Differential geometry of Grassmannian embeddings of based loop groups

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Abstract: In this article we study differential geometric properties of the most basic infinite-dimensional manifolds arising from fermionic $(1 + 1)$-dimensional quantum field theory: the restricted Grassmannian and the group of based loops in a compact simple Lie group. We determine the Riemann curvature tensor and the (linearly) divergent expression corresponding to the Ricci curvature of the restricted Grassmannian after proving that the latter manifold is an isotropy irreducible Hermitian symmetric space. Using the Gauss equation of the embedding of a based loop group into the restricted Grassmannian we show that the (conditional) Ricci curvature of a based loop group is proportional to its metric. Furthermore we explicitly derive the logarithmically divergent behaviour of several differential geometric quantities arising from this embedding.

Keywords: Restricted Grassmannian, based loop groups, infinite dimensional differential geometry, submanifold geometry, regularized Ricci curvature.


Introduction

The theory of infinite dimensional Lie groups and manifolds differs from its finite dimensional counterpart especially in the analytical problems arising; these range from foundational (in view of the lack of a general inverse function theorem on Fréchet spaces, compare, e.g., [13] and [6]) to more specific ones as, e.g., analytic subtleties stemming from the problem of defining curvature quantities as the traces of certain operators associated to geometrical data. The latter problem arises notably for infinite dimensional manifolds coming from the theory of quantized fields.

In this article we consider the most basic examples from fermionic $(1 + 1)$-dimensional quantum field theory: the restricted Grassmannian of a polarized complex separable Hilbert
space and based loops in a compact simple Lie group $K$, viewed as a quotient $LK/K$ of the free loop group $LK$ by the constant loops $K$. (For the relationship of these manifolds with theoretical physics see, e.g., [3, 12, 14, 16] and references therein.) It is well known that even in the $(1 + 1)$-dimensional case certain physical quantities are a priori ill defined, typically plagued by logarithmic divergences, and have to be “regularized.”

We study geometric objects obtained by taking traces, as, i.e., the Ricci curvature tensor, the second fundamental tensor and the mean curvature vector of an embedding, in the case of the restricted Grassmannian and the embedding of based loop groups therein. We employ classical differential geometric methods for homogeneous spaces and the doubly-infinite matrix approach to operator theory in Hilbert spaces.

Our first main result is the calculation of the Riemann curvature tensor of the restricted Grassmannian, after proving that the latter manifold is an isotropy irreducible Hermitian symmetric space. We also determine the “linear divergence” of its hypothetical Ricci curvature tensor.

After introducing a canonical “conditional” trace, defined by first summing over the Lie algebra of the complexification of the compact group $K$, we find that based loop groups are “conditionally” minimal inside the restricted Grassmannian. The “extrinsic contributions” as well as the “normal corrections” (see below in this introduction and especially Section 4 for the precise definitions) in the description of the conditional Ricci curvature of $LK/K$ by means of the Gauss equations of the embedding into the restricted Grassmannian both lead to logarithmically divergent traces. Nevertheless these divergences cancel and thus the conditional Ricci curvature of $LK/K$ is rigorously defined and turns out to be proportional to the metric, i.e., the based loop groups are Kähler–Einstein manifolds with respect to the metric induced from the restricted Grassmannian. This last result was previously obtained by D. Freed by defining directly, i.e., without making use of any embedding, the “$H^{1/2}$-metric” on $LK/K$ and considering the operators involved in the definition of the Ricci curvature as pseudodifferential operators on the circle [4].

Let us briefly describe the content of the various sections of this paper. The first section recalls some basic material from the theory of homogeneous Riemannian manifolds and contains the aforementioned results on the restricted Grassmannian. As a by-product we obtain the well-known fact that the finite dimensional complex Grassmannians $G_n(C^N)$ for $1 \leq n \leq N - 1$ have Kähler–Einstein constant equal to $N$. This observation “explains” the highly divergent nature of the formal expression for the Ricci curvature of the restricted Grassmannian.

In Section 2 we derive an explicit formula for the second fundamental tensor of orbits of subgroups of the group of isometries of a Riemannian symmetric space.

The third section contains the general set-up of the embedding of based loop groups into the restricted Grassmannian and some preparatory material such as the definition of useful bases of loop algebras and formulae for the Kähler structure on $LK/K$.

In Section 4 we recall how to calculate the Ricci curvature of an embedded submanifold via the Gauss equations. We define here what we call the “conditional trace” and also the ensuing doubly-infinite matrices, whose entries coming from the Riemann curvature of the restricted Grassmannian, respectively stemming from the second fundamental tensor will be referred to as the “(partially summed) extrinsic contributions,” respectively the “(partially summed) normal corrections.”
Relying on the second, respectively the first section, we determine the former in Section 5 and the latter in Section 6.

Finally Section 7 is devoted to the proof that the trace corresponding to the unconditional Ricci curvature of $LK/K$ yields a logarithmically divergent expression, whereas the conditional Ricci curvature is finite and proportional to its metric.

In the Appendix we discuss the notion of linear and logarithmic divergence of operators which we employ throughout the text, relying on [2] for the latter.

1. The restricted Grassmannian as a symmetric space

In this section we first recall some standard notations and facts from the theory of homogeneous Riemannian manifolds and notably symmetric spaces. Furthermore we prove that the restricted Grassmannian is an isotropy irreducible Riemannian symmetric space and derive explicit formulae for its Riemann curvature tensor. This allows one to observe that the formal expression for its Ricci curvature is “linearly divergent.”

Given a Lie group acting smoothly—from the left—on a manifold $M$, we can associate to each $x$ in the Lie algebra $g$ the “fundamental vector field of the $G$-action” $\dot{x}$ by setting

$$
\tau(x)_p := \frac{d}{dt} \bigg|_{t=0} (\exp(tx) \cdot p) \quad \forall p \in M.
$$

Let us recall that the map $\tau : g \to \mathfrak{X}(M)$, $\mathfrak{X}(M)$ denoting the vector fields on $M$, is an anti-homomorphism of Lie algebras.

In the case of a transitive action the projection $\pi : G \to G/H \cong M$, $g \mapsto gH$, where $H$ is the stabilizer of a point $o$ in $M$, induces a projection $\mathfrak{g} = T_o G \to T_o G/H$, which will be denoted by $\pi$ as well and which fulfills $\pi(x) = \tau(x)_o$ for all $x$ in $\mathfrak{g}$. In the sequel we shall often use the identification $T_o G/H \cong g/\mathfrak{h}$, which is induced from the (infinitesimal) projection $\pi$. Moreover, given a direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the restriction $\pi|_{\mathfrak{m}} : m \to g/\mathfrak{h}$ is an isomorphism whose inverse we shall always denote by $s$.

If we assume in addition that the vector space $\mathfrak{m}$ is $\text{Ad}(H)$-invariant and carries an $\text{Ad}(H)$-invariant positive definite scalar product $g^m$, the data

$$(M = G/H, \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g^m)$$

constitute a “reductive homogeneous Riemannian manifold,” since $g^m$ clearly gives rise to a $G$-invariant metric on $M$. A subclass of these manifolds is given by the “Riemannian symmetric spaces,” which are defined as homogeneous manifolds $(M, g)$ such that for each point $p$ in $M$, there is an isometry $\sigma$ fulfilling $\sigma^2 = \text{Id}_M$ and having $p$ as an isolated fixed point (compare, e.g., [1, 8, 9] or [10]). They are always homogeneous under the action of their isometry group and the differential in $o$ of the isometry $\sigma_o$ yields the direct sum decomposition. More precisely, let $G$ be a group acting transitively and isometrically, and $H$ be the stabilizer of $o$ in $G$, and $\mathfrak{h}$ the Lie algebra of $H$, then we have $\mathfrak{h} = \{x \in \mathfrak{g} | \sigma(x) = x\}$ and we define an $\text{Ad}(H)$-invariant complement by $\mathfrak{m} := \{x \in \mathfrak{g} | \sigma(x) = -x\}$. The identification of $\mathfrak{m}$ with $g/\mathfrak{h} = T_o M$ gives us...
the Ad($H$)-invariant metric $g^m$ on $m$. Let us also recall that—at least in finite dimensions—the map $\pi \circ \exp : m \to G / H$ yields real analytic coordinates near $eH$ on $M \cong G / H$ and that the Riemannian exponential map $\text{Exp}_o$ at $o = eH$ fulfills
\[
\text{Exp}_o = \pi \circ \exp : m \cong T_e H G / H \to G / H.
\] (1.2)

We shall use in the sequel the following convention for the Riemannian curvature tensor of a Riemannian manifold $(M, g)$:
\[
R^M_p (u, v) \cdot w = (D_{[U, V]} - [D_U, D_V]) \cdot w,
\] (1.3)
where $D$ is the Levi-Civita connection associated to $g$; $u, v, w$ are in the tangent space at a point $p$ in $M$ and $U, V, W$ are arbitrary local extensions of the respective vectors to vector fields near $p$. Let us also state the fundamental and well-known

**Fact 1.1.** If $(M = G / H, g = \mathfrak{h} \oplus m, g^m)$ is a Riemannian symmetric space and $x, y, z$ are in $m$, then
\[
R^m_o (\pi(x), \pi(y)) \cdot \pi(z) = \pi([[x, y], z]).
\] (1.4)

It will be useful to read this formula directly on $m$ by setting $R^m (x, y) \cdot z := s (R^M_p (\pi(x), \pi(y)) \cdot \pi(z))$:
\[
R^m (x, y) \cdot z = [[x, y], z] \quad \forall x, y, z \in m.
\] (1.5)

Let us now recall the definition of the infinite dimensional Grassmann manifold we shall study in the rest of this section. Given a complex separable Hilbert space $F$ with inner product $\langle \cdot, \cdot \rangle_F$ and a closed, infinite dimensional complex subspace $F_+$ whose orthocomplement $F_- := (F_+)^\perp$ is infinite dimensional as well, the “restricted Grassmannian of the polarized Hilbert space $F = F_+ \oplus F_-$” is defined as
\[
\text{Gr} = \text{Gr}(F, F_+) = \{W \subset F \mid W \text{ is a closed complex subspace of } F \text{ such that} \}
\]
\[
p_+: W \to F_+ \text{ is Fredholm and } p_- : W \to F_- \text{ is Hilbert–Schmidt}
\]
(see the standard reference [14]). It is transitively acted upon by the so-called “restricted unitary group”
\[
G = U_{\text{res}} = U_{\text{res}}(F, F_+) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(F) \bigg| b \text{ and } c \text{ are Hilbert–Schmidt} \right\}.
\]

The isotropy group of the point $o = F_+$ is the connected, contractible group
\[
H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G \bigg| a \in U(F_+) \text{ and } b \in U(F_-) \right\} \cong U(F_+) \times U(F_-).
\]

Let us observe that $G$ is a real analytic Banach Lie group modelled on its Lie algebra
\[
g = u_{\text{res}} = u_{\text{res}}(F, F_+)
\]
\[
= \left\{ \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \beta \end{pmatrix} \in \mathcal{B}(F) \bigg| \alpha^* = -\alpha, \beta^* = -\beta \text{ and } \gamma \text{ is Hilbert–Schmidt} \right\}.
\]
Grassmannian embedding of loop groups

(\mathcal{B}(F) denotes the bounded linear operators on F) by means of its locally diffeomorphic real analytic exponential map given by the exponential series of operators. An \text{Ad}(H)-invariant complement of

\[ \mathfrak{h} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in \mathcal{B}(F) \mid \alpha^* = -\alpha, \beta^* = -\beta \right\} \]

is given by

\[ m = \left\{ \begin{pmatrix} 0 & -\gamma^* \\ \gamma & 0 \end{pmatrix} \in \mathcal{B}(F) \mid \gamma \text{ is Hilbert–Schmidt} \right\} \]

and m is canonically identified with \( \mathcal{L}^2(F_+, F_-) \), the space of complex linear Hilbert–Schmidt operators from \( F_+ \) to \( F_- \). The isotropy representation of H is given by

\[ \text{Ad} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : \gamma = b \circ \gamma \circ a^{-1} \quad \text{for} \ \gamma \in \mathcal{L}^2(F_+, F_-) \quad (1.6) \]

and

\[ g^m(\gamma, \delta) := 2 \text{Re} \text{tr}_{F_+}(\gamma^* \delta) \quad \text{for} \ \gamma, \delta \in \mathcal{L}^2(F_+, F_-) \quad (1.7) \]

defines an \text{Ad}(H)-invariant metric on m. Let us observe that \( \gamma^* \) denotes the adjoint of \( \gamma \) as a complex linear operator from \( F_+ \) to \( F_- \) (or from \( F \) to \( F \), being zero on \( F_- \)). \( \text{tr} \) denotes the trace over the complex Hilbert space \( F_+ \) or equivalently over \( F \) and \( \text{Re} \ z \) denotes the real part of a complex number \( z \). Since the product \( \gamma^* \delta \) is of trace class, this trace is well defined. Let us explicitly point out the following trivial but computationally important identity:

\[ g^m(\gamma, \delta) = 2 \text{Re} \text{tr}_{F_+}(\gamma^* \delta) = 2 \text{Re} \text{tr}_F(\gamma^* \delta) = 2 \text{Re} \text{tr}_F(\delta^* \gamma) = 2 \text{Re} \text{tr}_F(\delta^* \gamma). \quad (1.8) \]

We shall often abbreviate in the sequel \( g^m(\cdot, \cdot) \) by \( \langle \cdot, \cdot \rangle \) and distinguish all other scalar products by subscripts. Obviously multiplication by \( i \) on \( \mathcal{L}^2(F_+, F_-) \) yields an \text{Ad}(H)-invariant isometric complex structure \( J \) on the real vector space underlying \( \mathcal{L}^2(F_+, F_-) \). Since this complex structure comes in fact from global holomorphic coordinates, shifting \( g^m \) and \( J \) by left multiplication by \( U_{\text{reg}} \) at every point of \( \text{Gr} \) yields a Hermitian metric on this complex manifold. An easy calculation further shows that this metric is Kähler (compare [14] for more details on the Kähler structure of \( \text{Gr} \)). Let us now derive some useful basic facts about the differential geometry of the restricted Grassmannian.

**Theorem 1.2.** The reductive Riemannian homogeneous space

\( (\text{Gr}(F, F_+) = G/H, \mathfrak{g} = \mathfrak{h} \oplus m, g^m) \)

is a Riemannian symmetric space. Notably, the Riemannian exponential map \( \text{Exp}_o \) in \( o = F_+ = eH \) is given by \( \pi \circ \exp : m \cong T_eH G/H \to G/H \) and its Riemannian curvature by the formula

\[ R^m(\gamma, \delta) \cdot \epsilon = (-\gamma \delta^* + \delta \gamma^*) \epsilon + \epsilon (\gamma^* \delta - \delta^* \gamma) = -\gamma \delta^* \epsilon + \delta \gamma^* \epsilon + \epsilon \gamma^* \delta - \epsilon \delta^* \gamma \quad (1.9) \]

for \( \gamma, \delta, \epsilon \in \mathcal{L}^2(F_+, F_-) \cong m. \) Furthermore, the linear isotropy representation of \( H \) on \( m \) is irreducible.
**Remark.** It follows easily from the proof below that all involutions \( \sigma_p \) \((p \in \text{Gr})\) preserve the complex structure of \( \text{Gr} \), whence the restricted Grassmannian is actually a Hermitian symmetric space.

