NOTE

A Matrix Trace Inequality

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1. INTRODUCTION

Recently, Chang [1] proved a matrix trace inequality for Hermitian matrices $A$ and $B$,

$$
\text{tr}(AB)^{2^n} \leq \text{tr}(A^{2^n}B^{2^n}),
$$

where $\kappa$ is an integer, which is a continuation of the work of Bellman [2], Coope [3], Neudecker [4], and Yang [5].

The main results of this paper are the following inequalities:

THEOREM 1. Let $A$ and $B$ be positive semidefinite matrices of the same order; then for $n = 1, 2, \ldots$

$$
0 \leq \text{tr}(AB)^{2n} \leq (\text{tr } A)^{2n-1}(\text{tr } B^n),
$$

$$
0 \leq \text{tr}(AB)^{2n+1} \leq (\text{tr } A)(\text{tr } B)(\text{tr } A^n)(\text{tr } B^n).
$$

COROLLARY 1. If $A$ and $B$ are defined in Theorem 1, then

$$
0 \leq \text{tr}(AB)^n \leq (\text{tr } A)^n(\text{tr } B)^n.
$$

2. LEMMAS

LEMMA 1 [3]. If $A$ and $B$ are positive semidefinite matrices of the same order, then

$$
0 \leq \text{tr}(AB) \leq \text{tr } A \cdot \text{tr } B.
$$
**LEMMA 2.** If $A$ and $B$ are positive semidefinite matrices of the same order, then for $n = 1, 2, \ldots, (AB)^n B$ and $(BA)^n B$ are positive semidefinite matrices.

**Proof.** Let $A$ and $B$ be Hermitian matrices of the same order. If $A$ is positive semidefinite then $(AB)^2 = (AB)^2 A(AB)^n$ is clearly positive semidefinite. If $B$ is positive semidefinite then $(AB)^2 = (AB)^2 A(AB)^n$ is clearly positive semidefinite. Therefore, if both $A$ and $B$ are positive semidefinite $(AB)^k A$ is positive semidefinite, $k = 1, 2, \ldots$. 

### 3. PROOF OF THEOREM

**Proof of Theorem 1.** From Lemma 1, Lemma 2, and the equality $\text{tr}(CD) = \text{tr}(DC)$ for any square matrices $C, D$ of the same order, we obtain

$$\text{tr}(AB)^2 = \text{tr}\left(A(BA)^{2n-1}B\right) \leq (\text{tr} A)(\text{tr}(BA)^{2n-1}B) \quad (5)$$

$$\text{tr}(BA)^2 = \text{tr}\left(B(AB)^{2n-1}AB\right) = \text{tr}\left((AB)^2(A)^{2n-1}AB^2\right)$$

$$\leq \text{tr}\left((BA)^{2n-1}A\right)\text{tr} B^2 = \text{tr}(A(BA)^{2n-1}BA)\text{tr} B^2$$

$$= \text{tr}\left((BA)^{2n-1}AB^2\right)\text{tr} B^2$$

$$\leq \text{tr}\left((BA)^{2n-1}B\right)\text{tr} A^2 \text{tr} B^2. \quad (6)$$

By (5) and (6), we obtain

$$\text{tr}\left((BA)^{2n-1}B\right) \leq \text{tr}\left((BA)^{2n-1}B\right)\text{tr} A^2 \text{tr} B^2,$$

which means

$$\text{tr}\left((BA)^{2n-1}B\right) \leq \text{tr}(BAB)(\text{tr} A^2)^{n-1}(\text{tr} B^2)^{n-1}$$

$$= \text{tr}(A)^{n-1}(\text{tr} B^2)^{n-1}$$

$$\leq \text{tr} A(\text{tr} A^2)^{n-1}(\text{tr} B^2)^{n}. \quad (7)$$

From (5) and (7) we obtain (1).

Similarly,

$$\text{tr}(AB)^{2n+1} = \text{tr}(A(BA)^{2n}B) \leq \text{tr} A\text{tr}(BA)^{2n}B) \quad (8)$$

$$\text{tr}(BA)^{2n} = \text{tr}(B(AB)^{2n-1}AB) = \text{tr}( (AB)^{2n-1}AB^2)$$

$$\leq \text{tr}\left((BA)^{2n-1}A\right)\text{tr} B^2 = \text{tr}(A(AB)^{2n-1}BA)\text{tr} B^2$$

$$= \text{tr}\left((BA)^{2n-1}B\right)\text{tr} B^2 \leq \text{tr}\left((BA)^{2n-1}B\right)\text{tr} A^2 \text{tr} B^2. \quad (9)$$
By (8) and (9), we obtain
\[
\text{tr}\left((BA)^{2n}B\right) \leq \text{tr}A^2 \text{tr}B^2 \text{tr}\left((BA)^{2(n-1)}B\right),
\]
which means
\[
\text{tr}\left((BA)^{2n}B\right) \leq (\text{tr}A^2)^n (\text{tr}B^2)^n \text{tr}B.
\] (10)

From (8) and (10), we obtain (2). □

Remark. By Lemma 1, \(0 \leq \text{tr}A^2 \leq (\text{tr}A)^2\), \(0 \leq \text{tr}B^2 \leq (\text{tr}B)^2\), this, together with (1) and (2), we obtain (3).

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REFERENCES