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Infinitesimal non-crossing cumulants and free probability of type B

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Abstract

Free probabilistic considerations of type B first appeared in the paper of Biane, Goodman and Nica [P. Biane, F. Goodman, A. Nica, Non-crossing cumulants of type B, Trans. Amer. Math. Soc. 355 (2003) 2263–2303]. Recently, connections between type B and infinitesimal free probability were put into evidence by Belinschi and Shlyakhtenko [S.T. Belinschi, D. Shlyakhtenko, Free probability of type B: Analytic aspects and applications, preprint, 2009, available online at www.arxiv.org under reference arXiv:0903.2721]. The interplay between "type B" and "infinitesimal" is also the object of the present paper. We study infinitesimal freeness for a family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$ in an infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi')$ and we introduce a concept of *infinitesimal non-crossing cumulant functionals* for $(\mathcal{A}, \varphi, \varphi')$, obtained by taking a formal derivative in the formula for usual non-crossing cumulants. We prove that the infinitesimal freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is equivalent to a vanishing condition for mixed cumulants; this gives the infinitesimal counterpart for a theorem of Speicher from "usual" free probability. We show that the lattices $NC^{(B)}(n)$ of non-crossing partitions of type B appear in the combinatorial study of $(\mathcal{A}, \varphi, \varphi')$, in the formulas for infinitesimal cumulants and when describing alternating products of infinitesimally free random variables. As an application of alternating free products, we observe the infinitesimal analogue for the well-known fact that freeness is preserved under compression with a free projection. As another application, we observe the infinitesimal analogue for a well-known procedure used to construct free families of free Poisson elements. Finally, we discuss situations when the freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ in $(\mathcal{A}, \varphi)$

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can be naturally upgraded to infinitesimal freeness in $(\mathcal{A}, \varphi, \varphi')$, for a suitable choice of a "companion functional" $\varphi' : \mathcal{A} \to \mathbb{C}$.

Keywords: Free probability (of type B); Infinitesimal freeness; Infinitesimal non-crossing cumulants; Dual derivation system

1. Introduction

1.1. The framework of the paper

This paper is concerned with a form of free independence for noncommutative random variables, which can be called "freeness of type B" or "infinitesimal freeness", and occurs in relation to objects of the form

$$\begin{cases} (\mathcal{A}, \varphi, \varphi'), \text{ where } \mathcal{A} \text{ is a unital algebra over } \mathbb{C} \\ \quad \text{and } \varphi, \varphi' : \mathcal{A} \to \mathbb{C} \text{ are linear with } \varphi(1_{\mathcal{A}}) = 1, \varphi'(1_{\mathcal{A}}) = 0. \end{cases} \tag{1.1}$$

The motivation for considering objects as in (1.1) is three-fold.

(a) This framework generalizes the link-algebra associated to a noncommutative probability space of type B, in the sense introduced by Biane, Goodman and Nica [2]. One can thus take the point of view that (1.1) provides us with an enlarged framework for doing "free probability of type B". This point of view is justified by the fact that lattices of non-crossing partitions of type B do indeed appear in the underlying combinatorics – see e.g. Theorem 6.4 below, concerning alternating products of infinitesimally free random variables.

(b) It turns out to be beneficial to consolidate the functionals φ, φ' from (1.1) into only one functional

$$\widetilde{\varphi} : \mathcal{A} \to \mathbb{G}, \qquad \widetilde{\varphi} := \varphi + \varepsilon\varphi', \tag{1.2}$$

where $\mathbb{G}$ denotes the two-dimensional Grassman algebra generated by an element ε which satisfies $\varepsilon^2 = 0$. Thus $\mathbb{G}$ is the extension of $\mathbb{C}$ defined as

$$\mathbb{G} = \{\alpha + \varepsilon\beta \mid \alpha, \beta \in \mathbb{C}\}, \tag{1.3}$$

with multiplication given by $(\alpha_1 + \varepsilon\beta_1) \cdot (\alpha_2 + \varepsilon\beta_2) = \alpha_1\alpha_2 + \varepsilon(\alpha_1\beta_2 + \beta_1\alpha_2)$, and the structure from (1.1) could equivalently be treated as

$$\begin{cases} (\mathcal{A}, \widetilde{\varphi}), \text{ where } \mathcal{A} \text{ is a unital algebra over } \mathbb{C} \\ \quad \text{and } \widetilde{\varphi} : \mathcal{A} \to \mathbb{G} \text{ is } \mathbb{C}\text{-linear with } \widetilde{\varphi}(1_{\mathcal{A}}) = 1. \end{cases} \tag{1.4}$$

The framework (1.4) was discussed in the PhD thesis of Oancea [6], under the name of "scarce[2] $\mathbb{G}$-probability space". Specifically, Chapter 7 of [6] considers a concept of $\mathbb{G}$-freeness for a family of unital subalgebras in a $\mathbb{G}$-probability space, which is defined via a vanishing condition for mixed $\mathbb{G}$-valued cumulants, and generalizes the concept of freeness of type B from [2].

[2] The adjective "scarce" is used in order to distinguish from the concept of "$\mathbb{G}$-probability space" from operator-valued free probability, where one would require the functional $\widetilde{\varphi}$ to be $\mathbb{G}$-linear.

(c) The recent paper [1] by Belinschi and Shlyakhtenko discusses a concept of "infinitesimal distribution" $(\mathbb{C}\langle X_1, \dots, X_k\rangle, \mu, \mu')$ which is exactly as in (1.1), with $\mathbb{C}\langle X_1, \dots, X_k\rangle$ denoting the algebra of polynomials in noncommuting indeterminates $X_1, \dots, X_k$. This remarkable paper brings forth the idea that interesting infinitesimal distributions arise when μ is the limit at 0 and μ' is the derivative at 0 for a family of k-variable distributions $(\mu_t : \mathbb{C}\langle X_1, \dots, X_k\rangle \to \mathbb{C})_{t\in T}$, where T is a set of real numbers having 0 as accumulation point. As we will show below, this ties in really nicely with the $\mathbb{G}$-valued cumulant considerations mentioned in (b); indeed, one could say that [1] puts the ε from (1.3) in its right place – it is a sibling of the ε's from calculus, only that instead of just having "ε^2 much smaller than ε" one goes for the radical requirement that $\varepsilon^2 = 0$.

Upon consideration, it seems that what goes best with the framework from (1.1) is the "infinitesimal" terminology from (c), which is in particular adopted in the next definition. Throughout the paper some terminology inspired from (a) and (b) will also be used, in the places where it is suggestive to do so (e.g. when talking about "soul companions for φ" in Section 1.3 below).

Definition 1.1. 1^o A structure $(\mathcal{A}, \varphi, \varphi')$ as in (1.1) will be called an *infinitesimal noncommutative probability space* (abbreviated as *incps*).

2^o Let $(\mathcal{A}, \varphi, \varphi')$ be an incps and let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. We will say that $\mathcal{A}_1, \dots, \mathcal{A}_k$ are *infinitesimally free* with respect to (φ, φ') when they satisfy the following condition:

$$\text{If } i_1, \dots, i_n \in \{1, \dots, k\} \text{ are such that } i_1 \neq i_2,\ i_2 \neq i_3,\ \dots,\ i_{n-1} \neq i_n,$$

$$\text{and if } a_1 \in \mathcal{A}_{i_1},\ \dots,\ a_n \in \mathcal{A}_{i_n} \text{ are such that } \varphi(a_1) = \cdots = \varphi(a_n) = 0,$$

$$\text{then } \varphi(a_1 \cdots a_n) = 0 \text{ and}$$

$$\varphi'(a_1 \cdots a_n) = \begin{cases} \varphi(a_1 a_n)\varphi(a_2 a_{n-1}) \cdots \varphi(a_{(n-1)/2}\, a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2}), \\ \quad \text{if } n \text{ is odd and } i_1 = i_n, i_2 = i_{n-1},\ \dots,\ i_{(n-1)/2} = i_{(n+3)/2}, \\ 0, \quad \text{otherwise}. \end{cases} \tag{1.5}$$

Recall that in the free probability literature it is customary to use the name *noncommutative probability space* for a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is linear with $\varphi(1_{\mathcal{A}}) = 1$. Thus the concept of *infinitesimal* noncommutative probability space is obtained by adding to $(\mathcal{A}, \varphi)$ another functional φ' as in (1.1). It is also immediate that Definition 1.1.2^o of infinitesimal freeness is obtained by adding the condition (1.5) to the "usual" definition for the freeness of $\mathcal{A}_1, \dots, \mathcal{A}_k$ in $(\mathcal{A}, \varphi)$ (as appearing e.g. in [10, Definition 2.5.1]).

Definition 1.1.2^o is a reformulation of the concept with the same name from Definition 13 of [1]. The relations with [1,2] are discussed more precisely in Section 2 (cf. Remarks 2.8, 2.9). Section 2 also collects a few miscellaneous properties of infinitesimal freeness that follow directly from the definition. Most notable among them is that one can easily extend to infinitesimal framework the well-known free product construction of noncommutative probability spaces $(\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$, as presented e.g. in Lecture 6 of [5]. More precisely: if $(\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k) =: (\mathcal{A}, \varphi)$ and if we are given linear functionals $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ such that $\varphi_i'(1_{\mathcal{A}}) = 0$, $1 \leqslant i \leqslant k$, then there exists a unique linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi' \mid \mathcal{A}_i = \varphi_i'$, $1 \leqslant i \leqslant k$, and such that $\mathcal{A}_1, \dots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. (See Proposition 2.4 below.) The resulting incps $(\mathcal{A}, \varphi, \varphi')$ can thus be taken, by definition, as the *free product* of $(\mathcal{A}_i, \varphi_i, \varphi_i')$ for $1 \leqslant i \leqslant k$.

1.2. Non-crossing cumulants for $(\mathcal{A}, \varphi, \varphi')$

An important tool in the combinatorics of free probability is the family of non-crossing cumulant functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n\geqslant 1}$ associated to a noncommutative probability space $(\mathcal{A}, \varphi)$. These functionals were introduced in [9]; for a detailed presentation of their basic properties, see Lecture 11 of [5]. For every $n \geqslant 1$, the multilinear functional $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ is defined via a certain summation formula over the lattice $NC(n)$ of non-crossing partitions of $\{1, \ldots, n\}$. We will review the formula for a general κ_n in Section 3.2, here we only pick a special value of n that we use for illustration, e.g. $n = 3$. In this special case one has

$$\begin{aligned}\kappa_3(a_1, a_2, a_3) = {}&\varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_2)\varphi(a_1a_3)\\ &- \varphi(a_3)\varphi(a_1a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3), \quad \forall a_1, a_2, a_3 \in \mathcal{A}.\end{aligned} \tag{1.6}$$

The expression on the right-hand side of (1.6) has 5 terms (premultiplied by integer coefficients[3] such as 1, −1, or 2), corresponding to the fact that $|NC(3)| = 5$.

Let now $(\mathcal{A}, \varphi, \varphi')$ be an incps as in Definition 1.1. Then in addition to the non-crossing cumulant functionals $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ associated to φ we will define another family of multilinear functionals $(\kappa'_n : \mathcal{A}^n \to \mathbb{C})_{n\geqslant 1}$, which involve both φ and φ'. For every $n \geqslant 1$, the functional κ'_n is obtained by taking a *formal derivative* in the formula for κ_n, where we postulate that the derivative of φ is φ' and we invoke linearity and the Leibnitz rule for derivatives. For instance for $n = 3$ the term $\varphi(a_1a_2a_3)$ on the right-hand side of (1.6) is derivated into $\varphi'(a_1a_2a_3)$, the term $\varphi(a_1)\varphi(a_2a_3)$ is derivated into $\varphi'(a_1)\varphi(a_2a_3) + \varphi(a_1)\varphi'(a_2a_3)$, etc, yielding the formula for κ'_3 to be

$$\begin{aligned}\kappa'_3(a_1, a_2, a_3) = {}&\varphi'(a_1a_2a_3) - \varphi'(a_1)\varphi(a_2a_3) - \varphi(a_1)\varphi'(a_2a_3)\\ &- \varphi'(a_2)\varphi(a_1a_3) - \varphi(a_2)\varphi'(a_1a_3) - \varphi'(a_3)\varphi(a_1a_2) - \varphi(a_3)\varphi'(a_1a_2)\\ &+ 2\varphi'(a_1)\varphi(a_2)\varphi(a_3) + 2\varphi(a_1)\varphi'(a_2)\varphi(a_3) + 2\varphi(a_1)\varphi(a_2)\varphi'(a_3).\end{aligned} \tag{1.7}$$

We will refer to the functionals κ'_n as *infinitesimal non-crossing cumulants* associated to $(\mathcal{A}, \varphi, \varphi')$. The precise formula defining them appears in Definition 4.2 below. The passage from the formula for κ_n to the one for κ'_n is related to a concept of *dual derivation system* on a space of multilinear functionals on $\mathcal{A}$, which is discussed in Section 7 of the paper.