**Proof.** The formulae for the isometric involutions \( \sigma_{eH} \) \((g \in G)\) are given exactly as in the case of finite dimensional Grassmannians (compare, e.g., [8]): let \( S \) be the operator \( \begin{pmatrix} 1 & 0 \\ 0 & S^{-1} \end{pmatrix} \) on \( F \) and \( \Sigma : G \rightarrow G \) given by \( \Sigma(g) = SgS^{-1} \). We can thus define \( \sigma_{eH}(gH) = \Sigma(g)H \). Obviously we have \( \Sigma(g_1 \cdot g_2) = \Sigma(g_1) \cdot \Sigma(g_2) \) and \( \Sigma(g) = g \) if and only if \( g \in H \). It now follows that \( \sigma := (\sigma_{eH})_{eH} \) yields the decomposition of \( g \) as \( h \oplus m \) with

\[
h = \{ x \in g | \sigma(x) = x \} \quad \text{and} \quad m = \{ x \in g | \sigma(x) = -x \}.
\]

Since \( \exp : g \rightarrow G \) and \( \pi \circ \exp : m \rightarrow G/H \) give locally real analytic coordinates, the map \( \pi \circ \exp : m \rightarrow G/H \) gives locally real analytic near \( eH \), and thus one can extend the usual finite dimensional arguments to our situation. Notably it follows that \( \sigma_{eH} \) is an isometry having \( eH \) as an isolated fixed point and that \( \sigma_{eH} \) is given by \( \Theta_g \circ \sigma_{eH} \circ \Theta_g^{-1} \), where \( \Theta_g \) denotes the diffeomorphism of \( G/H \) induced by left multiplication by \( g \). It follows in addition that \( \pi \circ \exp = \text{Exp}_p : m \cong T_{eH} G/H \rightarrow G/H \) and that the Riemannian curvature of \( \text{Gr}(F, F_+) \) at \( F_+ \) is given by the formula

\[
R^m\left( \begin{pmatrix} 0 & -y^* \\ y & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\delta^* \\ \delta & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\epsilon^* \\ \epsilon & 0 \end{pmatrix} \right) = \left[ \begin{pmatrix} 0 & -y^* \\ y & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\delta^* \\ \delta & 0 \end{pmatrix} \right], \begin{pmatrix} 0 & -\epsilon^* \\ \epsilon & 0 \end{pmatrix} \right].
\]

Identifying \( m \) with \( L^2(F_+, F_-) \) and calculating the lower left entry of the right-hand side immediately yields (1.9).

In order to prove the last assertion let us assume that \( \gamma \neq 0 \) is in an \( \text{Ad}(H) \)-invariant closed subspace \( V \) of \( L^2(F_+, F_-) \). Since Hilbert–Schmidt operators are compact there exist orthonormal sets \( \{ \psi_k \} \) \((k = 1, \ldots, N)\) and \( \{ \varphi_k \} \) \((k = 1, \ldots, N)\) \((N \text{ finite or countably infinite})\) in \( F_+ \) respectively \( F_- \) and positive real numbers \( \lambda_k \) such that \( \gamma = \sum_{k>0} \lambda_k \langle \psi_k, \cdot \rangle_{F_+} \varphi_k \), this meaning of course that \( \gamma(f) = \sum_{k>0} \lambda_k \langle \psi_k, f \rangle_{F_+} \varphi_k \) for all \( f \) in \( F_+ \) (see, e.g., [15, p. 203]). We define a unitary map \( b : F_- \rightarrow F_+ \) by setting \( b \varphi_1 = \varphi_1 \) and \( b|_{\{ \varphi_1 \}} = -1|_{\{ \varphi_1 \}} \). It follows that

\[
\gamma' = \text{Ad}(1, \lambda_1 \langle \psi_1, \cdot \rangle_{F_+} \varphi_1 - \sum_{k>1} \lambda_k \langle \psi_k, \cdot \rangle_{F_+} \varphi_k \) and (2\lambda_1)^{-1}(\gamma + \gamma') = \langle \psi_1, \cdot \rangle_{F_+} \varphi_1 \text{ are in } V.
\]

Since all rank one operators are (up to a factor) in the \( \text{Ad}(H) \)-orbit of \( \langle \psi_1, \cdot \rangle_{F_+} \varphi_1 \) all finite rank operators are in \( V \). The latter operators being dense in \( L^2(F_+, F_-) \), we conclude that \( V = L^2(F_+, F_-) \). \qed

**Remark.** Let us observe that the finite dimensional complex Grassmannian \( G_n(\mathbb{C}^N) \) \((1 \leq n \leq N - 1)\) can be studied along the same lines. One merely considers \( F = \mathbb{C}^N \), \( F_+ = \mathbb{C}^n \oplus \{0\} \) (the space generated by the first \( n \) vectors of the canonical basis), \( F_- = (F_+)^\perp = \{0\} \oplus \mathbb{C}^{N-n} \) and \( G = U(\mathbb{C}^N) \). The isotropy is then

\[
H \cong U(\mathbb{C}^n) \times U(\mathbb{C}^{N-n}) \quad \text{and} \quad m \cong \text{End}_\mathbb{C}(\mathbb{C}^n \oplus \{0\}) \oplus \mathbb{C}^{N-n} \cong \text{End}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^{N-n}).
\]

Formulæ (1.6) and (1.7) hold of course and we arrive at the following well-known result

**Corollary 1.3.** The finite dimensional complex Grassmannians \( G_n(\mathbb{C}^N) \) are Riemann symmetric spaces with Riemann curvature tensor given by (1.9), isotropy irreducible and Kähler–Einstein.
Proof. Since all other assertions follow immediately from Theorem 1.2, it remains to show the last one. Since $G_n(\mathbb{C}^N) = G/H$ is an isotropy irreducible Riemannian homogeneous space, an easy application of Schur’s lemma implies that $\text{Ric} = \lambda \cdot g$ (where $g$ is the Riemannian metric on $G/H$ induced from $g^m$), i.e., $G_n(\mathbb{C}^N)$ is Einstein (compare [5, p. 112]). □

In order to show that the last property in the preceding corollary does not pertain to the infinite dimensional restricted Grassmannian, we shall explicitly exhibit the divergence of the formal expression for the Ricci tensor of $\text{Gr}(F, F_+)$.

Let us first “coordinateitize” $T_F \text{Gr}(F, F_+) \cong m \cong L^2(F_+, F_-)$ by introducing a complex Hilbert basis $\{e_k | k \in \mathbb{Z}\}$ of $F$ such that $F_+$ respectively $F_-$ is generated by $\{e_k | k \geq 0\}$ respectively $\{e_k | k < 0\}$ (the introduction of the vector $e_0$ will be convenient later on since the relevant polarizations will come from Fourier decomposition of vector valued $L^2$-functions on the circle). Furthermore, we define complex linear rank one operators $E_{p,q}$ on $F$, for $p, q$ in $\mathbb{Z}$, by setting

$$E_{p,q}(f) = \langle e_q, f \rangle_F \cdot e_p.$$  \hfill (1.10)

It follows that their adjoints satisfy $(E_{p,q})^* = E_{q,p}$ and

$$g^m(E_{-k,l}, E_{-r,s}) = g^m(iE_{-k,l}, iE_{-r,s}) = 2 \cdot \delta_{k,r} \cdot \delta_{l,s},$$

$$g^m(E_{-k,l}, iE_{-r,s}) = 0.$$  \hfill (1.11)

This implies—upon recalling that $\gamma = i\gamma$ on $m$—that

$$\left\{ \frac{1}{\sqrt{2}} E_{-k,l}, \frac{1}{\sqrt{2}} iE_{-k,l} \bigg| k > 0, l \geq 0 \right\}$$  \hfill (1.12)

constitutes a Hilbert basis of the real Hilbert space $(m, g^m)$.

Let us recall that given a point $p$ in a Riemannian manifold $(M, g)$ with Riemann curvature $R^M_p$ (compare (1.3) for our convention), we can define, for $x, y$ in $T_pM$, an operator $A = A(x, y) : T_pM \to T_pM$, $A(z) = R^M_p(x, y)z$, for all $z$ in $T_pM$. Taking its trace one obtains, at least in finite dimensions, the Ricci tensor:

$$\text{Ric}^M_p(x, y) := \text{tr}_{T_pM}(A(x, y)) = \sum_a g^M_p(R^M_p(x, z_a)y, z_a)$$  \hfill (1.13)

where $g^M_p$ is the Riemannian metric of $M$ at the point $p$ and $\{z_a\}$ is any orthonormal basis of $(T_pM, g^M_p)$.

We proceed to calculate the coefficients of the Riemann curvature (with respect to the basis (1.12)) at the point $F_+$ of the restricted Grassmannian.

Lemma 1.4. For $a, b, c, d \in \{1, i\}$, $k, p, r, t > 0$ and $l, q, u, v \geq 0$ one has

$$g^m(R^m_aE_{-k,l}, cE_{-r,u}bE_{-p,q}, dE_{-t,v})$$

$$= (\delta_{k,p}\delta_{l,u}\delta_{q,v} + \delta_{k,r}\delta_{l,t}\delta_{p,u}\delta_{q,v}) \times 2 \text{Re} \{-abc\}$$

$$+ (\delta_{k,p}\delta_{l,u}\delta_{q,v} + \delta_{k,t}\delta_{l,r}\delta_{p,u}\delta_{q,v}) \times 2 \text{Re} \{ab\}.$$  \hfill (1.11)

Proof. The formula follows easily by direct computation using (1.7), (1.9) and (1.11). □
Turning to the question whether the restricted Grassmannian is an Einstein manifold, we specialize to \( a = b, k = p \) and \( l = q \) and obtain an operator

\[
A = A(a E_{-k,l}, a E_{-k,l}) : m \to m, \quad A \gamma = R^m(a E_{-k,l}, \gamma) a E_{-k,l}
\]

for \( \gamma \in m \). Formula (1.9) or the calculation below easily imply that \( A \) is continuous. The matrix coefficients of \( A \) with respect to the real Hilbert basis defined above are

\[
A_{(d,-t,v),(c,-r,u)} = \left( A \cdot \left( \frac{c}{\sqrt{2}} E_{-r,u}, \frac{d}{\sqrt{2}} E_{-t,v} \right) \right)
\]

for \( c, d \in \{1, i\}, r, t > 0 \) and \( u, v \geq 0 \).

**Corollary 1.5.** The matrix coefficients of \( A \) are given by

\[
A_{(d,-t,v),(c,-r,u)} = \alpha_{(c,-r,u)} \times (\delta_{c,d} \cdot \delta_{r,t} \cdot \delta_{u,v}),
\]

where

\[
\alpha_{(c,-r,u)} = \delta_{k,r} + \delta_{l,u} + \delta_{k,r} \cdot \delta_{l,u} \cdot (-2a^2c^2).
\]

Explicitly we have

\[
\alpha_{(c,-r,u)} = \begin{cases} 0 & \text{for } k \neq r \text{ and } l \neq u, \text{ and} \\ 1 & \text{for either } k = r \text{ or } l = u \text{ ("or" being exclusive here), and} \\ 4(1 - \delta_{c,u}) & \text{for } k = r \text{ and } l = u. 
\end{cases}
\]

**Proof.** The formulae follow easily from Lemma 1.4. \( \square \)

**Remark.** Since by the above corollary we have \( 0 \leq \alpha_{(c,-r,u)} \leq 4 \), it follows that the sectional curvature of the restricted Grassmannian is non-negative: let

\[
K^m(x, y) = \frac{\langle R^m(x, y) x, y \rangle}{\langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2}
\]

for \( x, y \in m \) be the sectional curvature of the plane \( \mathbb{R} \cdot x + \mathbb{R} \cdot y \) in \( T_x \text{Gr} \cong m \), then one has \( K^m(a E_{-k,l}, c E_{-r,u}) = \frac{1}{2} \alpha_{(c,-r,u)} \), which implies the non-negativity of \( K^m \) on \( m \). Since the last corollary shows that \( A \) is diagonal with respect to the basis (1.12) it is easy to analyze this operator:

**Proposition 1.6.** Let \( a \in \{1, i\}, k > 0 \text{ and } l \geq 0 \) be fixed and \( A = A(a E_{-k,l}, a E_{-k,l}) \) be defined by formula (1.14). Then the operator \( A : m \to m \) has the following properties:

(i) \( A \) is a bounded real linear operator.

(ii) \( A \) is self-adjoint and non-negative.

(iii) \( A \) has pure point spectrum, \( \sigma(A) = \{0, 1, 4\} \), and for the corresponding eigenspaces one has \( \dim_{\mathbb{R}} \text{Eig}(A, 0) = \dim_{\mathbb{R}} \text{Ker} A = \infty, \dim_{\mathbb{R}} \text{Eig}(A, 1) = \infty \text{ and } \dim_{\mathbb{R}} \text{Eig}(A, 4) = 1. \)

(iv) \( A \) is not compact.

**Proof.** The first assertion either follows directly from (1.9) or from the boundedness of the \( \alpha_{(c,-r,u)} \). The formulae for the \( \alpha_{(c,-r,u)} \) obtained in Corollary 1.5. imply immediately the second and third statements as well.
Since the multiplicity of the non-zero eigenvalue 1 is infinite, the operator $A$ cannot be compact. □

Obviously we have

**Corollary 1.7.** The operator $A = A(aE_{-k,l}, aE_{-k,l})$ is not trace class on $m$, i.e., the series

$$
\sum_{c \in \{1, \ldots, n\}, r > 0, u \geq 0} \alpha(c, r, u) = \sum_{r, u} g^m \left( R^m \left( aE_{-k,l}, \frac{c}{\sqrt{2}} E_{-r,u} \right) aE_{-k,l}, \frac{c}{\sqrt{2}} E_{-r,u} \right),
$$

which formally gives the Ricci tensor of $\text{Gr}(F, F_+)$ evaluated on the tangent vectors $aE_{-k,l}, aE_{-k,l}$ in $F_+$ and calculated with respect to the Hilbert basis $\{(c/\sqrt{2})E_{-r,u}\}$, diverges.

**Proof.** Since $A = |A|$ is non compact by Proposition 1.6., a fortiori it cannot be of trace class. □

It seems to be rather difficult to find a reasonable method of “regularizing” the Ricci tensor of $\text{Gr}(F, F_+)$. The operator $A$ is said to be linearly divergent (see the Appendix for the precise definition).

Let us now apply the results of Corollary 1.5 to recover the well-known finite dimensional case.

**Corollary 1.8.** The complex Grassmannian $G_n(\mathbb{C}^N)$ $(1 \leq n \leq N - 1)$, with the $U(N, \mathbb{C})$-invariant Riemannian metric induced from (1.8), has Kähler–Einstein constant equal to $N$.

