(g\). If \(x \in X\), \(G_x\) denotes the graph with vertex set \(X \setminus \{x\}\) and edge set \(\{e \in E | x \notin e\}\). \(|G|\) denotes the cardinality of \(X\). Further definitions may be found in [2] or [6].  

The Petersen graph (see Fig. 2) is bridgeless (that is, it has edge connectivity at least 2), trivalent, of girth 5 and has chromatic index 4. In this paper we prove:  

* The authors announced the case \(k = 3\) of this paper in abstract 691-05-11 of the Notices Am. Math. Soc. Notices 19 (1972) A-33.
If $k \geq 3$ and there exists a regular graph with valency $k$, edge connectivity $k$ and chromatic index $k + 1$, then there exists such a graph of any girth $g \geq 4$.

**Lemma 1.** Let $G$ be a graph with $n$ vertices. Suppose all vertices of $G$ have valency $k$ except $x_1, \ldots, x_s$ which have valency $l_1, \ldots, l_s$. If we can color the edges of $G$ in $k$ colors $c_1, \ldots, c_k$ and if $n_i$ is the number of vertices in $\{x_1, \ldots, x_s\}$ which are contained in an edge of color $c_i$, then

(a) $n_i \equiv n-s \pmod{2}$;
(b) $\sum_{i=1}^{k} n_i = \sum_{j=1}^{s} l_j$;
(c) if $\sum_{j=1}^{s} l_j < k$, then all $n_i$ are even and $\sum_{j=1}^{s} l_j$ is even;
(d) if $\sum_{j=1}^{s} l_j = k$ and $k$ is odd, then $n_i = 1$ for all $i$;
(e) if $\sum_{j=1}^{s} l_j = k$ and $n-s$ is odd, then $n_i = 1$ for all $i$.

**Proof.** Let $a_i$ be the number of edges in $G$ of color $c_i$. Clearly,

\[(*) \quad 2a_i = n-s + n_i.\]

Hence (a) is obvious. If we sum (*) over $i = 1, \ldots, k$ and use the fact that twice the number of edges is the sum of the valencies, we have

\[\sum_{i=1}^{k} (n-s+n_i) = 2 \sum_{i=1}^{k} a_i = (n-s)k + \sum_{j=1}^{s} l_j,\]

which yields (b). If $\sum_{j=1}^{s} l_j < k$, then some $n_i = 0$ and (c) follows from (a). If $\sum_{j=1}^{s} l_j = k$ and $k$ is odd, then all $n_i$ are odd. Thus no $n_i = 0$ and (d) follows easily. The condition $n-s$ odd, together with (a) implies that no $n_i = 0$, hence (e) also follows.

**Corollary 2.** Suppose $G$ is a regular graph of valency $k$ which has two vertex disjoint subgraphs $H_1$ and $H_2$ with the following properties:

(i) $e_1, \ldots, e_{k-2p}$ (with $0 < p < \frac{1}{2} k$) are the only edges of $G$ which contain vertices in both $H_1$ and $H_2$;
(ii) $e_i \cap e_j = \emptyset$ if $i \neq j$;
(iii) all edges of $G$ except $e_1, \ldots, e_{k-2p}$ are either in $H_1$ or $H_2$;
(iv) if $k$ is even, the number of vertices in $H_1$ is odd.

Then $G$ has chromatic index $k + 1$. 
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\[
\text{Fig. 1.}
\]

**Proof.** Let \(n\) be the number of vertices in the graph \(H\) formed by \(H_1\) and the edges \(e_1, \ldots, e_{k-2p}\). If \(k\) is even, then \(n - k + 2p\) is odd by the hypothesis (iv). Hence \(H\) cannot have chromatic index \(k\) by Lemma 1 (a), (c). If \(k\) is odd, \(k - 2p\) is odd, so again \(H\) cannot have chromatic index \(k\) by Lemma 1 (c). But by the Theorem of Vizing \([12]\) (see also \([6; 7; 13]\)), the only other possibility for the chromatic index of \(G\) is \(k + 1\).