The role of infinitesimal non-crossing cumulants in the study of infinitesimal freeness is described in the next theorem.

Theorem 1.2. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. The following statements are equivalent*:

(1) *$\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free.*
(2) *For every $n \geqslant 2$, for every $i_1, \ldots, i_n \in \{1, \ldots, k\}$ which are not all equal to each other, and for every $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$, one has that $\kappa_n(a_1, \ldots, a_n) = \kappa'_n(a_1, \ldots, a_n) = 0$.*

[3] The meaning of these coefficients is that they are special values of the Möbius function of $NC(3)$, as reviewed more precisely in Section 3.

Theorem 1.2 provides an infinitesimal version for the basic result of Speicher which describes the usual freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ in $(\mathcal{A}, \varphi)$ in terms of the cumulants κ_n (cf. [5, Theorem 11.16]).

In the remaining part of this subsection we point out some other interpretations of the formula defining κ_n' (all corresponding to one or another of the points of view (a), (b), (c) listed at the beginning of Section 1.1). The easy verifications required by these alternative descriptions of κ_n' are shown at the beginning of Section 4.

First of all, one can consider, as in [1], the situation when φ, φ' in (1.1) are obtained as the *infinitesimal limit* of a family of functionals $\{\varphi_t \mid t \in T\}$. Here T is a subset of $\mathbb{R}$ which has 0 as an accumulation point, every φ_t is linear with $\varphi_t(1_{\mathcal{A}}) = 1$, and we have

$$\varphi(a) = \lim_{t \to 0} \varphi_t(a) \quad \text{and} \quad \varphi'(a) = \lim_{t \to 0} \frac{\varphi_t(a) - \varphi(a)}{t}, \quad \forall a \in \mathcal{A}. \tag{1.8}$$

(Note that such families $\{\varphi_t \mid t \in T\}$ can in fact always be found, e.g. by simply taking $\varphi_t = \varphi + t\varphi'$, $t \in (0, \infty)$.) In such a situation, the formal derivative which leads from κ_n to κ_n' turns out to have the same effect as a "$\frac{d}{dt}$" derivative. Consequently, we get the alternative formula

$$\kappa_n'(a_1, \ldots, a_n) = \left[\frac{d}{dt} \kappa_n^{(t)}(a_1, \ldots, a_n)\right]\Bigg|_{t=0}, \tag{1.9}$$

where $\kappa_n^{(t)}$ denotes the nth non-crossing cumulant functional of φ_t.

Second of all, it is possible to take a direct combinatorial approach to the functionals κ_n', and identify precisely a set of non-crossing partitions which indexes the terms in the summation defining $\kappa_n'(a_1, \ldots, a_n)$. This set turns out to be

$$NCZ^{(B)}(n) := \bigl\{\tau \in NC^{(B)}(n) \bigm| \tau \text{ has a zero-block}\bigr\}, \tag{1.10}$$

where $NC^{(B)}(n)$ is the lattice of non-crossing partitions of type B of $\{1, \ldots, n\} \cup \{-1, \ldots, -n\}$ (see Section 3.1 for a brief review of this). Hence in a terminology focused on types of non-crossing partitions, one could call the functionals κ_n and κ_n' "non-crossing cumulants of type A and of type B", respectively. The idea put forth here is that, in some sense, summations over $NCZ^{(B)}(n)$ appear as "derivatives for summation over $NC(n)$". A more refined formula supporting this idea is shown in Proposition 7.6 below, in connection to the concept of dual derivation system.

In the case $n = 3$ that we are using for illustration, the 10 terms appearing on the right-hand side of (1.7) are indexed by the 10 partitions with zero-block in $NC^{(B)}(3)$. For example, the partitions corresponding to the first three terms and the last term from (1.7) are depicted in Fig. 1. The relation between a partition τ and the corresponding term is easy to follow: the zero-block Z of τ produces the $\varphi'(\cdots)$ factor, and every pair $V, -V$ of non-zero-blocks of τ produces a $\varphi(\cdots)$ factor.

Finally (third of all) one can also give a description of κ_n' which corresponds to the "$\mathbb{G}$-valued" point of view appearing as (b) on the list from Section 1.1. This goes as follows. Let $\widetilde{\varphi} = \varphi + \varepsilon\varphi' : \mathcal{A} \to \mathbb{G}$ be as in (1.2), and consider the family of $\mathbb{C}$-multilinear functionals $(\widetilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G})_{n \geqslant 1}$ defined by the same summation formula as for the usual non-crossing cumulant functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n \geqslant 1}$, only that now we use $\widetilde{\varphi}$ instead of φ in the summations. So, for example, for $n = 3$ we have

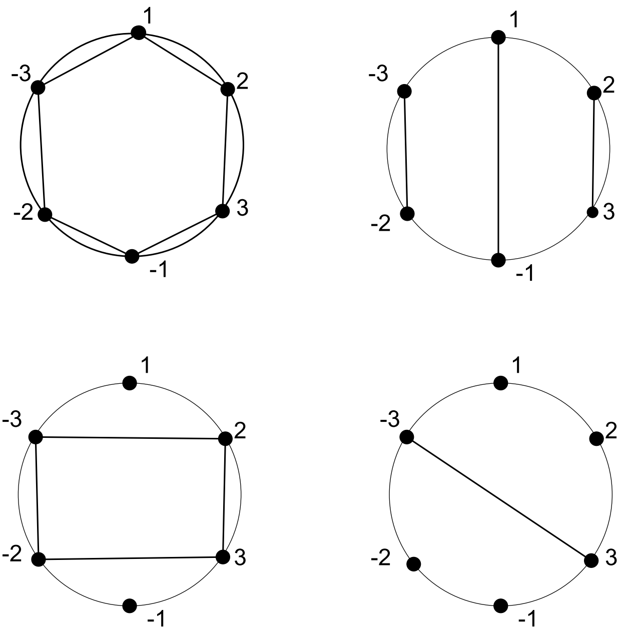

Fig. 1. Some partitions in $NCZ^{(B)}(3)$.

$$\begin{aligned}\widetilde{\kappa}_3(a_1,a_2,a_3) &= \widetilde{\varphi}(a_1a_2a_3) - \widetilde{\varphi}(a_1)\widetilde{\varphi}(a_2a_3) - \widetilde{\varphi}(a_2)\widetilde{\varphi}(a_1a_3)\\ &\quad - \widetilde{\varphi}(a_3)\widetilde{\varphi}(a_1a_2) + 2\widetilde{\varphi}(a_1)\widetilde{\varphi}(a_2)\widetilde{\varphi}(a_3) \in \mathbb{G}, \quad \forall a_1,a_2,a_3 \in \mathcal{A}. \end{aligned} \tag{1.11}$$

It then turns out that the functional κ_n' can be obtained by reading the ε-component of $\widetilde{\kappa}_n$.

We take the opportunity to introduce here a piece of terminology from the literature on Grassman algebras (see e.g. [3, pp. 1–2]): the complex numbers α, β which give the two components of a Grassman number $\gamma = \alpha + \varepsilon\beta \in \mathbb{G}$ will be called the *body* and respectively the *soul* of γ; it will come in handy throughout the paper to denote them[4] as

$$\alpha = \mathrm{Bo}(\gamma), \qquad \beta = \mathrm{So}(\gamma). \tag{1.12}$$

This notation will also be used in connection to a $\mathbb{G}$-valued function f defined on some set $\mathcal{S}$ – we define functions $\mathrm{Bo}\, f$ and $\mathrm{So}\, f$ from $\mathcal{S}$ to $\mathbb{C}$ by

$$(\mathrm{Bo}\, f)(x) = \mathrm{Bo}\big(f(x)\big), \qquad (\mathrm{So}\, f)(x) = \mathrm{So}\big(f(x)\big), \quad \forall x \in \mathcal{S}. \tag{1.13}$$

Returning then to the functionals $\widetilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G}$ from the preceding paragraph, their connection to the κ_n' (and also to the κ_n) can be recorded as

$$\mathrm{Bo}\,\widetilde{\kappa}_n = \kappa_n, \qquad \mathrm{So}\,\widetilde{\kappa}_n = \kappa_n', \quad \forall n \geqslant 1. \tag{1.14}$$

[4] Besides being amusing, "Bo" and "So" give a faithful analogue for the common notations "Re" and "Im" used when one introduces $\mathbb{C}$ as a 2-dimensional algebra over $\mathbb{R}$.

Due to (1.14), $\widetilde{\kappa}_n$ can be used as a simplifying tool in calculations with κ'_n (in the sense that it may be easier to run the corresponding calculation with $\widetilde{\kappa}_n$, in $\mathbb{G}$, and only pick soul parts at the end of the calculation). In particular, this will be useful when proving Theorem 1.2, since the condition $\kappa_n(a_1, \ldots, a_n) = \kappa'_n(a_1, \ldots, a_n) = 0$ from Theorem 1.2(2) amounts precisely to $\widetilde{\kappa}_n(a_1, \ldots, a_n) = 0$.

1.3. Using derivations to find "soul companions" for a given φ

When studying infinitesimal freeness it may be of interest to consider the situation where we have fixed a noncommutative probability space $(\mathcal{A}, \varphi)$ and a family $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \varphi)$. In this situation we can ask: how do we find interesting examples of functionals $\varphi' : \mathcal{A} \to \mathbb{C}$ with $\varphi'(1_{\mathcal{A}}) = 0$ and such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ become infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$? A nice name for such functionals φ' is suggested by the $\mathbb{G}$-valued point of view described in Section 1.1: since φ and φ' are the body part and respectively the soul part of the consolidated functional $\widetilde{\varphi} : \mathcal{A} \to \mathbb{G}$, one may say that we are looking for a suitable *soul companion* φ' for the given "body functional" φ (and in reference to the given subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$).

Let us note that the remark made at the end of Section 1.1 can be interpreted as a statement about soul companions. Indeed, this remark says that if $(\mathcal{A}, \varphi)$ is the free product of $(\mathcal{A}_1, \varphi_1), \ldots, (\mathcal{A}_k, \varphi_k)$, then a φ' from the desired set of soul companions is parametrized precisely by a family of linear functionals $\varphi'_i : \mathcal{A}_i \to \mathbb{C}$ such that $\varphi'_i(1_{\mathcal{A}}) = 0$, $1 \leqslant i \leqslant k$.

The point we follow here, with inspiration from [1], is that some interesting recipes to construct "soul companions" for a given $\varphi : \mathcal{A} \to \mathbb{C}$ arise from ideas pertaining to differentiability. This is intimately related to the fact that κ'_n is a formal derivative for κ_n, hence to equations of the form

$$d_n(\kappa_n) = \kappa'_n, \quad \forall n \geqslant 1,$$

where $(d_n)_{n \geqslant 1}$ is a dual derivation system on $\mathcal{A}$. Indeed, suppose we are given a derivation $D : \mathcal{A} \to \mathcal{A}$; then one has a natural dual derivation system associated to it, which acts by

$$(d_n f)(a_1, \ldots, a_n) = \sum_{m=1}^{n} f\big(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\big), \tag{1.15}$$

for $f : \mathcal{A}^n \to \mathbb{C}$ multilinear and $a_1, \ldots, a_n \in \mathcal{A}$. By using the d_n from (1.15), we obtain the following theorem.