**Proof.** From Corollary 1.3 we already know that $G_n(\mathbb{C}^N)$ is indeed Kähler–Einstein. So we are left only with the task of calculating $\lambda$ such that $\text{Ric}^m = \lambda \cdot g^m$, where $\text{Ric}^m$ is of course the Ricci tensor at the neutral point $F_+$, after identification of its tangent space with $m$. Decomposing $F = \mathbb{C}^N$ as the sum of $F_+ = \mathbb{C}^n \oplus \{0\} = \langle e_0, e_1, \ldots, e_{n-1}\rangle$ and $F_- = \{0\} \oplus \mathbb{C}^{N-n} = \langle e_{-1}, e_{-2}, \ldots, e_{-(N-n)}\rangle$, we can apply Corollary 1.5 to get

$$
\text{Ric}^m(aE_{-k,l}, aE_{-k,l}) = \sum_{r, u} A_{(c, -r, u), (c, -r, u)} = \sum_{r, u} \left( \sum_c A_{(c, -r, u), (c, -r, u)} \right)
$$

$$
= \sum_{r, u} \left( \sum_c \alpha_{(c, -r, u)} \right) = 2 \left( \sum_{r=1}^{N-n} \sum_{u=0}^{n-1} (\delta_{k,r} + \delta_{l,u}) \right) = 2N.
$$

Comparing to $g^m(aE_{-k,l}, aE_{-k,l}) = 2$ yields $\lambda = N$. □

**Remark.** Upon formally manipulating the divergent expression in the infinite dimensional case as in the proof of the last corollary one obtains

$$
\text{Ric}^m(aE_{-k,l}, aE_{-k,l}) = 2 \cdot \left( \sum_{r > 0} 1 + \sum_{u \geq 0} 1 \right)
$$

$$
= 2 \text{ times the cardinality of a complex Hilbert basis of } F.
$$
Since $\text{Gr}(F, F_+)$ is the closure of its dense subset $\lim_{N \to \infty} \left( \bigcup_{n=1}^{N-1} G_n(\mathbb{C}^N) \right)$ (see, e.g., [14]), the encountered divergence of the Einstein constant of $\text{Gr}(F, F_+)$ appears to be quite natural now in view of the above result for finite dimensional Grassmannians. Since the divergence stems from the contributions $\delta_{k,r}$ and $\delta_{l,a}$ (for fixed $k$ and $l$), Kähler–Einstein submanifolds of $\text{Gr}(F, F_+)$ should be complex submanifolds whose dimension equals “at most the square root” of the dimension of $\text{Gr}(F, F_+)$. Based loop groups will be shown in the remaining sections to provide such submanifolds.

2. The second fundamental tensor of orbits in symmetric spaces

In this section we will consider a Riemannian symmetric space $(M = G/H, \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g^m)$ and an orbit $M' = G' \cdot \mathfrak{h} H$ of a Lie subgroup $G'$ of $G$ in it. The principal result will be a formula for the second fundamental tensor of the embedding $M' \hookrightarrow M$ in terms of the infinitesimal data $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g^m, g' = \text{Lie} G')$.

We will frequently denote the components of a vector $v$ in a direct sum $V = E \oplus F$ by $v_E$ and $v_F$ in the sequel and of course continue to use the notations of the first section.

We prepare us with a useful formula for covariant derivatives of fundamental vector fields:

**Lemma 2.1.** Let $(M = G/H, \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g^m)$ be a Riemannian symmetric space and $D$ the Levi-Civita connection of $M$. Then

$$(D_X Y)_{\mathfrak{e} \mathfrak{h}} = \pi([\mathfrak{y}_h, x_m]) \quad \text{for all } x, y \in \mathfrak{g}. \quad (2.1)$$

**Proof.** Let us first recall (e.g., [1, formula (7.28 a)], or from [9]) that $(D_X Y)_{\mathfrak{e} \mathfrak{h}} = 0$ for all $x, y \in \mathfrak{m}$. By tensoriality in the entry $X$ we find for $x = x_h + x_m$ in $\mathfrak{g}$ and $y$ in $\mathfrak{m}$

$$(D_X Y)_{\mathfrak{e} \mathfrak{h}} = (D_{\tau(x_h)} Y)_{\mathfrak{e} \mathfrak{h}} + (D_{\tau(x_m)} Y)_{\mathfrak{e} \mathfrak{h}} = 0, \quad \text{since } \tau(x_h)_{\mathfrak{e} \mathfrak{h}} = 0.$$

The connection $D$ being torsion-free, we get for $x$ in $\mathfrak{m}$ and $y$ in $\mathfrak{g}$

$$(D_X Y)_{\mathfrak{e} \mathfrak{h}} = (D_Y X + [X, Y])_{\mathfrak{e} \mathfrak{h}} = ([X, Y])_{\mathfrak{e} \mathfrak{h}} = ([\tau(x), \tau(y)])_{\mathfrak{e} \mathfrak{h}}$$

$$= -\tau([x, y])_{\mathfrak{e} \mathfrak{h}} = -\pi([x, y]) = -\pi([x_m, y_h]).$$

Using again the tensoriality of $D$ in the first entry we arrive at the formula (2.1). \(\square\)

Let us recall that an embedding (or more generally an immersion) of a manifold $M'$ into a Riemannian manifold $(M, g^M, D = D^M)$ induces a Riemannian metric $g^{M'}$ with associated Levi-Civita connection $D' = D^{M'}$ on $M'$. The relation between the two covariant derivatives is usually described as follows. Let $u$ and $v$ be tangent vectors at a point $p$ in $M'$ and $U$ and $V$ be local extensions of them to $M$, tangent to $M'$. Then

$$B_p(u, v) := (D_U V)_{\mathfrak{e} M'} = \langle D_U V \rangle_p - \langle D'_{U|_{M'}} V \rangle_{p}.$$

is well defined and symmetric in $u$ and $v$. Thus we have a tensor field, “the second fundamental tensor (associated to the embedding $M' \hookrightarrow M$)” $B : TM' \otimes TM' \to (TM|_{M'})^\perp$, where $(TM|_{M'})^\perp$ denotes the orthocomplement of the tangent bundle $TM'$ inside the tangent bundle

We consider now more specifically a Lie subgroup $G'$ of $G$ with Lie algebra $\mathfrak{g}'$ in $\mathfrak{g}$ and its orbit $M' = G' \cdot eH$ in $M$. The tangent space of $M'$ is of course identified with $\mathfrak{g}'/\mathfrak{h}'$, where $\mathfrak{h}'$ is the Lie algebra of the stabilizer $H' = G' \cap H$ of $eH$ under the $G'$-action. Furthermore we define $m':= s(\mathfrak{g}'/\mathfrak{h}') \subset m$ as its isomorphic image (under the map $s = (\pi|_m)^{-1}$) in $m$. The normal space $(T_{eH}M)' = (\mathfrak{g}'/\mathfrak{h}')^\perp \subset \mathfrak{g}/\mathfrak{h} = T_{eH}M$ is then identified with $m'' = s((\mathfrak{g}'/\mathfrak{h}')^\perp)$, the orthocomplement of $m'$ in $m$ with respect to the metric $g^m$. Since $G'$ acts by isometries on the ambient manifold $M$ and transitively on $M'$ the second fundamental tensor is $G'$-invariant and determined by its value in the point $eH$ of $M'$. We conclude that the second fundamental tensor of the orbit $M'$ inside $M$ is completely characterized by the bilinear form

$$B': m' \times m' \to m'', B'(u, v) := s(B_{eH}(\pi(u), \pi(v))) \quad \text{for all } u, v \in m'. \quad (2.3)$$

We proceed to calculate a formula for this map:

**Proposition 2.2.** Let $(M = \mathbb{G}/\mathbb{H}, \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g^m)$ be a Riemannian symmetric space, $G'$ a Lie subgroup of $G$ and $M' = G' \cdot eH$. With the above notation we have

$$B'(u, v) = [y_h, x_m]_{m''} \quad \text{for all } u, v \in m' \text{ and } x, y \in \mathfrak{g}' \text{ such that } \pi(x) = u, \pi(y) = v. \quad (2.4)$$

**Proof.** Given $u = \pi(x), v = \pi(y)$ with $x$ and $y$ in $\mathfrak{g}'$, the fundamental vector fields $X = \tau(x)$ and $Y = \tau(y)$ provide extensions of the tangent vectors $\pi(x), \pi(y)$ in $T_{eH}M'$ to $M$ which are tangent to $M'$. It follows that $B_{eH}(\pi(x), \pi(y))$ equals $(D_X Y)_{eH}$, the component of $(D_X Y)_{eH}$ perpendicular to $T_{eH}M'$ in $T_{eH}M$.

Since $s$ is an isometry from $(T_{eH}M, g^M_{eH})$ to $(m, g^m)$ and $(m')^\perp = m''$ an obvious application of Lemma 2.1 yields

$$B'(u, v) = s((D_X Y)_{eH}) = s([\pi(y_h, x_m)]_{m''}) = [y_h, x_m]_{m''}. \quad \square$$

**Remarks.** (i) Either from its very definition or from a direct calculation using (2.4) one easily verifies that $B'$ is symmetric and depends only on the images of $x$ and $y$ in $\mathfrak{g}'/\mathfrak{h}'$ under the map $\pi$. However we stress the fact that the evaluation of the right-hand side of formula (2.4) demands to pick representatives in $\mathfrak{g}'$. In practice it is thus useful to start with $x$ and $y$ in $\mathfrak{g}'$ and reformulate (2.4) as follows:

$$B'(s \circ \pi(x), s \circ \pi(y)) = [y_h, x_m]_{m''} \quad \text{for all } x, y \in \mathfrak{g}'. \quad (2.5)$$

(ii) Formula (2.4) (or (2.5)) obviously yields a computational device to verify if a given homogeneous submanifold of a Riemannian symmetric space is minimal or totally geodesic. It is for example a simple exercise to check that $B'$ for a quadric in $\mathbb{P}_n(\mathbb{C})$ is non-vanishing, but that its trace is zero, i.e., one recovers the well-known fact that complex quadrics are minimal, but not totally geodesic in complex projective space. The result of the preceding proposition might thus be fruitful in extending the results of [7] on minimal orbits in the case that the ambient manifold is a Riemannian symmetric space (or possibly more generally a “naturally reductive homogeneous Riemannian manifold”).
(iii) Though the above proposition seems to be quite useful in finite dimensions, our main interest lies in the fact that it applies to orbits of Lie subgroups of $U_{\text{res}}(K, K_+)$ in $\text{Gr}(K, K_+)$, since the latter manifold enjoys—as shown in Section 1—all the usual properties of finite dimensional Riemannian symmetric spaces.

3. Embedding based loop groups into the restricted Grassmannian

Here we briefly review the by now classical embedding of the group of based loops into the restricted Grassmannian and give a precise account of the infinitesimal version of this embedding in terms of a natural basis of the loop algebra.

Let $K$ be a compact Lie group and $\varphi : K \to U(\mathbb{C}^d)$ be a unitary representation thereof. We define $F$ as the Hilbert space $L^2(S^1, \mathbb{C}^d)$, which is canonically isomorphic to $\mathbb{C}^d \otimes F$, where $F$ denotes the “scalar” space $L^2(S^1, \mathbb{C})$. The group $LK := C^\infty(S^1, K)$ of smooth loops in $K$ acts unitarily on $F$ by $j : LK \to U(F)$, $j(\varphi)(f)(t) := M_\varphi(f)(t) := \varphi(t) \cdot f(t)$ for $\varphi \in LK$, $f \in F$, $t \in S^1$. Identifying $S^1$ with $[0, 2\pi]/\sim$ (0 and $2\pi$ being equivalent) and setting $e_n(t) := \exp(int)$, we can Fourier decompose $f$ in $F$ as $\sum_{n \in \mathbb{Z}} f_n e_n$ with $f_n$ in $\mathbb{C}^d$ and analogously in the scalar case. Defining $F_+$ respectively $F_-$ as the spaces $\{f \in F | f_n = 0$ for $n < 0\}$ respectively $\{f \in F | f_n = 0$ for $n \geq 0\}$ we get a polarization $F = F_+ \oplus F_-$ (and similarly in the scalar case $F^s = F^s_+ \oplus F^s_-$). We collect the following basic facts, which can be easily derived from [14]:

**Proposition 3.1.** Let $K$ be a compact Lie group and $\varphi : K \to U(\mathbb{C}^d)$ an unitary representation. Then (i) the operators $M_\varphi$ are in $U_{\text{res}}(F, F_+)$, i.e.,

$$j : LK \to U_{\text{res}}(F, F_+), \quad j(\varphi) = M_\varphi.$$ 

If furthermore $\mathfrak{k} = \text{Lie } K$ is simple and the induced Lie algebra representation $\varphi : \mathfrak{k} \to \mathfrak{u}(d)$ is irreducible it follows that

(ii) the stabilizer of $F_+$ in $LK$ is $K$, the subgroup of constant loops, and

(iii) the orbit map $LK \to j(LK) \cdot F_+$ induces a real analytic embedding

$$LK / K \cong j(LK) \cdot F_+ \hookrightarrow \text{Gr}(F, F_+)$$

such that the pullback of the Riemannian metric of $\text{Gr}(F, F_+)$ to $T_e K LK / K \cong L\mathfrak{k} / \mathfrak{k} \cong C^\infty(S^1, \mathfrak{k})/\mathfrak{k}$ is the $H^{1/2}$ metric.

**Remarks.** (i) Since we will assume the condition that $\mathfrak{k}$ is simple and that the representation is already irreducible on the Lie algebra level in the remainder of the text, the map $\varphi : \mathfrak{k} \to \mathfrak{u}(d)$ is injective and its image is contained in $\mathfrak{su}(d)$. For simplicity of notations we will not distinguish between $\mathfrak{k}$ and its image $\varphi(\mathfrak{k})$ in the sequel.

(ii) Since $LK$ is real analytically diffeomorphic to a product of $K$ and its subgroups of loops fulfilling $\varphi(0) = \varphi(2\pi) = e$, the “based loop group,” $LK / K$ is diffeomorphic to this subgroup. Nevertheless we shall stick to the homogeneous space description.

(iii) Let us also recall that, if $\mathfrak{k} = \text{Lie } K$, then $L\mathfrak{k} = C^\infty(S^1, \mathfrak{k})$ is the Lie algebra of $LK$.

(iv) We shall not give a proof of the above proposition, but we shall give some more details on the metric aspect in the third remark after Lemma 3.3.
In order to derive concrete formulae for the differential of the embedding $LK/K \hookrightarrow \text{Gr}$ we will construct appropriate “bases” for $L\mathfrak{k}$. Let us fix the following real scalar product on $\text{End}_\mathbb{C}(\mathbb{C}^d)$:

$$
\langle A, B \rangle_d := \text{Re} \, \text{tr}_d(A^*B),
$$

(3.2)

where $\text{tr}_d$ denotes the complex trace of a complex linear endomorphism of $\mathbb{C}^d$. The complexification $\mathfrak{t}^\mathbb{C} = \mathfrak{t} \oplus i\mathfrak{t}$ of $\mathfrak{t}$ in $\text{End}_\mathbb{C}(\mathbb{C}^d)$ is then a direct orthogonal sum, since obviously $\mathfrak{t} \perp i\mathfrak{t}$ with respect to $\langle \cdot, \cdot \rangle_d$. We fix a real basis

$$
\{ a^\sigma \ | \ \sigma = 1, \ldots, \text{dim } \mathfrak{t} \}
$$

of $\mathfrak{t}$ fulfilling $\text{tr}_d((a^\sigma)^*a^\sigma) = \delta_{\sigma,\epsilon}$, forming a fortiori an orthonormal basis of $(\mathfrak{t}, \langle \cdot, \cdot \rangle_d)$. We remark that $(a^\sigma)^* = -a^\sigma$, $\text{tr}_d(a^\sigma) = 0$ and that $[a^\sigma, ia^\sigma | \sigma = 1, \ldots, \text{dim } \mathfrak{t}]$ constitutes a real orthonormal basis of $(\mathfrak{t}^\mathbb{C}, \langle \cdot, \cdot \rangle_d)$. Upon observing that $L\mathfrak{k}$ sits inside $L\mathfrak{t}^\mathbb{C} \cong \mathfrak{t}^\mathbb{C} \otimes \mathbb{C}^\infty(S^1, \mathbb{C})$, we can apply Fourier decomposition to the elements of $L\mathfrak{t}$ to obtain the following.