**Example 3.** Let \(x_1, \ldots, x_{k-2} (k \geq 3)\) subdivide \(k - 2\) of the edges of the complete graph \(S_{k+1}\) on \(k+1\) vertices. Let \(G(x_1, \ldots, x_{k-2})\) be the graph formed by adding the complete graph on the vertices \(x_1, \ldots, x_{k-2}\) to \(S_{k+1}\). If \(G\) is the graph formed by taking two copies of \(G(x_1, \ldots, x_{k-2})\) and adding the edges joining each \(x_i\) to its counterpart (see Fig. 1), then by Corollary 2, \(G\) has chromatic index \(k + 1\).

Let \(G = (X, E)\) and \(H = (Y, F)\) be graphs, \(x \in X\) and \(d(x) = d\). Let:
\[
\{x_1, \ldots, x_d\}
\]
be the set of vertices of \(G\) adjacent to \(x\). If \(\{y_1, \ldots, y_d\} \subseteq Y\), we shall form a graph \(B = B(G, x_1, \ldots, x_d; H, y_1, \ldots, y_d)\) as follows:
- The vertex set of \(B\) is \((X \setminus \{x\}) \cup Y\), while the edge set of \(B\) is \(\{e \in E | x \notin e\} \cup F \cup \{\{x_1, y_1\}, \ldots, \{x_d, y_d\}\}\). We call this operation **blowing up the vertex** \(x\) of \(G\) **into the graph** \(H\) **using the vertices** \(y_1, \ldots, y_d\).

Let \(C^k_g\) denote the class of all regular graphs of valency \(k\), edge connectivity \(k\) and girth \(g\).

**Lemma 4.** \(C^k_g\) is non-empty when \(k \geq 2\) and \(g \geq 3\).
Proof. For each \(k\) and each \(g\) we will construct by double induction a graph \(G(k, g)\) with \(f(k, g)\) vertices in the class \(C^k\). It is clear that the complete graph on \(k + 1\) vertices is in the class \(C^k\) for all \(k \geq 2\). It is also clear that the cycle of length \(g\) is in the class \(C^2\) for all \(g \geq 3\). We assume that \(G(k, g)\) has been constructed for all \(g \leq g_0\) and that \(G(k, g_0 + 1)\) has been constructed for all \(k \leq k_0\). To construct \(G = G(k_0 + 1, g_0 + 1)\), we blow up each vertex of \(G(f(k_0, g_0 + 1), g_0)\) into \(G(k_0, g_0 + 1)\) using all vertices of \(G(k_0, g_0 + 1)\). Suppose some \(k_0\) edges \(e_1, ..., e_{k_0}\) disconnect \(G\) into two components \(G_1\) and \(G_2\). Some copy of \(G(k_0, g_0 + 1)\) must lie partially in \(G_1\) and partially in \(G_2\); otherwise, \(k_0\) edges separate \(G(f(k_0, g_0 + 1), g_0)\), a contradiction. Since it takes at least \(k_0\) edges to separate \(G(k_0, g_0 + 1)\), all of the edges \(e_1, ..., e_{k_0}\) must lie in the same copy of \(G(k_0, g_0 + 1)\). But each of the vertices of this copy of \(G(k_0, g_0 + 1)\) is attached to other vertices of \(G\) by edges which have not been removed. Thus we have shown that \(G\) has edge connectivity \(k_0 + 1\). It is now easy to verify that \(G \in C_{g_0+1}^k\).

Remark 5. Since any regular graph of odd valency has an even number of vertices and since

\[ f(k + 1, g + 1) = f(f(k, g + 1), g) f(k, g + 1), \]

\(f(k, g)\) is even except when \(k = 2\) and \(g\) is odd or \(g = 3\) and \(k\) is even.

Theorem 6. If \(k \geq 3\) and there is a graph \(D(k) \in \bigcup_{g=4}^{\infty} C^k_g\) which has an even number of vertices and chromatic index \(k + 1\), then each \(C^k_g\) for \(g \geq 4\) has such a graph.