Theorem 1.3. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps, and let $(\kappa_n)_{n \geqslant 1}$, $(\kappa'_n)_{n \geqslant 1}$ be the families of non-crossing and respectively of infinitesimal non-crossing cumulant functionals associated to this incps. Suppose $D : \mathcal{A} \to \mathcal{A}$ is a derivation with the property that $\varphi' = \varphi \circ D$. Then for every $n \geqslant 1$ and $a_1, \ldots, a_n \in \mathcal{A}$ one has*

$$\kappa'_n(a_1, \ldots, a_n) = \sum_{m=1}^{n} \kappa_n\big(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\big). \tag{1.16}$$

Moreover, when combined with Theorem 1.2, the formula for infinitesimal cumulants obtained in (1.16) has the following immediate consequence.

Corollary 1.4. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \varphi)$. Suppose we found a derivation $D : \mathcal{A} \to \mathcal{A}$ such that $D(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for every $1 \leqslant i \leqslant k$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$, where $\varphi' = \varphi \circ D$.*

For comparison, let us also look at the parallel statement arising in connection to infinitesimal limits. This is essentially the same as Remark 15 from [1], and goes as follows.

Proposition 1.5. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \varphi)$. Suppose we found a family of linear functionals $(\varphi_t : \mathcal{A} \to \mathbb{C})_{t \in T}$ with $\varphi_t(1_{\mathcal{A}}) = 1$ for every $t \in T$ (where $T \subseteq \mathbb{R}$ has 0 as an accumulation point), and such that the following conditions are fulfilled:*

(i) *$\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \varphi_t)$ for every $t \in T$.*
(ii) *$\lim_{t \to 0} \varphi_t(a) = \varphi(a)$, for every $a \in \mathcal{A}$.*
(iii) *The limit $\varphi'(a) := \lim_{t \to 0}(\varphi_t(a) - \varphi(a))/t$ exists, for every $a \in \mathcal{A}$.*

Then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$, where $\varphi' : \mathcal{A} \to \mathbb{C}$ is defined by condition (iii).

A natural example accompanying Proposition 1.5 comes in connection to $\boxplus$-convolution powers of joint distributions of k-tuples (cf. Example 8.9 below). In Section 8 we also discuss a couple of natural situations when Corollary 1.4 applies (cf. Example 8.7).

1.4. Outline of the rest of the paper

Besides the introduction, the paper has seven other sections. In Section 2 we collect some basic properties of infinitesimal freeness, and we discuss the relations between Definition 1.1 and the frameworks of [1,2]. Section 3 is a review of background concerning non-crossing partitions and non-crossing cumulants. In Section 4 we introduce the non-crossing infinitesimal cumulants, we verify the equivalence between their various alternative descriptions, and we prove Theorem 1.2.

Sections 5 and 6 address the topic of alternating products of infinitesimally free random variables. Section 5 uses this topic to illustrate a "generic" method to obtain infinitesimal analogues for known results in usual free probability: one replaces $\mathbb{C}$ by $\mathbb{G}$ in the proof of the original result, then one takes the soul part in the $\mathbb{G}$-valued statement that comes out. By using this method we obtain the infinitesimal versions of two important facts related to alternating products that were originally found in [4] – one of them is about compressions by free projections, the other concerns a method of constructing free families of free Poisson elements. In Section 6 we remember that the concept of incps has its origins in the considerations "of type B" from [2], and we look at how the essence of these considerations persists in the framework of the present paper. The main point of the section is that, when taking the soul part of the $\mathbb{G}$-valued formulas for alternating products of infinitesimally free random variables, one does indeed obtain nice analogues of type B (with summations over $NC^{(B)}(n)$) for the type A formulas. In particular, this offers another explanation for why the infinitesimal cumulant functional κ_n' can be described by using a summation formula over $NCZ^{(B)}(n)$.

In Section 7 we return to the point of view of treating κ_n' as a derivative of the usual non-crossing cumulant functional κ_n, and we discuss the related concept of dual derivation system on a unital algebra $\mathcal{A}$. Finally, Section 8 elaborates on the discussion about soul companions from the above Section 1.3. In particular, we show how the dual derivation system provided by a derivation $D : \mathcal{A} \to \mathcal{A}$ leads to the setting for infinitesimal freeness from Corollary 1.4. Section 8 (and the paper) concludes with a couple of examples related to the settings of Corollary 1.4 and of Proposition 1.5.

2. Basic properties of infinitesimal freeness

In this section we collect some basic properties of infinitesimal freeness, and we discuss the relations between Definition 1.1 and the frameworks from [1,2].

Definition 2.1. Here are some standard variations of Definition 1.1.

1^o The concept of infinitesimal freeness carries over to $*$-algebras. More precisely, we will use the name *$*$-incps* for an incps $(\mathcal{A}, \varphi, \varphi')$ where $\mathcal{A}$ is a unital $*$-algebra and where

(i) φ is positive definite, that is, $\varphi(a^*a) \geqslant 0$, $\forall a \in \mathcal{A}$;
(ii) φ' is selfadjoint, that is, $\varphi'(a^*) = \overline{\varphi'(a)}$, $\forall a \in \mathcal{A}$.

2^o Another standard variation of the definitions is that infinitesimal freeness can be considered for arbitrary subsets of $\mathcal{A}$ (which don't have to be subalgebras). So if $(\mathcal{A}, \varphi, \varphi')$ is an incps (respectively a $*$-incps) and if $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are subsets of $\mathcal{A}$, then we will say that $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are *infinitesimally free* (respectively *infinitesimally $*$-free*) when the unital subalgebras (respectively $*$-subalgebras) generated by $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are so.

Remark 2.2. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \varphi)$. Let us denote $\mathcal{M} := \mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$ (the subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$). It is easy to see (cf. Remark 2.5.2 in [10] or Examples 5.15 in [5]) that the way how φ acts on $\mathcal{M}$ can be reconstructed from the restrictions $\varphi \mid \mathcal{A}_i$, $1 \leqslant i \leqslant k$. The simplest illustration for how this works is provided by the formula

$$\varphi(ab) = \varphi(a)\varphi(b), \quad \forall a \in \mathcal{A}_{i_1},\ b \in \mathcal{A}_{i_2}, \quad \text{with } i_1 \neq i_2, \tag{2.1}$$

which is obtained by expanding the product and then collecting terms in the equation $\varphi((a - \varphi(a)1_{\mathcal{A}}) \cdot (b - \varphi(b)1_{\mathcal{A}})) = 0$.

A similar phenomenon turns out to take place when dealing with infinitesimal freeness: the way how φ' acts on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$ can be reconstructed from the restrictions of φ and of φ' to $\mathcal{A}_i$, $1 \leqslant i \leqslant k$. For example, the counterpart of Eq. (2.1) says that

$$\varphi'(ab) = \varphi'(a)\varphi(b) + \varphi(a)\varphi'(b), \quad \forall a \in \mathcal{A}_{i_1},\ b \in \mathcal{A}_{i_2}, \quad \text{where } i_1 \neq i_2. \tag{2.2}$$

This is obtained by expanding the product and then collecting terms in the equation $\varphi'((a - \varphi(a)1_{\mathcal{A}}) \cdot (b - \varphi(b)1_{\mathcal{A}})) = 0$ (which is a particular case of Eq. (1.5)), and by taking into account that $\varphi'(1_{\mathcal{A}}) = 0$.

We leave it as an easy exercise to the reader to verify that the similar calculation for an alternating product of 3 factors (which makes a more involved use of Eq. (1.5)) leads to the formula

$$\varphi'(a_1ba_2) = \varphi'(a_1a_2)\varphi(b) + \varphi(a_1a_2)\varphi'(b), \quad \text{for } a_1, a_2 \in \mathcal{A}_{i_1},\ b \in \mathcal{A}_{i_2}, \quad \text{with } i_1 \neq i_2. \tag{2.3}$$

Remark 2.3. *(Traciality.)* Another well-known fact in usual free probability is that if the unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$ are free in $(\mathcal{A}, \varphi)$ and if $\varphi \mid \mathcal{A}_i$ is a trace for every $1 \leqslant i \leqslant k$, then φ is a trace on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$. This too extends to the infinitesimal framework: if $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$ and if $\varphi \mid \mathcal{A}_i, \varphi' \mid \mathcal{A}_i$ are traces for every $1 \leqslant i \leqslant k$, then φ and φ' are traces on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$. Rather than writing an ad-hoc proof of this fact based directly on Definition 1.1, we find it more instructive to do this by using cumulants – see Proposition 4.11 below.

We next move to describing the free product of infinitesimal noncommutative probability spaces announced at the end of Section 1.1.

Proposition 2.4. *Let $(\mathcal{A}_1, \varphi_1), \ldots, (\mathcal{A}_k, \varphi_k)$ be noncommutative probability spaces, and consider the free product $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$ (as described e.g. in Lecture 6 of [5]). Suppose that for every $1 \leqslant i \leqslant k$ we are given a linear functional $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ such that $\varphi_i'(1_{\mathcal{A}}) = 0$. Then there exists a unique linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi' \mid \mathcal{A}_i = \varphi_i'$, $1 \leqslant i \leqslant k$, and such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.*

Proof. We start by reviewing a few basic facts and notations related to $(\mathcal{A}, \varphi)$. Each of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is identified as a unital subalgebra of $\mathcal{A}$, such that $\varphi \mid \mathcal{A}_i = \varphi_i$. For $1 \leqslant i \leqslant k$ we denote $\mathcal{A}_i^o = \{a \in \mathcal{A}_i \mid \varphi(a) = 0\}$, and for every $n \geqslant 1$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$ such that $i_1 \neq i_2, \ \ldots, \ i_{n-1} \neq i_n$ we put

$$\mathcal{W}_{i_1,\ldots,i_n} := \mathrm{span}\{a_1 \cdots a_n \mid a_1 \in \mathcal{A}_{i_1}^o, \ \ldots, \ a_n \in \mathcal{A}_{i_n}^o\}. \tag{2.4}$$

It is known that $\mathcal{W}_{i_1,\ldots,i_n}$ is canonically isomorphic to the tensor product $\mathcal{A}_{i_1}^o \otimes \cdots \otimes \mathcal{A}_{i_n}^o$, via the identification $a_1 \cdots a_n \simeq a_1 \otimes \cdots \otimes a_n$, for $a_1 \in \mathcal{A}_{i_1}^o, \ \ldots, \ a_n \in \mathcal{A}_{i_n}^o$. Moreover it is known that the spaces $\mathcal{W}_{i_1,\ldots,i_n}$ defined in (2.4) realize a direct sum decomposition of the kernel of φ. (See [5, pp. 81–84].)

Due to the direct sum decomposition mentioned above, we may define the required functional φ' by separately prescribing its behaviour at $1_{\mathcal{A}}$ and on each of the subspaces $\mathcal{W}_{i_1,\ldots,i_n}$. We put $\varphi'(1_{\mathcal{A}}) := 0$. We also prescribe φ' to be 0 on $\mathcal{W}_{i_1,\ldots,i_n}$ whenever n is even, and whenever n is odd but it is not true that $i_m = i_{n+1-m}$ for all $1 \leqslant m \leqslant (n-1)/2$. Suppose next that $n = 2m-1$, odd, and that the indices $i_1, \ldots, i_n$ are such that $i_1 = i_{2m-1}, i_2 = i_{2m-2}, \ \ldots, \ i_{m-1} = i_{m+1}$. By using the identification $\mathcal{W}_{i_1,\ldots,i_n} \simeq \mathcal{A}_{i_1}^o \otimes \cdots \otimes \mathcal{A}_{i_n}^o$ it is immediate that we can define a linear map on $\mathcal{W}_{i_1,\ldots,i_n}$ by the requirement that

$$a_1 \cdots a_{2m-1} \mapsto \varphi_{i_1}(a_1 a_{2m-1}) \varphi_{i_2}(a_2 a_{2m-2}) \cdots \varphi_{i_{m-1}}(a_{m-1} a_{m+1}) \cdot \varphi_{i_m}'(a_m),$$

for every $a_1 \in \mathcal{A}_{i_1}^o, \ \ldots, \ a_n \in \mathcal{A}_{i_n}^o$; we take this as the prescription for how φ' is to act on $\mathcal{W}_{i_1,\ldots,i_n}$.

Directly from Definition 1.1 it is immediate that, with $\varphi' : \mathcal{A} \to \mathbb{C}$ defined as in the preceding paragraph, $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. The uniqueness of φ' with this property is also immediate. $\square$

Definition 2.5. Let $(\mathcal{A}_1, \varphi_1, \varphi_1'), \ldots, (\mathcal{A}_k, \varphi_k, \varphi_k')$ be infinitesimal noncommutative probability spaces. We define their *free product* to be $(\mathcal{A}, \varphi, \varphi')$ where $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$ and where $\varphi' : \mathcal{A} \to \mathbb{C}$ is the functional provided by Proposition 2.4.