**Lemma 3.2.** Each element $\xi$ of $L\mathfrak{k}$ can be written as

(i) $\xi = a_0 \otimes e_\mathfrak{t} + \sum_{n \in \mathbb{Z}\setminus\{0\}} a_n \otimes e_n$, with $a_0 \in \mathfrak{k}$ and $a_n \in \mathfrak{t}^\mathbb{C}$ satisfying $a_{-n} = -a_n^*$ also as

(ii) $\xi = a_0 \otimes e_\mathfrak{t} + \sum_{n>0} a'_n \otimes (e_n + e_{-n}) + \sum_{n<0} a''_n \otimes (e_n - e_{-n})$, with $a_0, a'_n, a''_n \in \mathfrak{t}$.

(iii) Furthermore, the differential $j_* : L\mathfrak{k} \rightarrow U_{\text{res}}(F, F_+)$ of the map $j$ is given by

$$
j_*(\xi) = M_\xi = \sum_{k, l \in \mathbb{Z}} a_{k-l} \otimes |k| |l|,
$$

where Dirac’s “bra-ket” shorthand notation $|k| |l|$ stands for the operator $E_{k,l} = e_k \langle e_l, \cdot \rangle_{F^s} : F^s \rightarrow F^s$ already defined in Section 1. Furthermore a tensor product $A \otimes E_{k,l}$ with $A$ in $\text{End}_\mathbb{C}(\mathbb{C}^d)$ acts in the obvious way on $F \cong \mathbb{C}^d \otimes F^s$.

**Proof.** The first two assertions follow directly from Fourier decomposition and the condition that $\xi(t) \in \mathfrak{t}$ for all $t \in S^1$. Observing that the differential $j_* = M$ is also given by associating a multiplication operator, i.e., $j_*(\xi) = M_\xi$, the third part boils down to an easy Fourier mode calculation as well. □

Let us now introduce the closed subspace $L_0\mathfrak{k} = \{ \xi \in L\mathfrak{k} | \int_0^{2\pi} \xi(t) \, dt = 0 \}$ which is given by those $\xi$’s whose “zero mode” $a_0$ vanishes. The natural projection $L\mathfrak{k} \rightarrow L\mathfrak{k}/\mathfrak{t}$ identifies $L_0\mathfrak{k}$ isomorphically with the quotient space. Adhering to the notations in the preceding sections, we set henceforth $j(LK) = G' \subset G = U_{\text{res}}$. $j(L\mathfrak{k}) = g'$ etc. and define $g_0 = j(L_0\mathfrak{k})$. Thus the map

$$
\pi \circ j : L\mathfrak{k} \rightarrow T_{F_0}LK : F_+ \cong g'/\hbar',
$$

restricted to $L_0\mathfrak{k}$, identifies this space with $g'/\hbar'$. Using the section $s = (\pi |_{m_0})^{-1} : g'/\hbar' \rightarrow m \subset g$, we find that $m_0 = s \circ \pi \circ j(L_0\mathfrak{k}) \subset m \subset g$ (with $m \cong L^2(F_+, F_-) \cong \text{End}_\mathbb{C}(\mathbb{C}^d) \otimes L^2(F_+^*, F_-^*)$) is isomorphic to $L_0\mathfrak{k}$ as well. Let us equip $m$ with its natural metric (compare (1.7) and (1.8))

$$
g^m(A \otimes \gamma', B \otimes \delta') = 2 \text{Re} \left\{ \text{tr}_d(A^*B) \cdot \text{tr}_{F'}((\gamma')^*\delta') \right\}
$$

(3.3)

$$
= 2 \text{Re} \left\{ \text{tr}_d(AB^*) \cdot \text{tr}_{F'}((\gamma')^*\delta') \right\}
$$
where \( A, B \in \text{End}_C(\mathbb{C}^d) \) and \( \gamma^i, \delta^i \in L^2(F^+_n, F^-_n) \). If there is no danger of confusion we will sometimes drop the subindices of the traces in the sequel.

Furthermore we fix the following “basis” of \( L_0 \):\[
\xi_n^{\sigma, \nu} = va^\sigma \otimes (e_n + v^2 e_{-n})
\]
with \( \sigma = 1, \ldots, \dim \mathfrak{k}, n > 0, v \in \{1, i\} \).

With this set-up we can calculate our basic formula for the forthcoming curvature calculations.

**Lemma 3.3.** For \( \xi_n^{\sigma, \nu} \), with \( n \in \mathbb{Z}_+, \sigma = 1, \ldots, \dim \mathfrak{k} \) and \( v \in \{1, i\} \) we have
\[
M_n^{\sigma, \nu} = j(\xi_n^{\sigma, \nu}) = va^\sigma \otimes \left( \sum_{k \in \mathbb{Z}} |k\rangle \langle k - n| + v^2 \sum_{k \in \mathbb{Z}} |k\rangle \langle k + n| \right)
\]
in \( g^c \subset g = u_{\text{res}}(F, F_+) \), and
\[
\gamma_n^{\sigma, \nu} := s \circ \pi(M_n^{\sigma, \nu}) = v^3 a^\sigma \otimes \sum_{k=1}^{n} |\overline{n-k}\rangle \langle n-k|
\]
in \( m'_0 \subset m \). Furthermore
\[
g^m(\gamma_n^{\sigma, \nu}, \gamma_m^{r, \mu}) = 2n \times \delta_{n,m} \times \delta_{\sigma, r} \times \delta_{\nu, \mu}.
\]

**Proof.** The formulae for \( M_n^{\sigma, \nu} \) follow from Lemma 3.2 (iii). Projecting to the off-diagonal part yields \( \gamma_n^{\sigma, \nu} \in L^2(F_+, F_-) \). Let us calculate the following scalar product
\[
g^m(\gamma_n^{\sigma, 1}, \gamma_m^{r, 1}) = 2 \Re \left\{ \tr((a^\sigma)^* a^r) \times \tr\left( \left( \sum_{k=1}^{n} |n-k\rangle \langle -k| \left( \sum_{l=1}^{m} |l-1\rangle \langle m-l| \right) \right) \right\}
\]
\[
= 2 \langle a^\sigma, a^r \rangle_d \times \tr\left( \sum_{k=1}^{\min(n,m)} |n-k\rangle \langle m-k| \right) = 2 \times \delta_{\sigma, r} \times \delta_{n,m} \times n.
\]

Since \( \gamma_n^{\sigma, t} = (-i)\gamma_n^{\sigma, 1} \), the other scalar products follow immediately. \( \square \)

**Remarks.**

(i) In view of future use in similar, though more involved calculations, we introduce the notations \( p \wedge q := \min(p, q) \) and \( p \vee q := \max(p, q) \) for real numbers \( p \) and \( q \) and we also notice that in order to compute traces of operator products one first matches the inner bra-ket indices (here \( -k = -l \)); then taking the trace amounts to identifying their common range and evaluating its length (here \( n \wedge m \)). One gets a non-trivial result if and only if the remaining bra and ket vectors in the various summands coincide. This procedure will be referred to as *matching and tracing* in the sequel.

(ii) The above lemma shows as a by-product that \( m'_0 \) is invariant under the complex structure \( J \) on \( m \) given by \( J(y) = iy \). Thus we have the useful formula
\[
\gamma_n^{\sigma, t} = (-i)\gamma_n^{\sigma, 1} = -J(\gamma_n^{\sigma, 1}).
\]

This yields a torsion-free almost complex structure on \( LK/K \). Since we do not need them we shall not exhibit the holomorphic coordinates on \( LK/K \) inducing this \( J \).
(iii) For each $s$ in $\mathbb{R}$ there is a pre-Hilbert structure on $C^\infty(S^1, \mathfrak{g})$ given by the "$H^s$-Sobolev metric"

$$\left( \sum_n a_n \otimes e_n, \sum_m b_m \otimes e_m \right)_H := \text{tr}_d(a_0^*b_0) + \sum_{n \neq 0} \text{tr}_d(a_n^*b_n) \cdot |n|^{2s}.$$ 

The scalar product calculation in the above lemma shows that the pull-back of $g^m$ to $L_0\mathfrak{k}$ is exactly the $H^{1/2}$-Sobolev structure. It seems to be a general phenomenon of the mathematical models of $(1+1)$-dimensional quantum field theories that this borderline case is involved ($H^s(S^1, \mathbb{C}) \rightarrow C^0(S^1, \mathbb{C})$ for $s > 1/2$), compare, e.g., [14] for our case of "current algebras" and [12] for the case of string theory.

The closure of $m_0' = s \circ \pi \circ j(LK)$ in $(\mathfrak{m}, g^m)$ is a complex Hilbert space $m'$, which is isomorphic to $\{ \xi \in L^2(S^1, \mathfrak{m}) \mid \int_0^{2\pi} \xi(t) \, dt = 0 \text{ and } \| \xi \|_{H^{1/2}} < \infty \}$, since the metric $g^m$ pulls back to the $H^{1/2}$-Sobolev metric under the isomorphism

$$L_0\mathfrak{k} = \left\{ \xi \in L^2(S^1, \mathfrak{k}) \mid \int_0^{2\pi} \xi(t) \, dt = 0 \right\} \xrightarrow{s \circ \pi \circ j} m_0.'$$

Obviously the vectors $\tilde{\gamma}^\sigma_{n,v} = (1/\sqrt{2n}) \gamma^\sigma_{n,v}$ ($n \in \mathbb{Z}_+$, $\sigma \in \{1, \ldots, \dim \mathfrak{k}\}$, $v \in \{1, i\}$) form a Hilbert basis of $m'$.

Defining $E_n = \sum_{k=1}^n (-k) (n-k) = \sum_{k=0}^{n-1} (k-n) |k\rangle \langle k|$ and $\tilde{E}_n := (1/\sqrt{2n})E_n$ (in $L^2(F^1_+, F^1_-)$), we have

$$\tilde{\gamma}^\sigma_{n,v} = v^3 a^\sigma \otimes \tilde{E}_n = v^3 a^\sigma \otimes \frac{1}{\sqrt{2n}} \sum_{k=1}^n (-k) |n-k\rangle.$$  \hspace{1cm} (3.6)

The description of $m \cong L^2(F_+, F_-)$ as $\text{End}(\mathbb{C}^d) \otimes_\mathbb{C} L^2(F_+, F_-)$ is reflected by a compatible canonical complex Hilbert space isomorphism $m' \cong \mathfrak{k} \otimes \ell_2^\mathbb{C}(\mathbb{Z}_+)$, where $\ell_2^\mathbb{C}(\mathbb{Z}_+)$ is of course generated by the Hilbert basis $\{\tilde{E}_n\}$. Going to the real Hilbert space structure point of view, we find $m' \cong \mathfrak{k} \otimes_\mathbb{R} \ell_2^\mathbb{R}(\mathbb{Z}_+)$. We shall denote the space $\ell_2^\mathbb{R}(\mathbb{Z}_+) = ((\tilde{E}_n| n \geq 0))_\mathbb{R}$ by $\ell_2^\mathbb{R}(\mathbb{Z}_+)$ for brevity in the sequel.

4. The Ricci tensor of a submanifold via the Gauss equations

In this section we recall how the Gauss equations can be used to calculate the Ricci tensor of a submanifold. We apply this to the based loop space in the restricted Grassmannian and obtain an explicit description of the $LK$-invariant Ricci tensor of $LK/K$ in $eK$ as the trace of a doubly infinite matrix. The determination of the entries of this matrix and the calculation of its trace norm and its trace will be carried out in the subsequent sections.

Given an embedding of a manifold $M'$ in a Riemannian manifold $(M, g)$, the induced metric $g' = g|_{M'}$ yields a Levi-Civita connection $D'$ and its Riemann curvature tensor $R'$ on $M'$. Then the Gauss equations express the components of $R'$ at a point $x$ of $M'$ in terms of the Riemann curvature $R_x$ of $M$ in $x$ and the second fundamental tensor $B_x$ (compare (2.2)): for all $a, b, c, d$
in $T_xM' \subset T_xM$ we have
\begin{align}
g^*_x(R'(a, b)c, d) &= g_x(R_x(a, b)c, d) \\
&+ \left\{ g_x(B_x(a, c), B_x(b, d)) - g_x(B_x(a, d), B_x(b, c)) \right\}. \quad (4.1)
\end{align}
(Note that there is a relative minus sign, e.g., in [9] and [11] due to a sign convention for the Riemann curvature tensor differing from (1.3)).

If $\{e_\alpha\}$ is an orthonormal basis of $(T_xM', g'_x)$, setting $b = d = e_\alpha$ and summing over $\alpha$ in the Gauss equation (4.1) yields, at least in finite dimensions, a formula for the Ricci tensor of $M'$ at $x$ (compare (1.13)). We have, for all $a, b$ in $T_xM'$
\begin{align}
\text{Ric}'_x(a, b) &= \sum_\alpha g^*_x(R'(a, e_\alpha)b, e_\alpha) = \sum_\alpha g_x(R_x(a, e_\alpha)b, e_\alpha) \\
&+ \left\{ \sum_\alpha g_x(B_x(a, b), B_x(e_\alpha, e_\alpha)) - \sum_\alpha g_x(B_x(a, e_\alpha), B_x(b, e_\alpha)) \right\}. \quad (4.2)
\end{align}

The basic idea is now to evaluate (4.2) at the point $x = F_+$ for the embedding $LK/K = LK \cdot F_+ \hookrightarrow \text{Gr}(F, F_+)$. Since $LK$ acts isometrically on $\text{Gr}(F, F_+)$ and transitively on $LK/K$, this already fixes the Ricci tensor of $LK/K$. Thus we have first to consider the following equation, which corresponds to (4.1):
\begin{align}
\langle R'(u, v)w, z \rangle &= \langle R(u, v)w, z \rangle + \left\{ \langle B'(u, w), B'(v, z) \rangle - \langle B'(u, z), B'(v, w) \rangle \right\}, \quad (4.3)
\end{align}
where $u, v, w, z$ are in $m'$, $\langle \cdot, \cdot \rangle = g^m$, $R'$ is the Riemann curvature of $LK/K$ in $p = F_+$ viewed as a map $(m')^{\otimes 3} \rightarrow m'$, $R$ is the Riemann curvature of $\text{Gr}(F, F_+)$ at $p = F_+$ viewed as a map $(m)^{\otimes 3} \rightarrow m$ and $B' = B_p$ is interpreted as a map $(m')^{\otimes 2} \rightarrow m'' := (m')^\perp \subset m$. Using the real Hilbert basis $\tilde{\gamma}^{\ell, \epsilon}_{\ell, \epsilon}$ of $m'$ we find the equation
\begin{align}
\langle R'(u, \tilde{\gamma}^{\ell, \epsilon}_p v), \tilde{\gamma}^{\ell, \epsilon}_{q} \rangle &= \langle R(u, \tilde{\gamma}^{\ell, \epsilon}_p v), \tilde{\gamma}^{\ell, \epsilon}_{q} \rangle \\
&+ \left\{ \langle B'(u, v), B'(\tilde{\gamma}^{\ell, \epsilon}_p, \tilde{\gamma}^{\ell, \epsilon}_q) \rangle - \langle B'(u, \tilde{\gamma}^{\ell, \epsilon}_q), B'(v, \tilde{\gamma}^{\ell, \epsilon}_p) \rangle \right\}. \quad (4.4)
\end{align}
Fixing $u$ and $v$ in $m'$ or in $m'_0$ defines (via matrix elements) an operator on $m' \cong \ell^C \otimes \ell^2(\mathbb{Z}_+)$, whose trace, provided it exists, is $\text{Ric}'(u, v)$, the Ricci tensor of $LK/K$ at $p = F_+$, evaluated on $u$ and $v$.