Proof. If \(g \geq 3\), let \(x_1, ..., x_k\) be adjacent to a vertex \(x\) of the graph \(G(k, g)\) of Lemma 4. We blow up each vertex of \(D(k)\) into \(G(k, g)_x\) using the vertices \(x_1, ..., x_k\). Clearly, the resulting graph \(C(k, g)\) is a member of \(C^k_g\). Suppose some \(C(k, g)\) has chromatic index \(k\). Consider the graph \(G\) formed by a fixed copy of \(G(k, g)_x\) and the edges \(e_1, ..., e_k\) joining this copy to other copies of \(G(k, g)_x\) in \(C(k, g)\). The number of vertices in \(G\) is \(f(k, g) + k - 1\). Since \(f(k, g) - 1\) is odd by Remark 5 when \(k \geq 3\) and \(g \geq 4\), Lemma 1 (d) and (e) imply that all the edges \(e_1, ..., e_k\) have a different color. Applying this to each blown up vertex
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Fig. 2.

of $D(k)$ allows us to color the edges of $D(k)$ with $k$ colors since $e_1, \ldots, e_k$ are just those edges which meet at $y$ in $D(k)$. By Vizing's Theorem, this contradiction shows that $C(k, g)$ must have chromatic index $k + 1$. Since

$$|C(k, g)| = (f(k, g) - 1) |D(k)|,$$

$|C(k, g)|$ is even if and only if $|D(k)|$ is even.

**Corollary 7.** Let $k$ be an even integer at least 4 or $k = 3$. For every $g \geq 3$, there exists a graph in $C^k_g$ which has chromatic index $k + 1$.

**Proof.** If $k = 3$, let $D(3)$ be the Petersen graph and apply Theorem 6. If $k$ is even, let $D(k)$ be the complete graph on $k + 1$ vertices. This graph has girth 3 and by Lemma 1 (a) with $s = 0$, it has chromatic index $k + 1$ (see also [1]). In this case, if $g \geq 4$, we can perform the same construction as in the proof of Theorem 6 to arrive at a $C(k, g)$ with an odd number of vertices and chromatic index $k + 1$.

**Remark 8.** Erdős and Sachs construct in [4] regular graphs of valency $k$ and girth $g$ for each $k \geq 2$ and $g \geq 3$, but we are not able to prove that they have edge connectivity $k + 1$ (see also [9]). However, they do prove that any such graph with the minimum number of vertices has edge connectivity at least $\lfloor \frac{1}{2} (k + 1) \rfloor$. Our construction in the proof of Lemma 4 is the same as in [8].
Remark 9. One can often obtain a much smaller graph than that obtained in the proof of Theorem 6 by blowing up a carefully selected subset of the vertices of $D(k)$ into $G(k, g)_k$. For example, when $k = 3$, we need only to blow up the circled vertices of the Petersen graph in Fig. 2.

Remark 10. Lemma 1 is a generalization of a theorem of Blanche Descartes [3] and, in fact, the idea for the construction which we use in the proof of Theorem 6 is found in that paper. (An argument of this type also appears in [5].) Tutte conjectures in [10; 11] that all bridgeless trivalent graphs with chromatic index 4 are subcontractible to the Petersen graph. (This conjecture obviously implies, and is possibly equivalent to, the Four Color Conjecture.) Each of the graphs $C(k, g)$ constructed in the proof of Theorem 6 is subcontractible to $L(k)$, the Petersen graph in case $k = 3$ in the proof of Corollary 7. Lemma 1 has an easy generalization to hypergraphs (for the definition, see [2]), but we do not include this since hypergraphs of rank $r$ and valency $k$ can easily have chromatic index $r(k - 1)/2 + 1$, when $k < r$ (the duals of block designs, for example). V. Faber and L. Lovasz have asked if the Theorem of Vizing can be generalized to yield $(r - 1)k + 1$ as an upper bound to the chromatic index of hypergraphs of rank $r < k$.

Added in proof. G.H.J. Meredith has recently shown that there exists a regular graph of valency $k$, edge $k$, girth 4, chromatic index $k + 1$ with an even number of vertices for all $k \geq 3$ except possibly $k = 5$. Our Theorem 6 thus shows that there exists such a graph of any girth $g \geq 4$.

References

References