Remark 2.6. In the context of Proposition 2.4, suppose that $(\mathcal{A}_1, \varphi_1), \ldots, (\mathcal{A}_k, \varphi_k)$ are $*$-probability spaces. Then so is the free product $(\mathcal{A}, \varphi)$ (see [5, Theorem 6.13]). If moreover each of the functionals $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ given in Proposition 2.4 is selfadjoint, then it is easily checked that the resulting functional $\varphi' : \mathcal{A} \to \mathbb{C}$ is selfadjoint too. Hence if in Definition 2.5 each of $(\mathcal{A}_i, \varphi_i, \varphi_i')$ is a $*$-incps, then the free product $(\mathcal{A}, \varphi, \varphi')$ is a $*$-incps as well.

Example 2.7. For an illustration of the above, we look at a simple example where the spaces $\mathcal{W}_{i_1,\ldots,i_n}$ are all 1-dimensional. Consider the k-fold free product group $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ and let φ be the canonical trace on the group algebra $\mathcal{A} := \mathbb{C}[\mathbb{Z}_2 * \cdots * \mathbb{Z}_2]$. So $\mathcal{A}$ is a unital $*$-algebra freely generated by k unitaries $u_1, \ldots, u_k$ of order 2, and has a linear basis $\mathcal{B}$ given by

$$\mathcal{B} = \{1_{\mathcal{A}}\} \cup \left\{ u_{i_1} \cdots u_{i_n} \;\middle|\; \begin{array}{l} n \geqslant 1,\ 1 \leqslant i_1, \ldots, i_n \leqslant k, \\ \text{with } i_1 \neq i_2,\ \ldots,\ i_{n-1} \neq i_n \end{array} \right\}. \tag{2.5}$$

The linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ acts on the basis $\mathcal{B}$ by

$$\varphi(1_{\mathcal{A}}) = 1, \quad \text{and} \quad \varphi(b) = 0, \quad \forall b \in \mathcal{B} \setminus \{1_{\mathcal{A}}\}.$$

It is easy to verify (see e.g. Lecture 6 in [5]) that we have $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$, where for $1 \leqslant i \leqslant k$ we denote $\mathcal{A}_i = \text{span}\{1_{\mathcal{A}}, u_i\}$ (2-dimensional $*$-subalgebra of $\mathcal{A}$), and where $\varphi_i := \varphi \mid \mathcal{A}_i$. The direct sum decomposition of $\mathcal{A}$ with respect to this free product structure simply has

$$\mathcal{W}_{i_1,\ldots,i_n} = \text{1-dimensional space spanned by } u_{i_1} \cdots u_{i_n},$$

for every $n \geqslant 1$ and every alternating sequence $i_1, \ldots, i_n$ as described in (2.5).

Now let $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ be linear functionals such that $\varphi_i'(1_{\mathcal{A}}) = 0$, $1 \leqslant i \leqslant k$. Clearly, these functionals are determined by the values

$$\varphi_1'(u_1) =: \alpha_1', \quad \ldots, \quad \varphi_k'(u_k) =: \alpha_k'.$$

The free product extension $\varphi' : \mathcal{A} \to \mathbb{C}$ then acts by

$$\varphi'(u_{i_1} \cdots u_{i_n}) = \begin{cases} \alpha_{i_m}', & \text{if } n = 2m-1 \text{ and } i_1 = i_{2m-1},\ \ldots,\ i_{m-1} = i_{m+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Note that formula (2.6) looks particularly nice in the case when $k = 2$ – indeed, in this case the requirement that $i_1 = i_{2m-1},\ \ldots,\ i_{m-1} = i_{m+1}$ is automatically satisfied whenever $n = 2m-1$ and $i_1, \ldots, i_n$ are as in (2.5).

Remark 2.8. *(Relation to [1]).* Definition 13 of [1] introduces a concept of infinitesimal freeness for unital subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ in an incps $(\mathcal{A}, \mu, \mu')$. As explained there (immediately following to Definition 13), this amounts to two requirements: that $\mathcal{A}_1, \mathcal{A}_2$ are free in $(\mathcal{A}, \mu)$, and that they satisfy the following additional condition:

$$\mu'\big((p_1-\mu(p_1)1_{\mathcal{A}})\cdots(p_n-\mu(p_n)1_{\mathcal{A}})\big)$$
$$=\sum_{m=1}^{n}\mu\big((p_1-\mu(p_1)1_{\mathcal{A}})\cdots\mu'(p_m)\cdots(p_n-\mu(p_n)1_{\mathcal{A}})\big) \tag{2.7}$$

for $p_1\in\mathcal{A}_{i_1},\ \dots,\ p_n\in\mathcal{A}_{i_n}$, where $i_1\neq i_2,\ \dots,\ i_{n-1}\neq i_n$. By denoting $p_m-\mu(p_m)1_{\mathcal{A}}=:q_m$ and by taking into account that $\mu'(q_m)=\mu'(p_m)$, $1\leqslant m\leqslant n$, one sees that condition (2.7) is equivalent to its particular case requesting that

$$\mu'(q_1\cdots q_n)=\sum_{m=1}^{n}\mu(q_1\cdots q_{m-1}q_{m+1}\cdots q_n)\cdot\mu'(q_m) \tag{2.8}$$

for $q_1\in\mathcal{A}_{i_1},\dots,q_n\in\mathcal{A}_{i_n}$, where $i_1\neq i_2,\dots,i_{n-1}\neq i_n$ and where $\mu(q_1)=\dots=\mu(q_n)=0$.

But now, let $\mathcal{A}_1,\mathcal{A}_2$ be unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A},\mu)$. A standard calculation from usual free probability (see e.g. [5, Lemma 5.18 on p. 73]) says that, with $q_1,\dots,q_n$ as in (2.8), one has $\mu(q_1\cdots q_{m-1}q_{m+1}\cdots q_n)=0$ unless it is true that $m-1=n-m$ and that $i_{m-1}=i_{m+1},\ i_{m-2}=i_{m+2},\ \dots,\ i_1=i_n$; moreover, if the latter conditions are satisfied, then

$$\mu(q_1\cdots q_{m-1}q_{m+1}\cdots q_n)=\mu(q_{m-1}q_{m+1})\mu(q_{m-2}q_{m+2})\cdots\mu(q_1q_n).$$

This clearly implies that the sum on the right-hand side of (2.8) has at most one term which is different from 0; and moreover, when such a term exists, it is exactly as described in Eq. (1.5) of Definition 1.1.

Hence, modulo an immediate reformulation, the concept of infinitesimal freeness from [1] is the same as the one used in this paper (which justifies the fact that we are calling it by the same name).

Remark 2.9. (Relation to [2]). A *noncommutative probability space of type B* is defined in [2] as a system $(\mathcal{A},\varphi,\mathcal{V},f,\Phi)$, where $(\mathcal{A},\varphi)$ is a noncommutative probability space, $\mathcal{V}$ is a complex vector space, $f:\mathcal{V}\to\mathbb{C}$ is a linear functional, and $\Phi:\mathcal{A}\times\mathcal{V}\times\mathcal{A}\to\mathcal{V}$ is a two-sided action. We will write for short $a\xi b$ and respectively $a\xi$, ξb instead of $\Phi(a,\xi,b)$ and respectively $\Phi(a,\xi,1_{\mathcal{A}})$, $\Phi(1_{\mathcal{A}},\xi,b)$, for $a,b\in\mathcal{A}$ and $\xi\in\mathcal{V}$. Let $\mathcal{A}_1,\dots,\mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ and let $\mathcal{V}_1,\dots,\mathcal{V}_k$ be linear subspaces of $\mathcal{V}$, such that $\mathcal{V}_i$ is closed under the two-sided action of $\mathcal{A}_i$, $1\leqslant i\leqslant k$. Definition 7.2 of [2] introduces a concept of what it means for $(\mathcal{A}_1,\mathcal{V}_1),\dots,(\mathcal{A}_k,\mathcal{V}_k)$ to be free in $(\mathcal{A},\varphi,\mathcal{V},f,\Phi)$. This amounts to two requirements: that $\mathcal{A}_1,\dots,\mathcal{A}_k$ are free in $(\mathcal{A},\varphi)$, and that the following additional condition is satisfied:

$$f(a_m\dots a_1\xi b_1\dots b_n)=\begin{cases}\varphi(a_1b_1)\cdots\varphi(a_nb_n)f(\xi),\\ \quad\text{if } m=n \text{ and } i_1=j_1,\ \dots,\ i_n=j_n,\\ 0,\quad\text{otherwise,}\end{cases} \tag{2.9}$$

holding for $m,n\geqslant 0$ and $a_1\in\mathcal{A}_{i_1},\ \dots,\ a_m\in\mathcal{A}_{i_m}$, $b_1\in\mathcal{A}_{j_1},\ \dots,\ b_n\in\mathcal{A}_{j_n}$, $\xi\in\mathcal{V}_h$, where any two consecutive indices among $i_m,\dots,i_1,h,j_1,\dots,j_n$ are different from each other, and where $\varphi(a_m)=\dots=\varphi(a_1)=0=\varphi(b_1)=\dots=\varphi(b_n)$.

Now, to $(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi)$ as above one associates a *link-algebra*, which is simply the direct product $\mathcal{M} = \mathcal{A} \times \mathcal{V}$ endowed with the natural structure of complex vector space and with multiplication

$$(a, \xi) \cdot (b, \eta) = (ab, a\eta + \xi b), \quad \forall a, b \in \mathcal{A},\ \xi, \eta \in \mathcal{V}. \tag{2.10}$$

If we define $\psi, \psi' : \mathcal{M} \to \mathbb{C}$ by

$$\psi\big((a, \xi)\big) := \varphi(a), \qquad \psi'\big((a, \xi)\big) := f(\xi), \quad \forall (a, \xi) \in \mathcal{M}, \tag{2.11}$$

then $(\mathcal{M}, \psi, \psi')$ becomes an incps. Let again $\mathcal{A}_1, \dots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ and $\mathcal{V}_1, \dots, \mathcal{V}_k$ be linear subspaces of $\mathcal{V}$ such that $\mathcal{V}_i$ is closed under the two-sided action of $\mathcal{A}_i$, $1 \leqslant i \leqslant k$. Then $\mathcal{M}_1 := \mathcal{A}_1 \times \mathcal{V}_1, \ \dots, \ \mathcal{M}_k := \mathcal{A}_k \times \mathcal{V}_k$ are unital subalgebras of the link-algebra $\mathcal{M}$, and we claim that

$$\left(\begin{array}{c} (\mathcal{A}_1, \mathcal{V}_1), \dots, (\mathcal{A}_k, \mathcal{V}_k) \\ \text{are free in } (\mathcal{A}, \varphi, \mathcal{V}, f, \Phi), \\ \text{in the sense of [2]} \end{array}\right) \quad \Leftrightarrow \quad \left(\begin{array}{c} \mathcal{M}_1, \dots, \mathcal{M}_k \text{ are inf. free} \\ \text{in } (\mathcal{M}, \psi, \psi'), \text{ in the} \\ \text{sense of Definition 1.1} \end{array}\right). \tag{2.12}$$

In order to prove the implication "$\Leftarrow$" in (2.12), we only have to write

$$f(a_m \dots a_1 \xi b_1 \dots b_n) = \psi'\big((a_m, 0_{\mathcal{V}}) \cdots (a_1, 0_{\mathcal{V}}) \cdot (0_{\mathcal{A}}, \xi) \cdot (b_1, 0_{\mathcal{V}}) \cdots (b_n, 0_{\mathcal{V}})\big)$$

and then invoke Eq. (1.5). For the implication "$\Rightarrow$", consider some elements $(a_1, \xi_1) \in \mathcal{M}_{i_1}, \ \dots, \ (a_n, \xi_n) \in \mathcal{M}_{i_n}$ where $i_1 \neq i_2, \ \dots, \ i_{n-1} \neq i_n$ and where $\psi((a_1, \xi_1)) = \cdots = \psi((a_n, \xi_n)) = 0$ (which just means that $\varphi(a_1) = \cdots = \varphi(a_n) = 0$). By using how the multiplication on $\mathcal{M}$ and how ψ' are defined, we see that

$$\psi'\big((a_1, \xi_1) \cdots (a_n, \xi_n)\big) = \sum_{m=1}^{n} f(a_1 \cdots a_{m-1} \xi_m a_{m+1} \cdots a_n). \tag{2.13}$$

But because of (2.9), at most one term in the sum on the right-hand side of (2.13) can be different from 0; moreover such a term can only occur for $m = (n+1)/2$, if (n is odd and) $i_1 = i_{2m-1}, \ \dots, \ i_{m-1} = i_{m+1}$. Finally, if the latter equalities of indices are satisfied, then the unique term left in the sum from (2.13) is $\varphi(a_1 a_{2m-1}) \cdots \varphi(a_{m-1} a_{m+1}) f(\xi_m)$, and the conditions defining the infinitesimal freeness of $\mathcal{M}_1, \dots, \mathcal{M}_k$ in $(\mathcal{M}, \psi, \psi')$ follow.