In Proposition 7.3 we shall calculate the logarithmically divergent behaviour of the trace norm of the operator defined by (4.4) and show explicitly that its divergences disappear upon summing over the Lie algebra indices first. This behaviour agrees with the results of Freed ([4]) obtained by directly defining the “$H^{1/2}$-metric” on $LK/K$ and considering the relevant operators as pseudodifferential operators on the circle.

One has thus to “condition” the trace by the prescription of summing first over the indices corresponding to $\ell$. In order to ensure a weak form of minimality of $LK/K$ in $\text{Gr}$ as well we shall in fact condition by taking immediately the trace over $\ell^C = \ell \otimes \mathbb{C}$. (Compare Remark (i) after Corollary 6.3.)

Thus we get for $u, v$ in $m'$ intermediate operators
\begin{align}
\text{Ric}'^T(u, v) : \ell^2(\mathbb{Z}_+) &\rightarrow \ell^2(\mathbb{Z}_+), \\
\text{Ric}'(u, v) := \text{tr}_{\ell^C}(z \mapsto R'(u, z)v),
\end{align}
where $\mathfrak{k}^\mathbb{C}$ is considered as a real vector space and $\text{tr}_{\mathfrak{k}^\mathbb{C}}$ denotes the real trace over $\mathfrak{k}^\mathbb{C}$. Using the Hilbert basis $\{ \tilde{E}_p \}$ of $\mathfrak{sl}(\mathbb{Z}_+)$, this operator can be written as a doubly-infinite matrix with entries

$$
\widehat{\text{Ric}}_{q,p}(u, v) = \sum_{\epsilon=1}^{\dim \mathfrak{g}} \sum_{\epsilon \in [1, \ell]} \{ R'(u, \tilde{\gamma}^{\epsilon,k}_p) v, \tilde{\gamma}^{\epsilon,k}_q \}
$$

with $p,q \in \mathbb{Z}_+$. Application of (4.4) then leads to the following “partially summed Gauss equations”:

$$
\widehat{\text{Ric}}_{q,p}(u, v) = \frac{1}{2\sqrt{pq}} \sum_{\epsilon \in [1, \ell]} \{ R(u, \gamma^{\epsilon,k}_p) v, \gamma^{\epsilon,k}_q \}
$$

$$
+ \frac{1}{2\sqrt{pq}} \sum_{\epsilon \in [1, \ell]} \{ B'(u, v), B'(\gamma^{\epsilon,k}_p, \gamma^{\epsilon,k}_q) \}
$$

$$
- \frac{1}{2\sqrt{pq}} \sum_{\epsilon \in [1, \ell]} \{ B'(u, \gamma^{\epsilon,k}_q), B'(v, \gamma^{\epsilon,k}_p) \}.
$$

We shall sometimes refer to the first term in the right-hand side of (4.6) as the extrinsic contribution and to the second term as the normal correction.

Next, our strategy will consist in computing first the matrix elements

$$
\widehat{\text{Ric}}_{p,q}(\gamma^{\sigma,v}_n, \gamma^{\tau,\mu}_m).
$$

After estimating the trace norm of the corresponding operator $\widehat{\text{Ric}}(\gamma^{\sigma,v}_n, \gamma^{\tau,\mu}_m)$, we derive the Ricci tensor of $LK/K$ and the Kähler–Einstein constant $\lambda$ such that

$$
\text{tr}_{\mathfrak{g}(\mathbb{Z}_+)}(\widehat{\text{Ric}}(u, v)) = \lambda \cdot \langle u, v \rangle.
$$

For the sake of readability we split the calculation into three different sections.

5. Partially summed extrinsic contributions to $\text{Ric}^{LK/K}$

The goal of this section is to calculate the extrinsic contribution to $\text{Ric}^{LK/K}$:

$$
\widehat{\text{Ric}}_{p,q}(u, v) := \sum_{\epsilon \in [1, \ell]} \{ R(u, \tilde{\gamma}^{\epsilon,k}_p) v, \tilde{\gamma}^{\epsilon,k}_q \}, \quad \text{for } u = \tilde{\gamma}^{\sigma,v}_n \text{ and } v = \tilde{\gamma}^{\tau,\mu}_m.
$$

Before stating and proving the result, we shall introduce a “Casimir element” $C := -\sum_{\epsilon=1}^{\dim \mathfrak{g}} a^\epsilon \otimes a^\epsilon$ associated to the orthonormal basis $\{ a^\epsilon \}$ of $(\mathfrak{k}, \langle \cdot, \cdot \rangle_\mathfrak{k})$. Since $\mathfrak{k}$ is simple, $C$ lies in the centre of the universal enveloping algebra $\mathfrak{U}(\mathfrak{k})$ of $\mathfrak{k}$ and thus for all irreducible representations $\vartheta$ of $\mathfrak{k}$, we find by Schur’s lemma a number $P_C(\vartheta)$ such that

$$
- \sum_{\epsilon} \vartheta (a^\epsilon) \circ \vartheta (a^\epsilon) = P_C(\vartheta) \cdot 1,
$$

where $1$ is the identity of the vector space carrying the representation $\vartheta$. Recalling that $Q$ is an irreducible representation we have
Proposition 5.1. The extrinsic contribution to the partially summed Gauss equations (4.6) is given by

$$
\mathcal{Ric}_{q,p}(\gamma^{\sigma,v}_n, \gamma_{m}^{\tau,\mu}) = \frac{1}{2\sqrt{pq}} \sum_{\epsilon,\kappa} \langle R(\gamma^{\sigma,v}_n, \gamma^{\epsilon,\kappa}_p) \gamma^{\tau,\mu}_m, \gamma^{\epsilon,\kappa}_q \rangle
$$

$$
= \frac{4}{\sqrt{pq}} \times (n \land p \land m \land q) \times P_C(q) \times \delta_{n+q,m+p} \times \delta_{\sigma,\tau} \times \delta_{\nu,\mu} \tag{5.2}
$$

Proof. The structure of the proof is as follows. In Step 1 we write the general term of the extrinsic contribution to the Ricci curvature of $LK/K$ as a sum of four traces. Setting $\epsilon = \epsilon'$ and $\kappa = \kappa'$ allows to simplify and to group the terms into two types. Summation over $\kappa$ in $\{1, i\}$ (Step 2) will eliminate one of the two types. Finally we sum over $\epsilon$ and arrive at (5.2).

Step 1. Recalling that $R(\gamma, \delta)\epsilon = -\gamma^s\delta + \delta^s\gamma + \epsilon \gamma^s \delta - \epsilon \delta^s \gamma$ (eq. (1.9)) and using the definition of $\langle \cdot, \cdot \rangle$, we find

$$
\langle R(\gamma^{\sigma,v}_n, \gamma^{\epsilon,\kappa}_p) \gamma^{\tau,\mu}_m, \gamma^{\epsilon,\kappa}_q \rangle = 2 \text{Re} \langle R(\gamma^{\sigma,v}_n, \gamma^{\epsilon,\kappa}_p) \gamma^{\tau,\mu}_m \cdot (\gamma^{\epsilon,\kappa}_q)^* \rangle
$$

$$
= -2 \text{Re} \left\{ \langle (\gamma^{\sigma,v}_n) \cdot (\gamma^{\epsilon,\kappa}_p)^* \cdot \gamma^{\tau,\mu}_m \cdot (\gamma^{\epsilon,\kappa}_q)^* \rangle + \langle (\gamma^{\tau,\mu}_m) \cdot (\gamma^{\epsilon,\kappa}_p)^* \cdot \gamma^{\sigma,v}_n \cdot (\gamma^{\epsilon,\kappa}_q)^* \rangle \right\} + 2 \text{Re} \left\{ \langle (\gamma^{\tau,\mu}_m) \cdot (\gamma^{\epsilon,\kappa}_p)^* \cdot (\gamma^{\sigma,v}_n)^* \cdot (\gamma^{\epsilon,\kappa}_q)^* \rangle \right\}.
$$

Let us explain how to calculate these traces by evaluating the first one: this will furnish the general pattern. Recalling that $\gamma^{\sigma,v}_n = v^3 a^\sigma \otimes E_n$ (compare (3.6)), we find, by repeated matching and tracing

$$
\text{tr}_F \left\{ (\gamma^{\sigma,v}_n \cdot (\gamma^{\epsilon,\kappa}_p)^*) \cdot (\gamma^{\tau,\mu}_m \cdot (\gamma^{\epsilon,\kappa}_q)^*) \right\}
$$

$$
= v^3 \mu R^2 \times \text{tr}_F(\langle a^\sigma a^\tau a^\epsilon \rangle \times \text{tr}_F(\langle E_n E_p E_m E_q \rangle)
$$

$$
= v^3 \mu R^2 \times \text{tr}_F(\langle a^\sigma a^\tau a^\epsilon \rangle \times (n \land p \land m \land q) \times \delta_{m+n,p+q}.
$$

Thus we get

$$
\langle R(\gamma^{\sigma,v}_n, \gamma^{\epsilon,\kappa}_p) \gamma^{\tau,\mu}_m, \gamma^{\epsilon,\kappa}_q \rangle
$$

$$
= -2 \text{Re} \left\{ v^3 \mu R^2 \right\} \times \text{tr}_F(\langle a^\sigma a^\tau a^\epsilon \rangle \times (n \land p \land m \land q) \times \delta_{m+n,p+q} \tag{5.3}
$$

$$
+ 2 \text{Re} \left\{ v^3 \mu R^2 \right\} \times \text{tr}_F(\langle a^\sigma a^\tau a^\epsilon \rangle \times (n \land p \land m \land q) \times \delta_{n+q,m+p}.
$$

Step 2. Setting $\epsilon = \epsilon'$ and summing over $\kappa = \kappa'$ in $\{1, i\}$ makes the first two summands of (5.3) vanish. Moreover we have, working in the representation $g$,

$$
\sum_{\epsilon} a^\epsilon \bar{a}^\epsilon = -P_C(g) \cdot 1.
$$
so we are left with
\[
\sum_{\epsilon} \sum_{\kappa} \langle R(y_n^{\sigma,v}, y_p^{\epsilon,k}) y_m^{\tau,\mu}, y_q^{\epsilon,k} \rangle
\]
\[
= 4 \text{Re} \{ \sigma^3 \mu^3 \} \times \text{tr}_d((a^\sigma)^* a^\tau + (a^\tau)^* a^\sigma) \times P_C(q) \times (n \wedge p \wedge m \wedge q) \times \delta_{n+q,m+p}
\]
\[
= 4 \delta_{v,\mu} \times 2 \delta_{\alpha,\tau} \times P_C(q) \times (n \wedge p \wedge m \wedge q) \times \delta_{n+q,m+p}.
\]

Multiplying by \(1/(2\sqrt{pq})\) gives the right-hand side of (5.2), whereas a direct check shows that
\[
(n \wedge p \wedge m \wedge q) \times \delta_{n+q,m+p} = (n \wedge p + n \wedge m - n) \times \delta_{n+q,m+p}. \quad \square
\]

**Remark.** Given a simple compact Lie algebra \(k\) and an irreducible unitary representation \(\theta\), one can determine \(P_C(\theta)\). We do not need the result, but we nevertheless observe that
\[
\text{tr} \sum_{\alpha} \sum_{\beta} A_{\alpha,\beta} \delta_{\alpha,\beta}. \quad \text{for } k = SU(d) \text{ and } \theta \text{ the fundamental representation of } K \text{ on } \mathbb{C}^d.
\]

We also observe that the operator \(\text{Ric}(y_n^{\sigma,v}, y_m^{\tau,\mu})\) is not of trace class on \(\ell^2(\mathbb{Z}^+)\). More precisely, we have the following:

**Corollary 5.2.** Let \(A := \text{Ric}(y_n^{\sigma,v}, y_m^{\tau,\mu})\). Then \(|A| := \sqrt{A^*A}\) is a compact diagonal operator, reading explicitly
\[
|A| := \sqrt{A^*A} = \sum_{p \geq 1 \wedge (n-m+1)} \beta(p) \tilde{E}_p(\tilde{E}_p, \cdot)^2;
\]
where
\[
0 \leq \beta(p) = \frac{n \wedge m \wedge p \wedge (p+m-n)}{\sqrt{p(p+m-n)}} \times (4P_C(q) \times \delta_{v,\mu} \times \delta_{\alpha,\tau})
\]

The trace norm of \(A\) is then given by
\[
\|A\|_1 = \sum_{p \geq 1 \wedge (n-m+1)} \beta(p)
\]
whence, for \(\sigma = \tau, v = \mu\), the above series is asymptotically equivalent to \(\sum n(n \wedge m) P_C(q) \frac{1}{n}\), i.e., it is logarithmically divergent (see the Appendix).

**Proof.** Recalling that
\[
\tilde{E}_n = \frac{1}{\sqrt{2n}} \sum_{k=0}^{n-1} |k-n\rangle \langle k|
\]
with \(n \geq 1\) form a Hilbert basis of \(\ell^2(\mathbb{Z}^+)\), we can write the operator \(A := \text{Ric}(y_n^{\sigma,v}, y_m^{\tau,\mu})\) on \(\ell^2(\mathbb{Z}^+)\) as follows:
\[
A = \sum_{q,p \neq 1} A_{q,p} \tilde{E}_q(\tilde{E}_p, \cdot)^2;
\]
where \(\langle \cdot , \cdot \rangle^2\) denotes the scalar product in \(\ell^2(\mathbb{Z}^+)\) and “\(\cdot\)” stands for
\[
A\phi = \sum_{q,p \geq 1} A_{q,p} \tilde{E}_q(\tilde{E}_p, \phi)^2 \quad \text{for all } \phi \in \ell^2(\mathbb{Z}^+).\]
The matrix coefficients $A_{q,p} = \langle \hat{E}_q, A \hat{E}_p \rangle_\Omega = \hat{\text{Ric}}_{q,p}(\gamma_{n}^{\sigma,\nu}, \gamma_{m}^{\tau,\mu})$ are of course given by (5.2). The conclusion follows then by directly calculating the operators $A^*A$ and $|A|$.

**Remark.** We observe that the Dixmier traces of the above $|A|$ respectively $\hat{\text{Ric}}(\gamma_{n}^{\sigma,\nu}, \gamma_{m}^{\tau,\mu})$ are given by $4(n \wedge m)P_{c}(\hat{q})$ respectively $4nP_{c}(\hat{q})$ (cf. [2] and the Appendix).

