Journal of Mathematical Analysis and Applications **250,** 372–374 (2000)
doi:10.1006/jmaa.2000.7068, available online at http://www.idealibrary.com on IDEAL®

NOTE

A Matrix Trace Inequality

Xiaojing Yang

Department of Mathematics, Tsinghua University, Beijing, People's Republic of China

Submitted by William F. Ames

Received January 4, 2000

1. INTRODUCTION

Recently, Chang [1] proved a matrix trace inequality for Hermitian matrices A and B,

$$\operatorname{tr}(AB)^{2^{\kappa}} \leq \operatorname{tr}(A^{2^{\kappa}} B^{2^{\kappa}}),$$

where κ is an integer, which is a continuation of the work of Bellman [2], Coope [3], Neudecker [4], and Yang [5].

The main results of this paper are the following inequalities:

THEOREM 1. *Let A and B be positive semidefinite matrices of the same order; then for $n = 1, 2, \ldots$*

$$0 \leq \operatorname{tr}(AB)^{2n} \leq (\operatorname{tr} A)^2 (\operatorname{tr} A^2)^{n-1} (\operatorname{tr} B^2)^n, \tag{1}$$

$$0 \leq \operatorname{tr}(AB)^{2n+1} \leq (\operatorname{tr} A)(\operatorname{tr} B)(\operatorname{tr} A^2)^n (\operatorname{tr} B^2)^n. \tag{2}$$

COROLLARY 1. *If A and B are defined in Theorem 1, then*

$$0 \leq \operatorname{tr}(AB)^n \leq (\operatorname{tr} A)^n (\operatorname{tr} B)^n. \tag{3}$$

2. LEMMAS

LEMMA 1 [3]. *If A and B are positive semidefinite matrices of the same order, then*

$$0 \leq \operatorname{tr}(AB) \leq \operatorname{tr} A \cdot \operatorname{tr} B. \tag{4}$$

0022-247X/00 $35.00

LEMMA 2. *If A and B are positive semidefinite matrices of the same order, then for $n = 1, 2, \ldots, (AB)^n B$ and $(BA)^n B$ are positive semidefinite matrices.*

Proof. Let A and B be Hermitian matrices of the same order. If A is positive semidefinite then $(AB)^{2n}A = (AB)^n A (BA)^n$ is clearly positive semidefinite. If B is positive semidefinite then $(AB)^{2n+1}A = (AB)^n ABA (BA)^n$ is clearly positive semidefinite. Therefore, if both A and B are positive semidefinite $(AB)^k A$ is positive semidefinite, $k = 1, 2, \ldots$. ∎

3. PROOF OF THEOREM

Proof of Theorem 1. From Lemma 1, Lemma 2, and the equality $\operatorname{tr}(CD) = \operatorname{tr}(DC)$ for any square matrices C, D of the same order, we obtain

$$\operatorname{tr}(AB)^{2n} = \operatorname{tr}\big(A(BA)^{2n-1}B\big) \le (\operatorname{tr} A)\big(\operatorname{tr}(BA)^{2n-1}B\big) \tag{5}$$

$$\begin{aligned}\operatorname{tr}\big((BA)^{2n-1}B\big) &= \operatorname{tr}\big(B(AB)^{2(n-1)}AB\big) = \operatorname{tr}\big((AB)^{2(n-1)}AB^2\big)\\ &\le \operatorname{tr}\big((BA)^{2(n-1)}A\big)\operatorname{tr} B^2 = \operatorname{tr}\big(A(BA)^{2(n-1)-1}BA\big)\operatorname{tr} B^2\\ &= \operatorname{tr}\big((BA)^{2(n-1)-1}BA^2\big)\operatorname{tr} B^2\\ &\le \operatorname{tr}\big((BA)^{2(n-1)-1}B\big)\operatorname{tr} A^2 \operatorname{tr} B^2.\end{aligned} \tag{6}$$

By (5) and (6), we obtain

$$\operatorname{tr}\big((BA)^{2n-1}B\big) \le \operatorname{tr}\big((BA)^{2(n-1)-1}B\big)\operatorname{tr} A^2 \operatorname{tr} B^2,$$

which means

$$\begin{aligned}\operatorname{tr}\big((BA)^{2n-1}B\big) &\le \operatorname{tr}(BAB)(\operatorname{tr} A^2)^{n-1}(\operatorname{tr} B^2)^{n-1}\\ &= \operatorname{tr}(AB^2)(\operatorname{tr} A^2)^{n-1}(\operatorname{tr} B^2)^{n-1}\\ &\le \operatorname{tr} A(\operatorname{tr} A^2)^{n-1}(\operatorname{tr} B^2)^{n}.\end{aligned} \tag{7}$$

From (5) and (7) we obtain (1).

Similarly,

$$\operatorname{tr}(AB)^{2n+1} = \operatorname{tr}\big(A(BA)^{2n}B\big) \le \operatorname{tr} A \operatorname{tr}\big((BA)^{2n}B\big) \tag{8}$$

$$\begin{aligned}\operatorname{tr}\big((BA)^{2n}B\big) &= \operatorname{tr}\big(B(AB)^{2n-1}AB\big) = \operatorname{tr}\big((AB)^{2n-1}AB^2\big)\\ &\le \operatorname{tr}\big((AB)^{2n-1}A\big)\operatorname{tr} B^2 = \operatorname{tr}\big(A(AB)^{2(n-1)}BA\big)\operatorname{tr} B^2\\ &= \operatorname{tr}\big((BA)^{2(n-1)}BA^2\big)\operatorname{tr} B^2 \le \operatorname{tr}\big((BA)^{2(n-1)}B\big)\operatorname{tr} A^2 \operatorname{tr} B^2.\end{aligned} \tag{9}$$

By (8) and (9), we obtain

$$\operatorname{tr}\left((BA)^{2n}B\right) \leq \operatorname{tr} A^2 \operatorname{tr} B^2 \operatorname{tr}\left((BA)^{2(n-1)}B\right),$$

which means

$$\operatorname{tr}\left((BA)^{2n}B\right) \leq (\operatorname{tr} A^2)^n (\operatorname{tr} B^2)^n \operatorname{tr} B. \tag{10}$$

From (8) and (10), we obtain (2). ∎

Remark. By Lemma 1, $0 \leq \operatorname{tr} A^2 \leq (\operatorname{tr} A)^2$, $0 \leq \operatorname{tr} B^2 \leq (\operatorname{tr} B)^2$, this, together with (1) and (2), we obtain (3).

ACKNOWLEDGMENT

The author thanks the referee for his valuable suggestions.

REFERENCES

1. D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999), 721–725.
2. R. Bellman, Some inequalities for positive definite matrices, *in* "General Inequalities 2, Proceedings 2nd International Conference on General Inequalities" (E. F. Beckenbach, Ed.), pp. 89–90, Birkhäuser, Basel, 1980.
3. I. D. Coope, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994), 999–1001.
4. H. Neudecke, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992), 302–303.
5. Y. Yang, A matrix trace inequality, *J. Math. Anal. Appl.* **133** (1988), 573–574.