Hence, by focusing on the link-algebra, one can incorporate the freeness of type B from [2] into the framework of this paper.

3. Background on non-crossing partitions and non-crossing cumulants

3.1. Non-crossing partitions

Notation 3.1. We will use the standard conventions of notation concerning non-crossing partitions (as they appear for instance in Lecture 9 of [5]). So for a positive integer n we denote by $NC(n)$ the set of all non-crossing partitions of $\{1, \dots, n\}$. We will use the abbreviation "$V \in \pi$" for "V is a block of π", and the number of blocks of $\pi \in NC(n)$ will be denoted as $|\pi|$. On

$NC(n)$ we will consider the partial order given by *reverse refinement*; that is, for $\pi, \rho \in NC(n)$ we write "$\pi \leqslant \rho$" to mean that every block of ρ is a union of blocks of π. The minimal and maximal element of $(NC(n), \leqslant)$ are denoted by 0_n (the partition of $\{1, \ldots, n\}$ into n blocks of 1 element each) and respectively 1_n (the partition of $\{1, \ldots, n\}$ into 1 block of n elements). It is easy to see that $(NC(n), \leqslant)$ is a lattice, i.e. that every $\pi, \rho \in NC(n)$ have a join (smallest common upper bound) and a meet (largest common lower bound), which will be denoted by $\pi \vee \rho$ and $\pi \wedge \rho$, respectively.

Remark 3.2. A block W of a partition $\pi \in NC(n)$ is called an *interval-block* if it is of the form $W = [p, q] \cap \mathbb{Z}$ for some $1 \leqslant p \leqslant q \leqslant n$. Every non-crossing partition has interval-blocks, and it is actually easy to check that the following stronger statement holds: let π be in $NC(n)$, let V be a block of π, and let $i < j$ be two elements of V which are consecutive in V (in the sense that $(i, j) \cap V \neq \emptyset$). If $j \neq i + 1$ (hence the interval (i, j) contains some integers) then there exists an interval-block W of π such that $W \subseteq (i, j)$.

Notation 3.3. The lattice of non-crossing partitions of type B of $2n$ elements will be denoted by $NC^{(B)}(n)$. Following the paper of Reiner [8] where $NC^{(B)}(n)$ was introduced, it is customary to denote the $2n$ elements that are being partitioned as $1, \ldots, n$ and $-1, \ldots, -n$, taken in the order $1 < \cdots < n < -1 < \cdots < -n$. So let $NC(\pm n)$ denote the lattice of all non-crossing partitions of the ordered set $\{1, \ldots, n\} \cup \{-1, \ldots, -n\}$. ($NC(\pm n)$ is thus just a copy of $NC(2n)$, where one puts different labels on some of the $2n$ points that are being partitioned.) Then $NC^{(B)}(n)$ consists of those partitions $\tau \in NC(\pm n)$ which have the symmetry property that

$$(V \text{ is a block of } \tau) \quad \Rightarrow \quad (-V \text{ is a block of } \tau)$$

(with $-V := \{-v \mid v \in V\} \subseteq \{1, \ldots, n\} \cup \{-1, \ldots, -n\}$). $NC^{(B)}(n)$ inherits from $NC(\pm n)$ the partial order by reverse refinement, and is closed under the operations $\vee$, $\wedge$, hence is a sublattice of $NC(\pm n)$. Note also that $NC^{(B)}(n)$ contains the minimal and maximal elements of $NC^{(B)}(n)$, which will be denoted as $0_{\pm n}$ and $1_{\pm n}$, respectively.

A block Z of a partition $\tau \in NC^{(B)}(n)$ is called a *zero-block* when it satisfies the condition $Z = -Z$. The set $\{\tau \in NC^{(B)}(n) \mid \tau \text{ has zero-blocks}\}$ will be denoted by $NCZ^{(B)}(n)$. Due to the non-crossing property, it is immediate that every $\tau \in NCZ^{(B)}(n)$ has exactly one zero-block (hence it is justified to talk about "the zero-block" of τ).

Remark 3.4 *(Kreweras complementation).* An important ingredient in the study of the lattice $NC(n)$ is a special anti-automorphism $\mathrm{Kr} : NC(n) \to NC(n)$, called the *Kreweras complementation map* (see [5, pp. 147–148]). Since $NC(\pm n) \simeq NC(2n)$, one also has such a map Kr on $NC(\pm n)$. (All occurrences of Kreweras complementation maps in this paper will be denoted in the same way, by "Kr".) Moreover, the sublattice $NC^{(B)}(n) \subseteq NC(\pm n)$ turns out to be invariant under the Kr map of $NC(\pm n)$, hence one can talk about the Kreweras complementation map on $NC^{(B)}(n)$ as well. It is easily checked that $\mathrm{Kr} : NC^{(B)}(n) \to NC^{(B)}(n)$ maps the sets $NCZ^{(B)}(n)$ and $NC^{(B)}(n) \setminus NCZ^{(B)}(n)$ bijectively onto each other (see Section 1.2 of [2]).

Remark 3.5 *(Absolute value map).* Let $\mathrm{Abs} : \{1, \ldots, n\} \cup \{-1, \ldots, -n\} \to \{1, \ldots, n\}$ denote the absolute value map sending $\pm i$ to i for $1 \leqslant i \leqslant n$. In [2] it was observed that it makes sense to extend the concept of "absolute value" to non-crossing partitions. That is, for $\tau \in NC^{(B)}(n)$ it makes sense to define $\mathrm{Abs}(\tau) \in NC(n)$ to be the partition of $\{1, \ldots, n\}$ into blocks of the form

$\mathrm{Abs}(V)$, $V \in \tau$. Moreover, Section 1.4 of [2] puts into evidence the remarkable fact that the map $\mathrm{Abs} : NC^{(B)}(n) \to NC(n)$ so defined is an $(n+1)$-to-1 map, and explains precisely how to find the $n+1$ partitions in $\mathrm{Abs}^{-1}(\pi)$, for a given $\pi \in NC(n)$. A part of this result which is important for the present paper is that for every $\pi \in NC(n)$ and $V \in \pi$ there exists a unique $\tau \in NCZ^{(B)}(n)$ such that $\mathrm{Abs}(\tau) = \pi$ and such that the zero-block Z of τ has $\mathrm{Abs}(Z) = V$. Clearly, this can be rephrased by saying that we have a bijection

$$\begin{cases} NCZ^{(B)}(n) \;\to\; \bigl\{(\pi, V) \bigm| \pi \in NC(n),\ V \in \pi\bigr\} \\ \qquad\qquad\ \tau \;\mapsto\; \bigl(\mathrm{Abs}(\tau), \mathrm{Abs}(Z)\bigr), \\ \qquad\qquad\qquad \text{where } Z := \text{the unique zero-block of } \tau. \end{cases} \tag{3.1}$$

Moreover, for every $\pi \in NC(n)$, the $n+1-|\pi|$ partitions in $\mathrm{Abs}^{-1}(\pi)$ that are not accounted by (3.1) are all from $NC^{(B)}(n) \setminus NCZ^{(B)}(n)$, and are naturally indexed by the blocks of $\mathrm{Kr}(\pi)$. For the explanation of how this happens, we refer to the Remark on p. 2270 of [2].

Remark 3.6 *(Möbius functions).* We will use the notation "$\mathrm{Möb}^{(A)}$" for the Möbius functions of the lattices $NC(n)$. The value $\mathrm{Möb}^{(A)}(\pi, \rho)$ for $\pi \leqslant \rho$ in $NC(n)$ can be given explicitly, as a product of signed Catalan numbers (see [5, p. 163]). In the present paper we will not need the concrete values $\mathrm{Möb}^{(A)}(\pi, \rho)$, but only the Möbius inversion formula; this says that if we have two families of vectors $\{f_\pi \mid \pi \in NC(n)\}$ and $\{g_\pi \mid \pi \in NC(n)\}$ in the same vector space over $\mathbb{C}$, then the relations

$$g_\rho = \sum_{\pi \in NC(n),\, \pi \leqslant \rho} f_\pi, \quad \forall \rho \in NC(n) \tag{3.2}$$

are equivalent to

$$f_\rho = \sum_{\pi \in NC(n),\, \pi \leqslant \rho} \mathrm{Möb}^{(A)}(\pi, \rho) \cdot g_\pi, \quad \forall \rho \in NC(n). \tag{3.3}$$

We will use the notation "$\mathrm{Möb}^{(B)}$" for the Möbius functions of the lattices $NC^{(B)}(n)$. The explicit values $\mathrm{Möb}^{(B)}(\sigma, \tau)$ for $\sigma \leqslant \tau$ in $NC^{(B)}(n)$ can be read off from the considerations in Section 3 of [8]. Here we will only need a simple connection between the types A and B, saying that

$$\bigl(\sigma \leqslant \tau \text{ in } NCZ^{(B)}(n)\bigr) \Rightarrow \mathrm{Möb}^{(B)}(\sigma, \tau) = \mathrm{Möb}^{(A)}\bigl(\mathrm{Abs}(\sigma), \mathrm{Abs}(\tau)\bigr). \tag{3.4}$$

For the proof of (3.4) one observes that Abs gives a poset isomorphism between the intervals $[\sigma, \tau] \subseteq NC^{(B)}(n)$ and $[\mathrm{Abs}(\sigma), \mathrm{Abs}(\tau)] \subseteq NC(n)$, then uses the fact that the values $\mathrm{Möb}^{(B)}(\sigma, \tau)$ and $\mathrm{Möb}^{(A)}(\mathrm{Abs}(\sigma), \mathrm{Abs}(\tau))$ only depend on the isomorphism classes (in the category of posets) of these intervals.

3.2. Non-crossing cumulants, in the usual $\mathbb{C}$-valued setting

The following notation for "restrictions of n-tuples" will be used throughout the whole paper.

Notation 3.7. Let $(a_1, \dots, a_n)$ be an n-tuple of elements in a set $\mathcal{A}$, and let $V = \{v_1, \dots, v_m\}$ be a non-empty subset of $\{1, \dots, n\}$, with $v_1 < \cdots < v_m$. Then we denote

$$(a_1, \dots, a_n) \mid V := (a_{v_1}, \dots, a_{v_m}) \in \mathcal{A}^m. \tag{3.5}$$

Definition 3.8. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space. The multilinear functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n \geqslant 1}$ defined by

$$\kappa_n(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \Bigl(\text{Möb}^{(A)}(\pi, 1_n) \cdot \prod_{V \in \pi} \varphi_{|V|}\bigl((a_1, \dots, a_n) \bigm| V\bigr) \Bigr),$$
$$\text{for } n \geqslant 1 \text{ and } a_1, \dots, a_n \in \mathcal{A} \tag{3.6}$$

(with $\varphi_{|V|}$ as in Remark 3.10 below) are called the *non-crossing cumulant functionals* associated to $(\mathcal{A}, \varphi)$.

The importance of non-crossing cumulants for free probability theory comes from the following theorem, originally found in [9] (see also the detailed presentation in Lecture 11 of [5]).