6. The second fundamental tensor of the embedding $L^{K}/K \hookrightarrow \text{Gr}$ and the partially summed normal corrections to $\text{Ric}^{L^{K}/K}$

The principal result of this section, beside a remark on the “minimality” of $L^{K}/K$ in $\text{Gr}(F, F_+)$, will be the determination of the following “normal corrections” to the Ricci tensor of $L^{K}/K$ (compare eq. (4.6)):

$$
\hat{\hat{B}}_{q,p}^{t}(\gamma_{n}^{\sigma,\nu}, \gamma_{m}^{\tau,\mu}) := \frac{1}{2 \sqrt{pq}} \sum_{c \in k} \left( B'(\gamma_{n}^{\sigma,\nu}, \gamma_{m}^{\tau,\mu}), B'(\gamma_{p}^{\sigma,\nu}, \gamma_{q}^{\tau,\mu}) \right)
- \frac{1}{2 \sqrt{pq}} \sum_{c \in k} \left( B'(\gamma_{n}^{\sigma,\nu}, \gamma_{m}^{\tau,\mu}), B'(\gamma_{m}^{\sigma,\nu}, \gamma_{p}^{\tau,\mu}) \right).
\text{(6.1)}
$$

Since our arguments will be based on the formulae obtained in Section 2, we begin by evaluating (2.1) in our situation.

**Lemma 6.1.** (i) For $x$ and $y$ in $\mathfrak{u}_{\text{reg}}(F, F_+) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and

$$
x_{m} = \begin{pmatrix} 0 & -\gamma^{*} \\ \gamma & 0 \end{pmatrix}, \quad y_{h} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},
$$

one has

$$
[y_{h}, x_{m}] = \begin{pmatrix} 0 & -(\beta \gamma - \gamma \alpha)^{*} \\ \beta \gamma - \gamma \alpha & 0 \end{pmatrix}.
$$

(ii) If $x = M_{r}^{\sigma,\nu} = j(\xi_{r}^{\sigma,\nu})$ and $y = M_{r}^{\tau,\mu} = j(\xi_{r}^{\tau,\mu})$ in $\mathfrak{g} = j(Lk)$ with $r \leq s$, then

$$
[(M_{r}^{\sigma,\nu})_{h}, (M_{r}^{\tau,\mu})_{m}] = \begin{pmatrix} 0 & -(\beta \gamma - \gamma \alpha)^{*} \\ \beta \gamma - \gamma \alpha & 0 \end{pmatrix},
$$

where

$$
\beta \gamma - \gamma \alpha = \beta_{s}^{\sigma,\nu} \circ \gamma_{s}^{\sigma,\nu} - \gamma_{r}^{\sigma,\nu} \circ \alpha_{s}^{\tau,\mu}
= \mu^{3}v^{3}a^{\tau}a^{\sigma} \otimes \sum_{k=0}^{s-1} |k - (r + s)\rangle \langle k| - \mu^{3}v^{3}a^{\sigma}a^{\tau} \otimes \sum_{k=s}^{r+s-1} |k - (r + s)\rangle \langle k|.
$$

**Proof.** The first assertion follows by direct calculation. In order to prove (ii), we recall that on the one hand

$$
M_{r}^{\sigma,\nu} = va^{\sigma} \otimes \left( \sum_{k \in \mathbb{Z}} |k \rangle \langle k - r| + v^{2} \sum_{k \in \mathbb{Z}} |k + r\rangle \langle k| \right)
$$

...
by Lemma 3.3, and on the other hand

\[ M_r^{\sigma,v} = \left( \frac{\alpha_r^{\sigma,v} - (\gamma_r^{\sigma,v})^*}{\beta_r^{\sigma,v}} \right) \]

can be written as \( \gamma_r^{\sigma,v} - (\gamma_r^{\sigma,v})^* + \alpha_r^{\sigma,v} + \beta_r^{\sigma,v} \), where, e.g., \( \gamma_r^{\sigma,v} \) is interpreted as a map from \( F \) to itself, that is zero on \( F_- \) and takes values only in \( F_+ \) etc. Decomposition of \( M_r^{\sigma,v} \) into the corresponding pieces then yields

\[ \gamma_r^{\sigma,v} = v^3 a^\sigma \otimes \sum_{k=-r}^{r-1} |k\rangle \langle k+r| = v^3 a^\sigma \otimes \sum_{k=0}^{r-1} |k-r\rangle \langle k| \]

(as was shown in Lemma 3.3),

\[ \alpha_r^{\sigma,v} = v^3 a^\sigma \otimes \sum_{k=0}^{r-1} |k\rangle \langle k+r| + va^\sigma \otimes \sum_{k=r}^{r-1} |k\rangle \langle k-r| \]

and

\[ \beta_r^{\sigma,v} = v^3 a^\sigma \otimes \sum_{k=-r}^{r-1} |k\rangle \langle k+r| + va^\sigma \otimes \sum_{k=0}^{r-1} |k\rangle \langle k-r|. \]

Since by (i) \( [(M_r^{\sigma,v})_h, (M_r^{\sigma,v})_m] \) is fully characterized by \( \beta^{\tau,\mu}_r \circ \gamma_r^{\sigma,v} = \beta^{\tau,\mu}_r \circ \alpha_r^{\tau,\mu} \), we compute (recalling that \( r \leq s \))

\[ \beta^{\tau,\mu}_r \circ \gamma_r^{\sigma,v} = \left( \mu a^\tau \otimes \mu^2 \sum_{k=-s}^{r-1} |k\rangle \langle k+s| + \sum_{k<s}^{r-1} |k\rangle \langle k-s| \right) \circ \left( v^3 a^\sigma \otimes \sum_{l=-r}^{r-1} |l\rangle \langle l+r| \right) \]

\[ = \mu^3 v^3 a^\tau a^\sigma \otimes \sum_{k=0}^{r-1} |k-(r+s)| \langle k| \]

and similarly

\[ \gamma_r^{\sigma,v} \circ \alpha_r^{\tau,\mu} = \mu^3 v^3 a^\sigma a^\tau \otimes \sum_{k=s}^{r+s-1} |k-(r+s)| \langle k|. \]

We notice the simple but useful

**Corollary 6.2.** For all \( p, q \geq 1 \) and \( \epsilon, \epsilon' \) in \( \{1, \ldots, \dim \mathfrak{t}\} \), one has

\[ \sum_{\kappa \in \{1,\ldots,\dim \mathfrak{t}\}} B'(\gamma_p^{\epsilon,\kappa}, \gamma_q^{\epsilon',\kappa}) = 0. \]

**Proof.** Since \( B' \) is symmetric, it is enough to consider the case \( p \leq q \). Then it follows from (2.3), \( s \circ \pi(M^p_{\epsilon,\kappa}) = \gamma_p^{\epsilon,\kappa} \) and from (ii) of the preceding lemma that

\[ B'(\gamma_p^{\epsilon,\kappa}, \gamma_q^{\epsilon',\kappa}) = \left( (M^p_{\epsilon,\kappa})_h^*, (M^p_{\epsilon,\kappa})_m^* \right)_{\mathfrak{m}^*} \]

\[ = \left( \kappa^6 a^\epsilon a^{\epsilon'} \otimes \sum_{k=0}^{p-1} |k-(p+q)| \langle k| - a^{\epsilon'} a^\epsilon \otimes \sum_{k=q}^{p+q-1} |k-(p+q)| \langle k| \right)_{\mathfrak{m}^*} \]

\[ = \kappa^2 B'(\gamma_p^{\epsilon+1}, \gamma_q^{\epsilon'-1}) \]
Thus obviously the sum over \( \kappa \in \{1, i\} \) vanishes. □

Corollary 6.2 implies that (6.1) reduces to

\[
\hat{B}^{\prime}_{q,p}(\gamma^\sigma_n, \gamma^\tau_m) = -\frac{1}{2\sqrt{pq}} \sum_{\epsilon, \kappa} \left[ B^{\prime}(\gamma^\sigma_q, \gamma^\epsilon_m, \kappa), B^{\prime}(\gamma^\tau_p, \gamma^\epsilon_n, \kappa) \right].
\]

(6.2)

**Remark.** The above corollary reflects of course only the fact that for a complex submanifold \( M' \) of a Kähler manifold \( M \) one has

\[ B_x(J_x u, J_x v) = J_x(B_x(u, v)) = -B_x(u, v) \]

for \( x \in M', u, v \in T_xM' \) and \( J_x \) the complex structure in \( x \). (Compare, e.g., [11].)

We also observe the following

**Corollary 6.3.** For all \( p, \epsilon, \kappa \)

\[ B^{\prime}(\gamma^\epsilon_p, \gamma^\epsilon_p) = k^2(a^\epsilon)^2 \otimes \left\{ \sum_{k=0}^{p-1} |k - 2p\rangle\langle k| - \sum_{k=p}^{2p-1} |k - 2p\rangle\langle k| \right\}. \]

**Proof.** By Lemma 6.1 we know that

\[ [(M^\epsilon_p)_b, (M^\epsilon_p)_m] = k^2(a^\epsilon)^2 \otimes \left\{ \sum_{k=0}^{p-1} |k - 2p\rangle\langle k| - \sum_{k=p}^{2p-1} |k - 2p\rangle\langle k| \right\}. \]

So it remains to show that the right-hand side is already in \( m'' \). Since

\[ \{ A \otimes E_m = A \otimes \sum_{l=0}^{m-1} |l - m\rangle\langle l| \mid A \in \mathbb{C}^n, m \geq 1 \} \]

is a total set in \( m' \cong \mathbb{C}^n \otimes \ell^2(\mathbb{Z}_+) \) it suffices to show that

\[ [(M^\epsilon_p)_b, (M^\epsilon_p)_m], A \otimes E_m \} = 0 \]

for all \( A \) and for all \( m \geq 1 \). The last equality follows easily by a straightforward matching and tracing calculation. □

**Remarks.** (i) Corollary 6.3 implies that a “naive” definition of the mean curvature vector leads to a series which is easily seen to be not absolutely convergent

\[
\text{“tr}_m B'' = \sum_{p, \epsilon, \kappa} B^{\prime}(\gamma^\epsilon_p, \gamma^\epsilon_p) = \sum_{p, \epsilon, \kappa} \left( \frac{1}{2p} k^2(a^\epsilon)^2 \otimes \left\{ \sum_{k=0}^{p-1} |k - 2p\rangle\langle k| - \sum_{k=p}^{2p-1} |k - 2p\rangle\langle k| \right\} \right).
\]

This contrasts of course with the fact that in finite dimensions a complex submanifold of a Kähler manifold is always minimal, i.e., the trace of the second fundamental form vanishes at each point therein. On the other hand if we condition the trace by first summing over the indices
corresponding to $\ell^C$, Corollary 6.2 shows that $\sum_{k,\ell} B'(\gamma_{\ell,\ell}, \gamma_{\ell,\ell})$ vanishes for all $p \geq 1$ and thus a fortiori the mean curvature vector
\[
\sum_{p \geq 1} \left( \sum_{k,\ell} B'(\gamma_{\ell,\ell}, \gamma_{\ell,\ell}) \right)
\]
is zero. Therefore we recover, via the conditional trace approach, a form of minimality for $LK/K$ in $Gr(F, F_+)$. Let us point out that $LK/K$ should not be considered as a “weakly” totally geodesic submanifold of $Gr(F, F_+)$, since there is no averaging process involved in the definition of such manifolds.

(ii) Our strategy in the sequel will rely on the following simple observation: since $\{ \gamma_{p',q'} | p' \geq 1, \epsilon' = 1, \ldots, \dim \ell$ and $\kappa' \in \{1, i\}\}$ is a Hilbert basis of $m'$, we can calculate the scalar product of the $m''$-component of two elements $u$ and $v$ in $m$ by the following formula (recall that we are dealing with the real Hilbert space structure of $m$)
\[
\langle u_m, v_m \rangle = \langle u, v \rangle - \sum_{p',q'} \langle u_{p'}, v_{p'} \rangle \langle v_{p'}, u_{p'} \rangle.
\]
The above formula implies for $x, y, z, w$ in $g'$:
\[
\langle B'((\pi(x), \pi(y)), B'((\pi(z), \pi(w))) = \langle \{[y, x_m], [w, z_m]\} - \sum_{p',q',r,s} ([y, x_m], \gamma_{p',q'})([w, z_m], \gamma_{r,s} \rangle). \quad (6.3)
\]
In order to make use of (6.3) we need the following coefficient formula:

**Lemma 6.4.** For all $\sigma, \tau$ and $\mu, \nu$ and for $r \leq s$ one has
\[
\langle (M^{r,s}_{\sigma,\tau})_{\mu,\nu} | \gamma_{p',q'} \rangle = \frac{1}{\sqrt{2p'}} \times 2r \times \delta_{r+s,p'} \times \text{tr}_d ((\sigma')^* [\sigma^*]) \times \text{Re} \{ (\kappa')^3 \mu^3 \nu^3 \}
\]

**Proof.** Using Lemma (6.1), (ii) we find
\[
\langle \gamma_{p',q'} \rangle = \sum_{l=1}^{p'} (\kappa')^3 \cdot (\sigma')^* \langle \gamma_{p',q'} \rangle = 2 \text{Re} \text{tr} \left( ((\kappa')^3 \cdot (\sigma')^* \sum_{l=1}^{p'} (p' - l) \langle \gamma_{p',q'} \rangle \cdot (\mu^3 \nu^3) \right.
\]
\[
\times (a^* a^* \otimes \sum_{k=0}^{r-1} \langle k \rangle - (r + s) \langle k \rangle) \langle a^* a^* \otimes \sum_{k=s}^{r-1} \langle k \rangle \langle k \rangle)
\]
which easily leads to the desired assertion. □

We shall now give the general formula for scalar products of second fundamental vectors but first have to remark that we provisionally replace $\gamma$ by $M$ in the left-hand side in order to partially avoid notational clashes due to the abundance of indices. Since one can view $B'$ as a map from $(g')^{\otimes 2}$ to $m''$ this is consistent with hitherto used notations. Furthermore, let us stress that $a^3$ is a complex number, with $a$ being either 1 or $i$ and $a^\alpha$ etc. are elements of a fixed basis.
Let \( \mathfrak{f} \); in the same vein, \( d \) is another complex number being either 1 or \( i \), whereas the subscript \( d \) of \( \langle \cdot, \cdot \rangle_d \) refers to the dimension of \( \mathbb{C}^d \), the vector space carrying the representation \( \varrho \) of \( K \). We are in a position to state the following

**Lemma 6.5.** For \( \alpha, \beta, \gamma, \Delta \in \{1, \ldots, \dim \mathfrak{f}\} \), for \( a, b, c, d \in \{1, i\} \), and \( r \leq s, t \leq u \) one has

\[
B'(M^\sigma_{\alpha}, M^\beta_{\beta}, M^\gamma_{\gamma}, M^\Delta_{\Delta}) = \delta_{r+s, t+u} \times 2 \text{Re} \left\{ a^3 b^3 c^3 \delta_{r,s} \right\} \times \left[ (r \wedge t) - \text{tr}_d(a^\beta a^\alpha a^\gamma a^\Delta + a^a a^\beta a^\gamma a^\Delta) \right]
\]

**Proof.** (ii) after the proof of Corollary 6.3 implies that

\[
\text{left-hand side} = \left[ \langle (M^\sigma_{\alpha})_b, (M^\gamma_{\gamma})_m \rangle, \langle (M^\Delta_{\Delta})_b, (M^\beta_{\beta})_m \rangle \right]
\]

\[
- \sum_{p', q', k'} \left( \langle (M^\sigma_{\alpha})_b, (M^\beta_{\beta})_m \rangle, \gamma^\sigma_{p', k'} \right) \times \left( \langle (M^\Delta_{\Delta})_b, (M^\gamma_{\gamma})_m \rangle, \gamma^\gamma_{q', k'} \right)
\]

Lemma 6.1 allows us to calculate the first summand in the last right-hand side whereas Lemma 6.4 yields the second one (by a tedious but direct calculation). \( \square \)

This entails immediately the useful

**Corollary 6.6.** For all \( n, m, p, q \in \mathbb{Z}_+ \), \( \hat{B}'_{q,p}(\gamma_{n}^{\sigma}, \gamma_{m}^{\tau}) \) is proportional to \( \delta_{n+q,m+p} \).