Theorem 3.9. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. The following statements are equivalent*:

(1) *$\mathcal{A}_1, \dots, \mathcal{A}_k$ are free.*
(2) *For every $n \geqslant 2$, for every $i_1, \dots, i_n \in \{1, \dots, k\}$ which are not all equal to each other, and for every $a_1 \in \mathcal{A}_{i_1}, \ \dots, \ a_n \in \mathcal{A}_{i_n}$, one has that $\kappa_n(a_1, \dots, a_n) = 0$.*

Remark 3.10. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space. For every $n \geqslant 1$ let us consider the multiplication map

$$\text{Mult}_n : \mathcal{A}^n \to \mathcal{A}, \qquad \text{Mult}_n(a_1, \dots, a_n) = a_1 \cdots a_n, \tag{3.7}$$

and let us denote $\varphi_n := \varphi \circ \text{Mult}_n$. The multilinear functionals $(\varphi_n)_{n \geqslant 1}$ are called the *moment functionals* of $(\mathcal{A}, \varphi)$. For every $n \geqslant 1$ and $\pi \in NC(n)$ let us next define a multilinear functional $\varphi_\pi^{(A)} : \mathcal{A}^n \to \mathbb{C}$ by[5]

$$\varphi_\pi^{(A)}(a_1, \dots, a_n) := \prod_{V \in \pi} \varphi_{|V|}\bigl((a_1, \dots, a_n) \bigm| V\bigr), \quad a_1, \dots, a_n \in \mathcal{A}. \tag{3.8}$$

Then Definition 3.8 can be rephrased as saying that

$$\kappa_n = \sum_{\pi \in NC(n)} \text{Möb}^{(A)}(\pi, 1_n) \cdot \varphi_\pi^{(A)} \in \mathfrak{M}_n, \tag{3.9}$$

[5] The superscript "(A)" is used in anticipation of the fact that some multilinear functionals $\varphi_\tau^{(B)}$ with $\tau \in NC^{(B)}(n)$ will appear in Section 6 of the paper.

where $\mathfrak{M}_n$ denotes the vector space of multilinear functionals from $\mathcal{A}^n$ to $\mathbb{C}$. Moreover, if for every $\pi \in NC(n)$ we introduce (by analogy with (3.8)) a functional $\kappa_\pi^{(A)} \in \mathfrak{M}_n$ defined by

$$\kappa_\pi^{(A)}(a_1, \dots, a_n) := \prod_{V \in \pi} \kappa_{|V|}\big((a_1, \dots, a_n) \,\big|\, V\big), \quad a_1, \dots, a_n \in \mathcal{A}, \tag{3.10}$$

then it is not hard to see that the formula (3.9) for κ_n extends to

$$\kappa_\rho^{(A)} = \sum_{\pi \in NC(n),\, \pi \leqslant \rho} \text{Möb}^{(A)}(\pi, \rho) \cdot \varphi_\pi^{(A)}, \quad \forall \rho \in NC(n). \tag{3.11}$$

Thus for a given $n \geqslant 1$, the families of functionals $\{\kappa_\pi^{(A)} \mid \pi \in NC(n)\}$ and $\{\varphi_\pi^{(A)} \mid \pi \in NC(n)\}$ are exactly as in the above Remark 3.6. Eq. (3.11) and its equivalent counterpart which expresses $\varphi_\rho^{(A)}$ as the sum of the functionals $\{\kappa_\pi^{(A)} \mid \pi \leqslant \rho\}$ go under the name of *non-crossing moment-cumulant formulas* for $(\mathcal{A}, \varphi)$.

3.3. Non-crossing cumulants in the $\mathbb{G}$-valued setting

Remark 3.11. We will work with the Grassman algebra $\mathbb{G}$ from Section 1.1, and with the maps $\text{Bo}, \text{So} : \mathbb{G} \to \mathbb{C}$ defined in Section 1.2. It is immediate that the multiplication of $\mathbb{G}$ is commutative, and that the "body" map $\text{Bo} : \mathbb{G} \to \mathbb{C}$ is a homomorphism of unital algebras. Concerning how the "soul" map So behaves with respect to multiplication, we record the immediately verified formula

$$\text{So}(\gamma_1 \cdots \gamma_n) = \sum_{i=1}^{n} \Bigg(\text{So}(\gamma_i) \cdot \prod_{\substack{\leqslant j \leqslant n, \\ j \neq i}} \text{Bo}(\gamma_j) \Bigg), \quad \forall n \geqslant 1,\ \forall \gamma_1, \dots, \gamma_n \in \mathbb{G}. \tag{3.12}$$

Notation 3.12. *For the rest of this subsection we fix a pair* $(\mathcal{A}, \widetilde{\varphi})$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\widetilde{\varphi} : \mathcal{A} \to \mathbb{G}$ is $\mathbb{C}$-linear with $\widetilde{\varphi}(1_{\mathcal{A}}) = 1$. In connection to this $\widetilde{\varphi}$ we will repeat all the constructions of functionals described in Remark 3.10, with the only difference that the range space of these functionals is now $\mathbb{G}$. So for every $n \geqslant 1$ we put $\widetilde{\varphi}_n = \widetilde{\varphi} \circ \text{Mult}_n : \mathcal{A}^n \to \mathbb{G}$, where $\text{Mult}_n : \mathcal{A}^n \to \mathcal{A}$ is the same as in Equation (3.7). Then for every $\pi \in NC(n)$ we define $\widetilde{\varphi}_\pi : \mathcal{A}^n \to \mathbb{G}$ by

$$\widetilde{\varphi}_\pi(a_1, \dots, a_n) := \prod_{V \in \pi} \widetilde{\varphi}_{|V|}\big((a_1, \dots, a_n) \,\big|\, V\big), \quad a_1, \dots, a_n \in \mathcal{A}. \tag{3.13}$$

This is followed by defining a family of cumulant functionals $(\widetilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G})_{n \geqslant 1}$, where

$$\widetilde{\kappa}_n = \sum_{\pi \in NC(n)} \text{Möb}^{(A)}(\pi, 1_n) \cdot \widetilde{\varphi}_\pi, \quad n \geqslant 1. \tag{3.14}$$

Finally, for every $\pi \in NC(n)$ we define $\widetilde{\kappa}_\pi : \mathcal{A}^n \to \mathbb{G}$ by

$$\widetilde{\kappa}_\pi(a_1, \dots, a_n) := \prod_{V \in \pi} \widetilde{\kappa}_{|V|}\big((a_1, \dots, a_n) \,\big|\, V\big), \quad a_1, \dots, a_n \in \mathcal{A}. \tag{3.15}$$

It is easily seen that, exactly as in the $\mathbb{C}$-valued case from Remark 3.10, the families of functionals $\{\widetilde{\kappa}_\pi \mid \pi \in NC(n)\}$ and $\{\widetilde{\varphi}_\pi \mid \pi \in NC(n)\}$ are related by moment-cumulant formulas (i.e. by summation formulas as shown in Eqs. (3.2), (3.3) of Remark 3.6). We only record here the special case of moment-cumulant formula which expresses $\widetilde{\varphi}_{1_n}$ as a sum of cumulant functionals, and thus says that

$$\widetilde{\varphi}(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \widetilde{\kappa}_\pi(a_1, \ldots, a_n) \in \mathbb{G}, \quad \forall a_1, \ldots, a_n \in \mathcal{A}. \tag{3.16}$$

Remark 3.13. A natural question concerning $(\mathcal{A}, \widetilde{\varphi})$ is whether the analogue of Theorem 3.9 holds in this framework. As will be explained in detail in Remark 4.9 below, both conditions (1) and (2) from the statement of Theorem 3.9 can be faithfully transcribed in the context of $(\mathcal{A}, \widetilde{\varphi})$, but then they are no longer equivalent to each other – the implication (2) $\Rightarrow$ (1) still holds, but its converse does not.

In the remaining part of this subsection we will point out two other facts from the theory of usual non-crossing cumulants where (unlike for Theorem 3.9) both the statement and the proof can be transcribed without any problems from usual $\mathbb{C}$-valued framework to the $\mathbb{G}$-valued framework of $(\mathcal{A}, \widetilde{\varphi})$.

Proposition 3.14. *One has that $\widetilde{\kappa}_n(a_1, \ldots, a_n) = 0$ whenever $n \geqslant 2$, $a_1, \ldots, a_n \in \mathcal{A}$, and there exists $1 \leqslant m \leqslant n$ such that $a_m \in \mathbb{C}1_{\mathcal{A}}$.*

Proof. This is the analogue of Proposition 11.15 in [5]. It is straightforward (left to the reader) to see that the proof shown on p. 182 of [5] goes without any changes to the $\mathbb{G}$-valued framework. □

Proposition 3.15. *Let $x_1, \ldots, x_s$ be in $\mathcal{A}$ and consider some products of the form*

$$a_1 = x_1 \cdots x_{s_1}, \qquad a_2 = x_{s_1+1} \cdots x_{s_2}, \qquad \ldots, \qquad a_n = x_{s_{n-1}+1} \cdots x_{s_n},$$

where $1 \leqslant s_1 < s_2 < \cdots < s_n = s$. Then

$$\widetilde{\kappa}_n(a_1, \ldots, a_n) = \sum_{\substack{\pi \in NC(s) \text{ such} \\ \text{that } \pi \vee \theta = 1_s}} \widetilde{\kappa}_\pi(x_1, \ldots, x_s), \tag{3.17}$$

where $\theta \in NC(s)$ is the partition with interval blocks $\{1, \ldots, s_1\}$, $\{s_1 + 1, \ldots, s_2\}$, $\ldots$, $\{s_{n-1} + 1, \ldots, s_n\}$.

Proof. This is the analogue of Theorem 11.20 in [5], and the proof of this theorem (as shown on pp. 178–180 of [5]) goes without any changes to the $\mathbb{G}$-valued framework. □

4. Infinitesimal cumulants and the proof of Theorem 1.2

Notation 4.1. *Throughout this whole section we fix an incps $(\mathcal{A}, \varphi, \varphi')$.* We will use the notation "κ_n" for the non-crossing cumulant functionals associated to φ, as described in Section 3.2. Moreover, we will denote, same as in the introduction:

$$\widetilde{\varphi} = \varphi + \varepsilon\varphi' : \mathcal{A} \to \mathbb{G}$$

and we will consider the family of non-crossing cumulant functionals $(\widetilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G})_{n\geqslant 1}$ which are associated to $\widetilde{\varphi}$ as in Section 3.3.

Definition 4.2. For every $n \geqslant 1$, consider the multilinear functional $\kappa_n' : \mathcal{A}^n \to \mathbb{C}$ defined by the formula

$$\kappa_n'(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \sum_{V \in \pi} \Bigl[\text{Möb}(\pi, 1_n) \varphi'_{|V|}\bigl((a_1, \dots, a_n) \bigm| V\bigr) \cdot \prod_{\substack{W \in \pi \\ W \neq V}} \varphi_{|W|}\bigl((a_1, \dots, a_n) \mid W\bigr) \Bigr], \quad (4.1)$$

for $a_1, \dots, a_n \in \mathcal{A}$. The functionals κ_n' will be called *infinitesimal non-crossing cumulant functionals* associated to $(\mathcal{A}, \varphi, \varphi')$.

A moment's thought shows that Eq. (4.1) is indeed obtained from the formula (3.6) defining κ_n, where one uses the formal derivation procedure announced in Section 1.2 of the introduction.

We next make precise (in Propositions 4.3, 4.5 and Remark 4.4) the equivalence between Definition 4.2 and the other facets of κ_n' that were mentioned in Section 1.2.

Proposition 4.3. *Suppose that φ, φ' are the infinitesimal limit of a family $\{\varphi_t \mid t \in T\}$, in the sense described in Eq. (1.8). Let us use the notation $\kappa_n^{(t)}$ for the non-crossing cumulant functional of φ_t, for $t \in T$ and $n \geqslant 1$. Then for every $n \geqslant 1$ and every $a_1, \dots, a_n \in \mathcal{A}$ one has that*

$$\kappa_n(a_1, \dots, a_n) = \lim_{t \to 0} \kappa_n^{(t)}(a_1, \dots, a_n),$$

and

$$\kappa_n'(a_1, \dots, a_n) = \left[\frac{d}{dt} \kappa_n^{(t)}(a_1, \dots, a_n) \right] \Bigg|_{t=0}.$$

Proof. Fix $n \geqslant 1$ and $a_1, \dots, a_n \in \mathcal{A}$. For every $t \in T$ we have that

$$\kappa_n^{(t)}(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \text{Möb}^{(A)}(\pi, 1_n) \cdot \prod_{V \in \pi} \varphi_t\bigl((a_1, \dots, a_n) \bigm| V\bigr). \quad (4.2)$$

From (4.2) it is clear that $\lim_{t \to 0} \kappa_n^{(t)}(a_1, \dots, a_n) = \kappa_n(a_1, \dots, a_n)$. Moreover, it is immediate that the function of t appearing on the right-hand side of (4.2) has a derivative at 0; and upon using linearity and the Leibnitz formula to compute this derivative, one obtains precisely the formula (4.1) that defined $\kappa_n'(a_1, \dots, a_n)$. $\square$

Remark 4.4. As observed in Remark 3.5, the set $\{(\pi, V) \mid \pi \in NC(n), V \in \pi\}$ which indexes the sum on the right-hand side of Eq. (4.1) is the image of $NCZ^{(B)}(n)$ via the bijection $(\tau \in NCZ^{(B)}(n)$ with zero-block $Z) \mapsto (\text{Abs}(\tau), \text{Abs}(Z))$. When τ and (π, V) correspond to each other via this bijection, we have that $\text{Möb}^{(B)}(\tau, 1_{\pm n}) = \text{Möb}^{(A)}(\pi, 1_n)$ (cf. (3.4) in Remark 3.6); moreover, the rest of the product indexed by (π, V) on the right-hand side of Eq. (4.1) is precisely equal to $\varphi_\tau^{(B)}(a_1, \dots, a_n)$, where we anticipate here the notation $\varphi_\tau^{(B)}$ from Eq. (6.3). In conclusion, the change of variable from (V, π) to τ converts (4.1) into a summation formula which is a counterpart "of type B" for Eq. (3.9),

$$\kappa_n' = \sum_{\tau \in NCZ^{(B)}(n)} \text{Möb}^{(B)}(\tau, 1_{\pm n}) \cdot \varphi_\tau^{(B)}. \tag{4.3}$$

It is easy to see that (4.3) is equivalent to a plain summation formula which writes $\varphi'(a_1 \cdots a_n)$ in terms of cumulants (cf. Remark 6.5 below, where one also sees that the absence of terms indexed by partitions from $NC^{(B)}(n) \setminus NCZ^{(B)}(n)$ is caused by the fact that $\varphi'(1_{\mathcal{A}}) = 0$).

Proposition 4.5. *For every $n \geqslant 1$ one has that* $\text{Bo}\,\widetilde{\kappa}_n = \kappa_n$ *and* $\text{So}\,\widetilde{\kappa}_n = \kappa_n'$.

Proof. For the first statement we only have to take the body part on both sides of Eq. (3.14) and use the fact that $\text{Bo} : \mathbb{G} \to \mathbb{C}$ is a homomorphism of unital algebras. For the second statement we take soul parts in (3.14) and then use the multiplication formula (3.12). □

We now go to Theorem 1.2. Note that, in view of Proposition 4.5, the equalities "$\kappa_n(a_1, \dots, a_n) = \kappa_n'(a_1, \dots, a_n) = 0$" from condition (2) of Theorem 1.2 may be replaced with "$\widetilde{\kappa}_n(a_1, \dots, a_n) = 0$". We will prove Theorem 1.2 in this alternative form, which is stated below as Proposition 4.7.

Lemma 4.6. *Suppose that n is a positive integer and π is a partition in $NC(n)$, such that the following two properties hold*:

(i) *For every $1 \leqslant i \leqslant n-1$, the numbers i and $i+1$ do not belong to the same block of π.*
(ii) *π has at most one block of cardinality* 1.

Then n is odd, and π is the partition

$$\bigl\{\{1, n\}, \{2, n-1\}, \dots, \bigl\{(n-1)/2, (n+3)/2\bigr\}, \bigl\{(n+1)/2\bigr\}\bigr\}.$$

Proof. Clearly, condition (i) implies that π cannot have interval-blocks V with $|V| \geqslant 2$. By also taking (ii) and Remark 3.2 into account, we thus see that π has a unique interval-block V_o, of the form $V_o = \{p\}$ for some $1 \leqslant p \leqslant n$.

Let V be a block of π, distinct from V_o. We claim that

$$\bigl|V \cap [1, p)\bigr| \leqslant 1, \qquad \bigl|V \cap (p, n]\bigr| \leqslant 1. \tag{4.4}$$

Indeed, assume for instance that we had $|V \cap [1, p)| \geqslant 2$. Then we could find $i, j \in V$ such that $i < j < p$ and $(i, j) \cap V = \emptyset$. Note that $j \neq i+1$, due to condition (i); but then, as observed in Remark 3.2, the partition π must have an interval-block $W \cap (i, j)$, in contradiction to the fact that the unique interval-block of π is V_o.

For every block $V \neq V_o$ of π it then follows that $|V \cap [1, p)| = |V \cap (p, n]| = 1$. Indeed, if in (4.4) one of the sets $V \cap [1, p)$, $V \cap (p, n]$ would be empty, then it would follow that $|V| = 1$ and hypothesis (ii) would be contradicted.

The list of blocks of π which are distinct from V_o can thus be written in the form

$$\begin{cases} V_1 = \{i_1, j_1\}, \ \dots, \ V_m = \{i_m, j_m\}, \quad \text{where} \\ i_1 < p < j_1, \ \dots, \ i_m < p < j_m, \ \text{and } i_1 < i_2 < \dots < i_m. \end{cases} \tag{4.5}$$

Observe that in (4.5) we must have $j_1 > j_2 > \dots > j_m$. Indeed, if it was true that $j_s < j_t$ for some $1 \leqslant s < t \leqslant m$, then it would follow that $i_s < i_t < p < j_s < j_t$, and the blocks V_s, V_t would cross. Hence we have obtained $i_1 < \dots < i_m < p < j_m < \dots < j_1$; together with (4.5), this implies that $n = 2m + 1$ and that π is precisely the partition indicated in the lemma. □

Proposition 4.7. *Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. The following statements are equivalent*:

(1) *$\mathcal{A}_1, \dots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.*
(2) *For every $n \geqslant 2$, for every $i_1, \dots, i_n \in \{1, \dots, k\}$ which are not all equal to each other, and for every $a_1 \in \mathcal{A}_{i_1}, \ \dots, \ a_n \in \mathcal{A}_{i_n}$, one has that $\widetilde{\kappa}_n(a_1, \dots, a_n) = 0$.*

Proof. "(1) ⇒ (2)". We prove the required statement about cumulants by induction on n. For the base case $n = 2$, consider elements $a_1 \in \mathcal{A}_{i_1}$ and $a_2 \in \mathcal{A}_{i_2}$, where $i_1 \neq i_2$. By using the formulas which define κ_2 and κ_2' and by invoking Eqs. (2.1) and (2.2) from Remark 2.2 we find that

$$\begin{cases} \kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2) = 0 \quad \text{and} \\ \kappa_2'(a_1, a_2) = \varphi'(a_1 a_2) - \varphi'(a_1)\varphi(a_2) - \varphi(a_1)\varphi'(a_2) = 0, \end{cases}$$

hence $\widetilde{\kappa}_2(a_1, a_2) = \kappa_2(a_1, a_2) + \varepsilon \kappa_2'(a_1, a_2) = 0$.

We now prove the induction step: assume that the vanishing of mixed cumulants is already proved for $1, 2, \dots, n-1$, where $n \geqslant 3$. We consider elements $a_1 \in \mathcal{A}_{i_1}, \ \dots, \ a_n \in \mathcal{A}_{i_n}$ where not all indices $i_1, \dots, i_n$ are equal to each other, and we want to prove that $\widetilde{\kappa}_n(a_1, \dots, a_n) = 0$. By invoking Proposition 3.14 we may replace every a_m with $a_m - \varphi(a_m)1_{\mathcal{A}}$, $1 \leqslant m \leqslant n$, and therefore assume without loss of generality that $\varphi(a_1) = \dots = \varphi(a_n) = 0$. Observe that this implies $\widetilde{\varphi}(a_p)\widetilde{\varphi}(a_q) = (\varepsilon\varphi'(a_p)) \cdot (\varepsilon\varphi'(a_q)) = 0$, hence that

$$\widetilde{\kappa}_2(a_p, a_q) = \widetilde{\varphi}(a_p a_q) - \widetilde{\varphi}(a_p)\widetilde{\varphi}(a_q) = \widetilde{\varphi}(a_p a_q), \quad \forall 1 \leqslant p < q \leqslant n. \tag{4.6}$$

Another assumption that can be made without loss of generality is that $i_m \neq i_{m+1}$, $\forall 1 \leqslant m < n$. Indeed, if there exists $1 \leqslant m < n$ such that $i_m = i_{m+1}$, then we invoke the special case of Proposition 3.15 which states that

$$\widetilde{\kappa}_{n-1}(a_1, \dots, a_m a_{m+1}, \dots, a_n) = \widetilde{\kappa}_n(a_1, \dots, a_n) + \sum_{\substack{\pi \in NC(n) \text{ with } |\pi| = 2 \\ \pi \text{ separates } m \text{ and } m+1}} \widetilde{\kappa}_\pi(a_1, \dots, a_n). \tag{4.7}$$

The induction hypothesis gives us that the left-hand side and every term in the sum on the right-hand side of Eq. (4.7) are equal to 0, and it follows that $\widetilde{\kappa}_n(a_1, \dots, a_n)$ must be 0 as well.

Hence for the rest of the proof of this induction step we will assume that $\varphi(a_1) = \dots = \varphi(a_n) = 0$ and that $i_1 \neq i_2, \ \dots, \ i_{n-1} \neq i_n$. This makes $a_1, \dots, a_n$ be exactly as in Definition 1.1, so we get that $\varphi(a_1 \cdots a_n) = 0$ and that $\varphi'(a_1 \cdots a_n)$ is as described in Eq. (1.5). In terms of the functional $\widetilde{\varphi}$, we have

$$\widetilde{\varphi}(a_1 \cdots a_n) = \varepsilon\varphi'(a_1 \cdots a_n)$$
$$= \begin{cases} \varepsilon\varphi(a_1\, a_n)\varphi(a_2\, a_{n-1}) \cdots \varphi(a_{(n-1)/2}\, a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2}), \\ \quad \text{if } n \text{ is odd and } i_1 = i_n,\, i_2 = i_{n-1},\ \ldots,\ i_{(n-1)/2} = i_{(n+3)/2}, \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.8}$$

Now let us consider the relation (3.16), written in the equivalent form

$$\widetilde{\kappa}_n(a_1, \ldots, a_n) = \widetilde{\varphi}(a_1 \cdots a_n) - \sum_{\substack{\pi \in NC(n), \\ \pi \neq 1_n}} \widetilde{\kappa}_\pi(a_1, \ldots, a_n). \tag{4.9}$$

Observe that if a partition $\pi \in NC(n)$ has two distinct blocks $\{p\}, \{q\}$ of cardinality one, then the term indexed by π on the right-hand side of (4.9) vanishes, because it contains the subproduct $\widetilde{\kappa}_1(a_p)\widetilde{\kappa}_1(a_q) = \widetilde{\varphi}(a_p)\widetilde{\varphi}(a_q) = 0$. On the other hand if $\pi \in NC(n)$ has a block V which contains two consecutive numbers i and $i+1$, then the term indexed by π on the right-hand side of (4.9) vanishes as well, due to the induction hypothesis. Hence the sum subtracted on the right-hand side of (4.9) can only get non-zero contributions from partitions $\pi \in NC(n)$ which satisfy the hypotheses of Lemma 4.6; from the lemma it then follows that the sum in question is 0 for n even, and is equal to

$$\widetilde{\kappa}_2(a_1, a_n)\widetilde{\kappa}_2(a_2, a_{n-1}) \cdots \widetilde{\kappa}_2(a_{(n-1)/2}, a_{(n+3)/2}) \cdot \widetilde{\kappa}_1(a_{(n+1)/2}) \tag{4.10}$$

for n odd.