**Proof.** Recalling from (6.2) that

\[
\hat{B}'_{q,p}(\gamma_{n}^{\sigma}, \gamma_{m}^{\tau}) = -\frac{1}{2 \sqrt{pq}} \sum_{k, \xi} \left\{ B'(\gamma_{n}^{\sigma}, \gamma_{q}^{\xi}), B'(\gamma_{m}^{\tau}, \gamma_{p}^{\tau}) \right\}
\]

the assertion follows from Lemma 6.5 and the symmetry of \( B' \). \( \square \)

In order to calculate \( \hat{B}'_{q,p}(\gamma_{n}^{\sigma}, \gamma_{m}^{\tau}) \) we first observe that we can assume \( n \leq m \) without loss of generality, since

\[
\text{Ric}_{q,p}(\gamma_{n}^{\sigma}, \gamma_{m}^{\tau}) = \hat{\text{Ric}}_{q,p}(\gamma_{m}^{\tau}, \gamma_{n}^{\sigma})
\]

by the usual symmetries of the Riemann curvature tensor. Corollary 6.6 then implies that we have to consider only the following three cases:

- Case I: \( q \geq n, p \geq m \),
- Case II: \( q \geq n, p \leq m \),
- Case III: \( q \leq n, p \leq m \). We now achieve the goal of this section.

**Lemma 6.7.** For all \( p, q \) and for \( n \leq m \), the following identity holds

\[
\hat{B}'_{q,p}(\gamma_{n}^{\sigma}, \gamma_{m}^{\tau}) = \left\{ -\frac{4}{\sqrt{pq}} \times (n \wedge p) \times P_c(q) + \frac{2(n \wedge p)(m \wedge q)}{\sqrt{pq} \times (p + m)} \times P_c(\text{ad}) \right\}
\]

\[
\times \delta_{n+q,m+p} \times \delta_{\nu, \mu} \times \delta_{n, \tau}.
\]
**Proof.** The identity follows by distinguishing the Cases I–III and observing that the results can be cast as stated in the lemma. Since the calculations are similar in the three cases we consider only the “asymptotic situation for fixed $n$ and $m$,” i.e., Case I: $q \gg n$ and $p \gg m$. We nevertheless point out that the other two cases demand more than merely copying the calculations of this case.

Since $(-2\sqrt{pq}) \cdot \hat{B}_{q,p}(y_n^{\sigma,\nu}, y_{m}^{\tau,\mu}) = \sum_{\kappa,\xi} \langle B'(y_n^{\sigma,\nu}, y_q^{\varepsilon,\kappa}), B'(y_{m}^{\tau,\mu}, y_p^{\varepsilon,\xi}) \rangle$. Lemma 6.5 gives us an explicit formula for the summands in the right-hand side

$$\begin{align*}
\langle B'(y_n^{\sigma,\nu}, y_q^{\varepsilon,\kappa}), B'(y_{m}^{\tau,\mu}, y_p^{\varepsilon,\xi}) \rangle & = \delta_{n+q,m+p} \times 2 \cdot \text{Re} [v^{3} \kappa^{3} \mu^{3} \nu^{3}] \times [\text{tr}_d(a^{\nu} a^{\xi} a^{\xi} + a^{\nu} a^{\xi} a^{\nu})] \\
& - \delta_{n+q,m+p} \times \left( \frac{2mn}{n+q} \right) \times [a^{\nu}, a^{\xi}] \right)_{d} \\
& \times \sum_{\kappa'} \left( \text{Re}[\kappa^{3} v^{3} \kappa^{3}] \cdot \text{Re}[\kappa^{3} \mu^{3} \kappa^{3}] \right)
\end{align*}$$

Observing that first

$$\sum_{\xi} \text{tr}_d \left( (a^{\nu} a^{\xi} + a^{\xi} a^{\xi}) a^{\nu} a^{\xi} \right) = \text{tr}_d \left( (a^{\nu})^\dagger \left( \sum_{\xi} (-a^{\nu} a^{\xi}) a^{\xi} \right) \right) + \text{tr}_d \left( (a^{\xi})^\dagger \left( \sum_{\xi} (-a^{\nu} a^{\xi}) a^{\nu} \right) \right)$$

$$= 2 \delta_{\sigma,\tau} \cdot P_C(q),$$

secondly

$$\sum_{\xi} \langle a^{\nu}, -\text{ad}(a^{\xi}) \circ \text{ad}(a^{\nu}) a^{\nu} \rangle_{d} = P_C(\text{ad}) \cdot \delta_{\sigma,\tau}$$

and finally

$$\sum_{\kappa,\kappa' \in [1,i]} \left( \text{Re}[\kappa^{3} v^{3} \kappa^{3}] \cdot \text{Re}[\kappa^{3} \mu^{3} \kappa^{3}] \right) = 2 \cdot \delta_{\nu,\mu}$$

we conclude that

$$\sum_{\xi,\kappa} \langle B'(y_n^{\sigma,\nu}, y_q^{\varepsilon,\kappa}), B'(y_{m}^{\tau,\mu}, y_p^{\varepsilon,\xi}) \rangle = \delta_{n+q,m+p} \times 4n \times \delta_{\nu,\mu} \times 2 P_C(q) \delta_{\sigma,\tau}$$

$$- \delta_{n+q,m+p} \times \left( \frac{2mn}{n+q} \right) \times P_C(\text{ad}) \times \delta_{\sigma,\tau} \times 2 \delta_{\nu,\mu}.$$
Proposition 7.1. For all $n$ of divergence behaviour of the trace corresponding to a hypothetical unconditional Ricci tensor exists and show that it is proportional to the metric induced from the Grassmannian, i.e., that
\[
\text{Dividing by } -2\sqrt{pq} \text{ yields}
\]
\[
\hat{B}'_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu}) = \left\{ -\frac{4}{\sqrt{pq}} \times n \times P_C(\varrho) + \frac{2mn}{\sqrt{pq}(n+q)} \times P_C(\text{ad}) \right\} \times \delta_{n+q,m+p} \times \delta_{\sigma,\tau} \times \delta_{\nu,\mu}.
\]
Since in Case I, $q \geq n$ and $p \geq m$ (and in all cases $n \leq m$), this formula yields the assertion in this case. 

Remark. (i) We just notice that, in case II, one uses the easily proved identity
\[
2 \sum c \operatorname{tr}(a^e a^\sigma a^\tau) = [2 P_C(\varrho) - P_C(\text{ad})] \times \delta_{\sigma,\tau}.
\]
(ii) Since the first summand in the above formula for $\hat{B}'_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu})$ equals (minus) the extrinsic contribution $\hat{\text{Ric}}_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu})$ (compare Proposition 5.1) and the second will be shown to be of trace class in Section 7 below, Corollary 5.2 allows us to observe that the trace norm of $\hat{B}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu})$ is logarithmically divergent in general. This can be also ascertained by taking for instance $m = n, \sigma = \tau, \mu = \nu$: one easily finds for its Dixmier trace the value $-4n P_C(\varrho)$ (compare also the Appendix).

7. The Ricci curvature of $L/K$

In this section we prove that the (conditional) Ricci curvature of the based loop space $L/K$ exists and show that it is proportional to the metric induced from the Grassmannian, i.e., that $L/K$ is a Kähler–Einstein manifold. Furthermore we precisely describe in Proposition 7.3 the divergence behaviour of the trace corresponding to a hypothetical unconditional Ricci tensor of $L/K$.

We first combine the achievements of Sections 5 and 6.

Proposition 7.1. For all $n, m, p, q$ in $\mathbb{Z}_+, \sigma, \tau$ in $\{1, \ldots, \dim \mathfrak{g}\}$ and $\mu, \nu$ in $\{1, i\}$ one has
\[
\text{Ric}_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu}) = \frac{2(n \wedge p) \cdot (m \wedge q)}{\sqrt{pq} \cdot (p+m)} \times \delta_{n+q,m+p} \times (P_C(\text{ad}) \times \delta_{\sigma,\tau} \times \delta_{\nu,\mu}).
\] (7.1)

Proof. Let us recall from (4.6), (5.2) and (6.1)
\[
\text{Ric}_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu}) = \text{Ric}_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu}) + \hat{B}'_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu}).
\]
Considering first the case $n \leq m$, Lemmata 5.1 and 6.7 yield
\[
\text{Ric}_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu}) = \frac{2(n \wedge p) \cdot (m \wedge q)}{\sqrt{pq} \cdot (p+m)} \times \delta_{n+q,m+p} \times (P_C(\text{ad}) \times \delta_{\sigma,\tau} \times \delta_{\nu,\mu}).
\]
The case $m < n$ yields
\[
\text{Ric}_{q,p}(\gamma^\sigma_{\mu}, \gamma^\tau_{\nu}) = \frac{2(m \wedge q) \cdot (n \wedge p)}{\sqrt{pq} \cdot (q+n)} \times \delta_{n+q,m+p} \times (P_C(\text{ad}) \times \delta_{\sigma,\tau} \times \delta_{\nu,\mu}),
\]
which proves (7.1) for all $n, m$ in $\mathbb{Z}_+$. \(\square\)

We can now prove that the doubly-infinite matrices $(\widehat{\text{Ric}}_{q, p})_{q, p \geq 1}$ do indeed define trace class operators on $l^2(\mathbb{Z}_+)$:

**Lemma 7.2.** For all indices $n, m, \sigma, \tau, \nu, \mu$ the operator $\widehat{\text{Ric}}'_{\sigma, \tau}(y^{\sigma, \nu}_n, y^{\tau, \mu}_m)$ is of trace class and

$$
\|\widehat{\text{Ric}}'_{\sigma, \tau}(y^{\sigma, \nu}_n, y^{\tau, \mu}_m)\|_1 = \left( \sum_{p \geq 1 \wedge (n - m + 1)} 2(n \wedge p)((m - n) + (n \wedge p)) / \sqrt{p(p + m - n)(p + m)} \right) \times (P_C(\text{ad}) \times \delta_{\nu, \mu} \times \delta_{\sigma, \tau})
$$

$$
\leq 2 \times 2\sqrt{nm} \times (P_C(\text{ad}) \times \delta_{\nu, \mu} \times \delta_{\sigma, \tau}).
$$

**Proof.** Analogously to the proof of Corollary 5.2 we look upon the operator

$$
A := \widehat{\text{Ric}}'_{\sigma, \tau}(y^{\sigma, \nu}_n, y^{\tau, \mu}_m)
$$
on $l^2(\mathbb{Z}_+)$ as

$$
A = \sum_{q, p \geq 1} A_{q, p} \widehat{E}_q(\widehat{E}_p, \cdot),
$$

with $A_{q, p} = \widehat{\text{Ric}}'_{q, p}(y^{\sigma, \nu}_n, y^{\tau, \mu}_m)$ given by formula (7.1). A short calculation shows that

$$
A = \sum_{p \geq 1 \wedge (n - m + 1)} \alpha(p) \widehat{E}_{p+m-n}(\widehat{E}_p, \cdot),
$$

with

$$
0 \leq \alpha(p) = \frac{2(n \wedge p)((m - n) + (n \wedge p))}{\sqrt{p(p + m - n)(p + m)}} \times (P_C(\text{ad}) \times \delta_{\nu, \mu} \times \delta_{\sigma, \tau}).
$$

Furthermore one finds that

$$
|A| = \sqrt{A^*A} = \sum_{p \geq 1 \wedge (n - m + 1)} \alpha(p) \widehat{E}_p(\widehat{E}_p, \cdot).
$$

Thus

$$
\|A\|_1 = \text{tr}_{l^2}(\sqrt{A^*A}) = \sum_{p \geq 1 \wedge (n - m + 1)} \alpha(p)
$$

$$
= \left( \sum_{p \geq 1 \wedge (n - m + 1)} \frac{2(n \wedge p)((m - n) + (n \wedge p))}{\sqrt{p(p + m - n)(p + m)}} \right) \times (P_C(\text{ad}) \times \delta_{\nu, \mu} \times \delta_{\sigma, \tau}).
$$

Denoting the first factor of the last right-hand side as $t(n, m)$, we proceed to estimate it by distinguishing the cases $m \geq n$ and $m < n$. Since both are similar we work out here only the first one. Thus, in view of $1 \vee (n - m + 1) = 1$, we find

$$
t(n, m) = \sum_{p=1}^n \frac{2p(m - n + p)}{\sqrt{p(p + m - n)(p + m)}} + \sum_{p>n} \frac{2nm}{\sqrt{p(p + m - n)(p + m)}},
$$

The first sum can be easily estimated by $2n \leq 2\sqrt{nm}$. The second sum can be estimated by

$$
2nm \sum_{p>n} \frac{1}{p(p + m)} \leq 2nm \int_n^{+\infty} \frac{dx}{x(x + m)} = 2n \log \left( \frac{n + m}{n} \right) \leq 2n \sqrt{\frac{n}{m}}.
$$
Proposition 7.3. The operator $A = A(\gamma_n^{\sigma,\tau}, \gamma_m^{\sigma,\tau}) : m' \cong t \otimes \mathbb{R} C \otimes \mathbb{R} \ell^2(\mathbb{Z}_+) \to m'$, defined by $A(\xi) = R'(\gamma_n^{\sigma,\tau}, \xi) : \gamma_m^{\sigma,\tau},$ has logarithmically divergent trace.

Proof. Using the Gauss equation in the form (4.4) we can describe the operator $A$ by its matrix elements:

$$A(q,\epsilon',\kappa'),(p,\epsilon,\kappa) = \left\{ R'(\gamma_n^{\sigma,\tau}, \tilde{\gamma}_p^{\epsilon,\kappa}) \cdot \gamma_m^{\epsilon',\kappa'}, \tilde{\gamma}_q^{\epsilon,\kappa} \right\}$$

$$+ \left\{ B'(\gamma_n^{\sigma,\tau}, \gamma_m^{\epsilon',\kappa'}) \cdot (\gamma_p^{\epsilon,\kappa} \cdot \tilde{\gamma}_q^{\epsilon,\kappa'}) \right\}$$

$$- \left\{ B'(\gamma_n^{\sigma,\tau}, \tilde{\gamma}_q^{\epsilon',\kappa'}) \cdot (\gamma_m^{\epsilon,\kappa} \cdot \gamma_p^{\epsilon,\kappa}) \right\}.$$

(7.2)

By equation (5.3) one finds that the first summand of the right-hand side of (7.2) is given by

$$\frac{1}{\sqrt{pq}} \cdot \text{Re}[\nu^3 \mu^3 \kappa^3 \kappa^3] \cdot \text{tr}_d(a^\sigma a^\alpha a^\alpha a^\alpha + a^\gamma a^\alpha a^\alpha) \cdot (n \land p \land m \land q) \cdot \delta_{n+m,p+q}$$

plus a term proportional to $\delta_{n+m,p+q}$. Since the latter yields—for fixed $n$ and $m$—a finite rank operator it will not be relevant for the divergence of the trace of $A$. We formalize this by defining the following condition:

$$p \lor q \geq 2 \cdot (n \lor m), \quad (*)$$

which is of course fulfilled for all but a finite number of indices. For these indices $\delta_{n+m,p+q}$ is obviously always zero. Furthermore for indices satisfying ($*$) and $n + q = m + p$, one has $n \leq q$ and $m \leq p$, i.e., the factor $(n \land p \land m \land q) \times \delta_{n+q,m+p}$ always equals $(n \land m) \times \delta_{n+q,m+p}$.

Lemma 6.5 yields now that the second term of the right-hand side of (7.2) is proportional to $\delta_{n+m,p+q}$ for all indices and thus vanishes for indices fulfilling ($*$).

In order to calculate the third term of the right-hand side of (7.2) we observe first that Lemma 6.5 implies that it is always proportional to $\delta_{n+q,m+p}$. Under the condition ($*$), we have $n \leq q$ and $m \leq p$ and can thus directly apply the explicit formula of Lemma 6.5 for these indices

$$\left\{ B'(\gamma_n^{\sigma,\tau}, \tilde{\gamma}_q^{\epsilon',\kappa'}) \cdot B'(\gamma_m^{\epsilon,\kappa} \cdot \gamma_p^{\epsilon',\kappa}) \right\}$$

$$= \left( \frac{n \land m}{\sqrt{pq}} \right) \times \text{Re}[\nu^3 \mu^3 \kappa^3 \kappa^3] \times \text{tr}_d(a^\sigma a^\alpha a^\alpha a^\alpha + a^\gamma a^\alpha a^\alpha) \times \delta_{n+q,m+p}$$

$$+ C(\sigma, \tau, \nu, \mu, \epsilon', \kappa, \kappa') \times \left( \frac{nm}{\sqrt{pq}(p + m)} \right) \times \delta_{n+q,m+p},$$

where $C$ is a constant depending on the indicated finite parameter set.
The second term of the last right-hand side is easily shown to define a trace class operator on \( m' \cong \mathfrak{g} \otimes \mathbb{R} \ell^2(\mathbb{Z}_+) \) by an obvious extension of Lemma 7.2.

We arrive at the conclusion that \( A \) is given by a finite sum of trace class operators plus an operator \( A' \) whose matrix entries are zero for \( p \lor q < 2 \cdot (n \lor m) \) and as follows for \( p, q \) fulfilling (*)

\[
A'_{(q, e', s'), (p, e, s)} = \delta_{n+q, m+p} \times \left( \frac{n \land m}{\sqrt{pq}} \right) \times \text{Re} \left\{ v^3 p^3 k^3 \sum_{l} \right\}
\]

\[
\times \text{tr}_d (a^\sigma a^\tau a^\epsilon a^\tau + a^\tau a^\sigma a^\epsilon a^\tau - a^\sigma a^\tau a^\epsilon a^\tau - a^\tau a^\sigma a^\epsilon a^\tau)
\]

\[
= \delta_{q, p+m-n} \times \left( \frac{n \land m}{\sqrt{pq}} \right) \times \text{Re} \left\{ v^3 p^3 k^3 \sum_{l} \right\}
\]

\[
\times \text{tr}_d ([a^\sigma, a^\tau] \cdot [a^\epsilon, a^\tau])
\]

\[
= \delta_{q, p+m-n} \times \left( \frac{n \land m}{\sqrt{pq}} \right) \times \text{Re} \left\{ v^3 p^3 k^3 \sum_{l} \right\}
\]

\[
\times \langle \text{ad}([a^\sigma, a^\tau]) a^\epsilon, a^\tau \rangle_d.
\]

A straightforward calculation in the spirit of the proof of Lemma 7.2 gives us

\[
|A'| = \sqrt{(A')^*(A')} = T \otimes \text{Id}_\mathbb{C} \otimes \sum_{p \geq n + (m/n)} \left( \frac{m \land n}{\sqrt{p(p + m - n)}} \right) \tilde{E}_p (\tilde{E}_p, \cdot)_{\ell^2}
\]

(as an endomorphism of \( m' \cong \mathfrak{g} \otimes \mathbb{R} \mathbb{C} \otimes \mathbb{R} \ell^2(\mathbb{Z}_+) \)), where \( T : \mathfrak{g} \to \mathfrak{g} \) is the square root of the non-negative operator \( S^* S \) with \( S = \text{ad}([a^\sigma, a^\tau]) \).

Setting, e.g., \( n = m \) and \( \sigma \neq \tau \) we find

\[
\|A'\|_1 = 2n \times \text{tr}_\mathfrak{g}(T) \times \sum_{p \geq 2n} \frac{1}{p},
\]

which is logarithmically divergent, since \( \text{tr}_\mathfrak{g}(T) > 0 \) \( \square \)

Remarks. (i) The appearance of \([a^\epsilon, a^\tau]\) in all matrix elements of \( A' \) explains why the conditioning of the trace of \( A \) yields a trace class operator.

(ii) The operator \( |A'| \) above has Dixmier trace equal to \( 2(n \land m) \text{tr}_\mathfrak{g}(T) \) (see [2] and the Appendix).

We apply now Lemma 7.2 to prove the trace class property for general tangent vectors to \( LK/K \).

Lemma 7.4. Let \( u, v \) be in \( m'_0 = s(T_{F_2} LK \cdot F_2), \) then \( \tilde{\text{Ric}}(u, v) \) is of trace class on \( \ell^2(\mathbb{Z}_+) \).

Proof. Describing \( u \) and \( v \) as \( s \circ \pi \circ j(x) \) and \( s \circ \pi \circ j(y) \), respectively, for \( x \) and \( y \) in \( L_0 \mathfrak{g} = \{ \xi \in C^\infty(S^1, \mathfrak{g}) \mid \int \xi = 0 \} \), we recall from Section 3 that \( x = \sum x_n e_n \) and \( y = \sum y_n e_n \). Since \( x \) and \( y \) are smooth, we have

\[
\sum_{n, \sigma, \tau} |x_n|^2 |n|^{2k} < \infty \quad \forall k \geq 0
\]
and similarly for $y$. Recalling that $s \circ \pi \circ f : L_0 \to m'_0$ is an isomorphism, we find

$$u = \sum_{n} x_{n}^{\sigma, v} (s \circ \pi \circ f(\xi_{m}^{\sigma, v})) = \sum_{n} x_{n}^{\sigma, v} y_{n}^{\sigma, v}$$

and $v = \sum y_{m}^{\sigma, v} y_{m}^{\tau, \mu}$. It follows that

$$\|\tilde{\text{Ric}}'(u, v)\|_1 = \left\| \frac{1}{n} \sum_{n, \sigma, v} \sum_{m, \tau, \mu} x_{m}^{\sigma, v} y_{m}^{\tau, \mu} \tilde{\text{Ric}}'(y_{n}^{\sigma, v}, y_{m}^{\tau, \mu}) \right\|_1 \leq \sum_{n, \sigma, v} \sum_{m, \tau, \mu} |x_{n}^{\sigma, v}| |y_{m}^{\tau, \mu}| \|\tilde{\text{Ric}}'(y_{n}^{\sigma, v}, y_{m}^{\tau, \mu})\|_1$$

Using Lemma 7.2 we can now estimate

$$\|\tilde{\text{Ric}}'(u, v)\|_1 \leq 4 \times P_{c}(\text{ad}) \times \left( \sum_{\sigma, v} \left( \left( \sum_{n} |x_{n}^{\sigma, v}| \sqrt{n} \right) \cdot \left( \sum_{m} |y_{m}^{\sigma, v}| \sqrt{m} \right) \right) \right),$$

which is finite for $x$ and $y$ in $L\mathfrak{k}$. \[\square\]

**Remark.** It seems to be an interesting question whether it is possible to extend the latter property to the closure of the tangent space of $LK/K$ in Gr$(F, F_+)$, i.e., to have the slightly stronger estimate

$$\|\tilde{\text{Ric}}'(u, v)\|_1 \leq \text{cst} \times \sqrt{\sum_{n, \sigma, v} \left( \sum_{n} |x_{n}^{\sigma, v}|^2 n \right) \cdot \sqrt{\sum_{m, \tau, \mu} \left( \sum_{m} |y_{m}^{\tau, \mu}|^2 m \right)}} \leq \text{cst} \times \|u\|_m \cdot \|v\|_m.$$

The trace class property being established, it is now possible to calculate the Ricci tensor of $LK/K$:

**Theorem 7.5.** The Ricci tensor of the based loop space $LK/K$ is equal to $P_{c}(\text{ad})$ times the metric induced from the embedding into the Grassmannian, i.e.,

$$\text{Ric}^{LK/K} = P_{c}(\text{ad}) \cdot g^{LK/K}.$$

**Proof.** By the $LK$-invariance of the Ricci tensor, we only have to consider the point $F_+$ in $LK \cdot F_+ \cong LK/K$. Using the notation of the proof of Lemma 7.2 we have

$$\text{Ric}'(y_{n}^{\sigma, v}, y_{m}^{\tau, \mu}) = \text{tr}_{C_{2}(A)}(\tilde{\text{Ric}}'(y_{n}^{\sigma, v}, y_{m}^{\tau, \mu})) = \text{tr}_{C_{2}(A)}(A) = \sum_{p \geq 1} (\sum_{p \geq 1} \alpha(p) \tilde{E}_{p+m-n}(\tilde{E}_{p+1} \cdot \tilde{E}_{p}) \cdot \delta_{m,n}) \cdot \left( \sum_{p \geq 1} \frac{2(n \wedge p)^2}{p(p + n)} \right) \cdot \left( P_{c}(\text{ad}) \cdot \delta_{v,\mu} \cdot \delta_{\sigma, z} \right).$$

A simple telescopic series argument now shows that

$$\sum_{p \geq 1} \frac{2(n \wedge p)^2}{p(p + n)} = 2n$$
yielding
\[ \text{Ric}'(\gamma_n^{\sigma,v}, \gamma_m^{\tau,\mu}) = P_C(\text{ad}) \cdot g^m(\gamma_n^{\sigma,v}, \gamma_m^{\tau,\mu}). \]

Then, upon taking \( u = \sum x_n^{\sigma,v} \gamma_n^{\sigma,v} \) and \( v = \sum y_m^{\tau,\mu} \gamma_m^{\tau,\mu} \) in \( m_0 = s(T_{cK}LK/K) \) as in the preceding lemma, it follows that
\[ \text{Ric}'(u, v) = \text{tr} \left( \sum x_n^{\sigma,v} y_m^{\tau,\mu} \text{Ric}'(\gamma_n^{\sigma,v}, \gamma_m^{\tau,\mu}) \right) = \sum_{n, \sigma, v} (x_n^{\sigma,v} y_n^{\sigma,v} \cdot 2n \cdot P_C(\text{ad})) \]
\[ = P_C(\text{ad}) \cdot g^m(u, v). \]

Remarks. (i) Since the matrices \( a^\ell \) are normalized with respect to the ad-invariant metric \( \langle \cdot, \cdot \rangle_d \), which is proportional but not necessarily equal to the metric induced from the Killing form of \( \mathfrak{k} \), the same holds true for the resulting “Casimir element” \( C = -\sum_{\ell=1}^{\dim \mathfrak{k}} a^\ell \otimes a^\ell \), i.e., it would be more appropriate to refer to this Casimir element by \( C_0 \).  

(ii) For \( K = SU(d) \) and \( \mathfrak{g} \) the fundamental representation on \( \mathbb{C}^d \), one has \( P_C(\text{ad}) = 2d \).

Appendix

In this Appendix we give the definition of linear divergence (see Section 1) and recall that of logarithmic divergence.

Linear divergence

One might abstract from the details of the operator \( A \) defined by equation (1.14) in Section 1 and consider the following more general class of operators.

A bounded linear operator \( T \) on a separable Hilbert space is said to have “linearly diverging trace” (or, briefly, is “linearly divergent”) if there exists \( \lambda > 0 \) in the spectrum of \( |T| \) such that \( \dim \text{Eig}(|T|, \lambda) = \infty \).  

Obviously such an operator cannot be compact and thus is in no Schatten class \( \mathcal{L}^p \) (see, e.g., [17] or [2] for their definition for \( 1 \leq p < \infty \)). Thus regularization methods connected with the \( \mathcal{L}^p \)-property are not available for such operators. On the other hand the zeta function
\[ \zeta_{|T|}(s) = \sum_n \lambda_n^{-s}, \]
where \( \{\lambda_n\} \) are the non zero elements of the point spectrum of \( |T| \) repeated according to their multiplicity, diverges for all \( s \) in the complex plane, which seems to rule out the “zeta function approach” as well.

Logarithmic divergence

Consider the “interpolation ideal” defined by
\[ \mathcal{L}^{(1, \infty)}(H) = \{ T \in \mathcal{K}(H) \mid \sigma_N(T) = O(\log N) \} \]
where $\mathcal{K}(H)$ denotes the closed $C^*$-ideal of compact operators on a (separable) Hilbert space $H$, with $\sigma_N(T) = \sum_{n=0}^{N-1} \mu_n$, where $\{\mu_n\}_{n=0}^\infty$ denotes the sequence of eigenvalues of $|T|$ arranged in decreasing order and repeated according to their multiplicity (see [2]).

The elements of $\mathcal{L}^{(1,\infty)}(H)$ are called “logarithmically divergent operators” (or operators with “logarithmically divergent trace”). We further recall that if $T$ is positive then, upon defining its associated Riemann’s Zeta function

$$\zeta_T(s) := \text{tr}(T^s) = \sum_{n=0}^\infty \mu_n^s$$

(for Re $s > 1$), the Hardy–Littlewood Tauberian theorem states that $L = \lim_{s \to 1^+} (s - 1)\zeta_T(s)$ ($= \text{Res}_{s=1} \zeta_T(s)$) if after analytic continuation $s = 1$ is a simple pole of $\zeta_T(s)$ exists if and only if $\hat{L} = \lim_{N \to \infty} (1/\log N) (\sum_{n=0}^{N-1} \mu_n)$ exists and then $\hat{L} = L$. One then finds that this common value equals the “Dixmier trace” $\text{tr}_\omega(T)$, independently of the limiting procedure $\omega$ used to define the latter. In other words, $T$ is “measurable” ([2, pp. 307/308]).

Also observe that if $K$ is a finite dimensional Hilbert space, then $\mathcal{B}(K) \otimes \mathcal{L}^{(1,\infty)}(H) = \mathcal{L}^{(1,\infty)}(K \otimes H)$. In the left-hand side any cross norm is employed in defining the topological tensor product (see, e.g., [18, IV.2]) whereas for an element of the form $S \otimes A$, one has

$$\|S \otimes A\|_{\mathcal{L}^{(1,\infty)}} = \|S\|_{\mathcal{L}^1} \|A\|_{\mathcal{L}^{(1,\infty)}} = (\text{tr} |S|) \cdot \left( \sup_{N \geq 2} \frac{\sigma_N(|A|)}{\log N} \right),$$

upon taking the trace norm on the finite dimensional space $\mathcal{B}(K)$, and the standard norm on $\mathcal{L}^{(1,\infty)}(H)$.

In this paper we proved that $\hat{\text{Ric}}, \hat{\text{B}}, \text{A}'$ (whence $\text{Ric}'$) are logarithmically divergent (the operators $\text{A}'$ and $\text{Ric}'$ are dealt with by the preceding observation).

As for the actual values of the Dixmier traces (of positive operators) given in the main text, they are easily obtained from observing that

$$\lim_{N \to \infty} \frac{1}{\log N} \left( \sum_{n=0}^{N-1} \frac{1}{\sqrt{n(n+c)}} \right) = 1$$

for all $n_0 > (0 \vee -c)$.

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Grassmannian embedding of loop groups