Let us focus for a moment on the quantity that appeared in (4.10). The vanishing of mixed cumulants of order 2 (which is part of our induction hypothesis) implies that this quantity vanishes unless $i_1 = i_n$, $i_2 = i_{n-1}, \ldots, i_{(n-1)/2} = i_{(n+3)/2}$. In the case that the latter equalities of indices hold, we can continue (4.10) with

$$= \widetilde{\varphi}(a_1a_n)\widetilde{\varphi}(a_2a_{n-1}) \cdots \widetilde{\varphi}(a_{(n-1)/2}a_{(n+3)/2}) \cdot \widetilde{\varphi}(a_{(n+1)/2}) \quad \big(\text{due to (4.6)}\big)$$
$$= \varepsilon\varphi(a_1a_n)\varphi(a_2a_{n-1}) \cdots \varphi(a_{(n-1)/2}a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2}). \tag{4.11}$$

(The equality (4.11) holds because $\widetilde{\varphi}(a_{(n+1)/2}) = \varepsilon\varphi'(a_{(n+1)/2})$, and due to how the multiplication on $\mathbb{G}$ works.)

So all in all, what we have obtained is that

$$\widetilde{\kappa}_n(a_1, \ldots, a_n)$$
$$= \begin{cases} \widetilde{\varphi}(a_1 \cdots a_n) - \varepsilon\varphi(a_1a_n)\varphi(a_2a_{n-1}) \cdots \varphi(a_{(n-1)/2}a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2}), \\ \quad \text{if } n \text{ is odd and } i_1 = i_n,\, i_2 = i_{n-1},\ \ldots,\ i_{(n-1)/2} = i_{(n+3)/2}, \\ \widetilde{\varphi}(a_1 \cdots a_n), \quad \text{otherwise.} \end{cases} \tag{4.12}$$

By comparing Eqs. (4.12) and (4.8) we see that, in all cases, we have $\widetilde{\kappa}_n(a_1, \ldots, a_n) = 0$. This concludes the induction argument, and the proof of the implication (1) $\Rightarrow$ (2) of the proposition.

"(2) $\Rightarrow$ (1)". Consider indices $i_1, \ldots, i_n \in \{1, \ldots, k\}$ and elements $a_1 \in \mathcal{A}_{i_1},\ \ldots,\ a_n \in \mathcal{A}_{i_n}$ such that $i_1 \neq i_2,\ \ldots,\ i_{n-1} \neq i_n$ and such that $\varphi(a_1) = \cdots = \varphi(a_n) = 0$. We have to prove that $\varphi(a_1 \cdots a_n) = 0$ and that $\varphi'(a_1 \cdots a_n)$ is as described in formula (1.5) from Definition 1.1. To this end we consider the $\mathbb{G}$-valued moment $\widetilde{\varphi}(a_1 \cdots a_n) = \varphi(a_1 \cdots a_n) + \varepsilon\varphi'(a_1 \cdots a_n)$, and write it in terms of $\mathbb{G}$-valued cumulants as in Section 3.3:

$$\widetilde{\varphi}(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \prod_{V \in \pi} \widetilde{\kappa}_{|V|}\big((a_1, \ldots, a_n) \mid V\big). \tag{4.13}$$

An argument very similar to the one used in the proof of the implication (1) ⇒ (2) above shows that the sum on the right-hand side of (4.13) can only get non-zero contributions from partitions $\pi \in NC(n)$ which satisfy the hypotheses of Lemma 4.6. If n is even then there is no such partition, and we obtain $\widetilde{\varphi}(a_1 \cdots a_n) = 0$. If n is odd, then the sum in (4.13) reduces to only one term and we obtain that

$$\widetilde{\varphi}(a_1 \cdots a_n) = \widetilde{\kappa}_2(a_1, a_n)\widetilde{\kappa}_2(a_2, a_{n-1}) \cdots \widetilde{\kappa}_2(a_{(n-1)/2}, a_{(n+3)/2}) \cdot \widetilde{\kappa}_1(a_{(n+1)/2}). \tag{4.14}$$

Moreover, in the case when n is odd, the hypothesis that mixed cumulants vanish gives us that the right-hand side of (4.14) is equal to 0 unless we have $i_1 = i_n, \ \ldots, \ i_{(n-1)/2} = i_{(n+3)/2}$. And finally, if the latter equalities of indices hold, then the right-hand side of (4.14) gets converted into $\varepsilon\varphi(a_1 a_n)\varphi(a_2 a_{n-1}) \cdots \varphi(a_{(n-1)/2} a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2})$, by the same argument that led to (4.11) in the proof of the implication (1) ⇒ (2). The conclusion is that $\varphi(a_1 \cdots a_n) = 0$ (in all cases), and that $\varphi'(a_1 \cdots a_n)$ is as in Eq. (1.5), as required. □

Corollary 4.8. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_k$ be subsets of $\mathcal{A}$. The following statements are equivalent*:

(1) *$\mathcal{X}_1, \ldots, \mathcal{X}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.*
(2) *For every $n \geqslant 2$, for every $i_1, \ldots, i_n \in \{1, \ldots, k\}$ which are not all equal to each other, and for every $x_1 \in \mathcal{X}_{i_1}, \ \ldots, \ x_n \in \mathcal{X}_{i_n}$, one has that $\widetilde{\kappa}_n(x_1, \ldots, x_n) = 0$.*

Proof. This is a faithful copy of the proof giving the analogous result over $\mathbb{C}$ (cf. [5, Theorem 11.20]). For the reader's convenience, we repeat here the highlights of the argument. Let $\mathcal{A}_i$ denote the unital subalgebra of $\mathcal{A}$ generated by $\mathcal{X}_i$, $1 \leqslant i \leqslant k$. The infinitesimal freeness of $\mathcal{X}_1, \ldots, \mathcal{X}_k$ is by definition equivalent to the one of $\mathcal{A}_1, \ldots, \mathcal{A}_k$, hence to the fact that condition (2) from Proposition 4.7 holds. We must thus prove that ((2) in Proposition 4.7) is equivalent to ((2) in Corollary 4.8). The implication "⇒" is trivial. For "⇐" it suffices (by multilinearity of $\widetilde{\kappa}_n$ and Proposition 3.14) to prove that $\widetilde{\kappa}_n(a_1, \ldots, a_n) = 0$ when

$$a_1 = x_1 \cdots x_{s_1}, \qquad a_2 = x_{s_1+1} \cdots x_{s_2}, \qquad \ldots, \qquad a_n = x_{s_{n-1}+1} \cdots x_{s_n} \tag{4.15}$$

for $n \geqslant 2$ and $1 \leqslant s_1 < s_2 < \cdots < s_n$, where $x_1, \ldots, x_{s_1} \in \mathcal{X}_{i_1}, \ x_{s_1+1}, \ldots, x_{s_2} \in \mathcal{X}_{i_2}, \ \ldots,$ $x_{s_{n-1}+1}, \ldots, x_{s_n} \in \mathcal{X}_{i_n}$, and where the indices $i_1, \ldots, i_n$ are not all equal to each other. But for $a_1, \ldots, a_n$ as in (4.15), Proposition 3.15 gives us the cumulant $\widetilde{\kappa}_n(a_1, \ldots, a_n)$ as a sum of cumulants $\widetilde{\kappa}_\pi(x_1, \ldots, x_{s_n})$; and a direct combinatorial analysis (exactly as on p. 186 of [5]) shows that all the latter cumulants vanish because of condition (2) from Corollary 4.8. □

Remark 4.9. Since the functional $\widetilde{\varphi} : \mathcal{A} \to \mathbb{G}$ and its associated cumulants $\widetilde{\kappa}_n$ play such a central role in the proof of Theorem 1.2, it is natural to ask: can't one actually characterize infinitesimal freeness by the same kind of moment condition as in the definition of usual freeness, with the only modification that one now uses $\widetilde{\varphi}$ instead of φ? To be precise, consider the following condition which a family of unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_k \subseteq \mathcal{A}$ may or may not satisfy:

$$\begin{cases} \text{For every } n \geqslant 1 \text{ and } 1 \leqslant i_1, \dots, i_n \leqslant k \text{ such that } i_1 \neq i_2, \ \dots, \ i_{n-1} \neq i_n, \\ \text{and every } a_1 \in \mathcal{A}_{i_1}, \ \dots, \ a_n \in \mathcal{A}_{i_n} \text{ such that } \widetilde{\varphi}(a_1) = \cdots = \widetilde{\varphi}(a_n) = 0, \\ \text{one has that } \widetilde{\varphi}(a_1 \cdots a_n) = 0. \end{cases} \tag{4.16}$$

Isn't then condition (4.16) equivalent to infinitesimal freeness?

On the positive side it is immediate, directly from Definition 1.1, that (4.16) is indeed implied by infinitesimal freeness. However, the converse statement is not true: it may happen that (4.16) is satisfied and yet $\mathcal{A}_1, \dots, \mathcal{A}_k$ are not infinitesimally free. What causes this to happen is that one cannot generally "center" an element $a \in \mathcal{A}$ with respect to $\widetilde{\varphi}$ (the scalars available are from $\mathbb{C}$, and there may be no $\lambda \in \mathbb{C}$ such that $\widetilde{\varphi}(a - \lambda 1_{\mathcal{A}}) = 0$). This limits the scope of condition (4.16), and makes it insufficient for recomputing $\widetilde{\varphi}$ on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$ from the restrictions $\widetilde{\varphi} \mid \mathcal{A}_i$, $1 \leqslant i \leqslant k$.

For a simple concrete example showing how (4.16) may fail to imply infinitesimal freeness, suppose we are in the situation from Example 2.7, with $\mathcal{A} = \mathbb{C}[\mathbb{Z}_2 * \cdots * \mathbb{Z}_2]$ and where $\mathcal{A}_1 = \mathrm{span}\{1_{\mathcal{A}}, u_1\}, \dots, \mathcal{A}_k = \mathrm{span}\{1_{\mathcal{A}}, u_k\}$ are the k copies of $\mathbb{C}[\mathbb{Z}_2]$ canonically embedded into $\mathcal{A}$. Suppose moreover that the linear functionals $\varphi, \varphi' : \mathcal{A} \to \mathbb{C}$ are such that $\widetilde{\varphi} = \varphi + \varepsilon\varphi'$ satisfies

$$\widetilde{\varphi}(1_{\mathcal{A}}) = 1, \qquad \widetilde{\varphi}(u_1) = \cdots = \widetilde{\varphi}(u_k) = \varepsilon. \tag{4.17}$$

Then, no matter how $\widetilde{\varphi}$ acts on words of length $\geqslant 2$ made with $u_1, \dots, u_k$, it will be true that $\mathcal{A}_1, \dots, \mathcal{A}_k$ satisfy condition (4.16) with respect to $\widetilde{\varphi}$; this is due to the simple reason that the restrictions $\widetilde{\varphi} \mid \mathcal{A}_i$ $(1 \leqslant i \leqslant k)$ are one-to-one. But on the other hand, Remark 2.2 tells us that if $\mathcal{A}_1, \dots, \mathcal{A}_k$ are to be infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$, then $\widetilde{\varphi}$ is uniquely determined by (4.17); for example, the formulas given for illustration in Eqs. (2.1), (2.2) imply that we must have $\widetilde{\varphi}(u_1 u_2) = \widetilde{\varphi}(u_1)\widetilde{\varphi}(u_2) = \varepsilon^2 = 0$. Hence any choice of $\widetilde{\varphi}$ as in (4.17) and with $\widetilde{\varphi}(u_1 u_2) \neq 0$ provides an example for how condition (4.16) does not imply infinitesimal freeness.

We conclude this section by establishing the fact about traciality that was announced in Remark 2.3.

Lemma 4.10. *Let $\mathcal{B}$ be a unital subalgebra of $\mathcal{A}$, and suppose that $\widetilde{\varphi} \mid \mathcal{B}$ is a trace. Then*

$$\widetilde{\kappa}_n(b_1, b_2, \dots, b_n) = \widetilde{\kappa}_n(b_2, b_n, \dots, b_1), \quad \forall n \geqslant 2, \ b_1, \dots, b_n \in \mathcal{B}. \tag{4.18}$$

Proof. Let Γ be the cyclic permutation of $\{1, \dots, n\}$ defined by $\Gamma(1) = 2, \ \dots, \ \Gamma(n-1) = n$, $\Gamma(n) = 1$. It is easy to see (cf. [5, Exercise 9.41 on p. 153]) that Γ induces an automorphism of the lattice $NC(n)$ which maps $\pi = \{V_1, \dots, V_p\} \in NC(n)$ to $\Gamma \cdot \pi := \{\Gamma(V_1), \dots, \Gamma(V_p)\}$.

Now let some $b_1, \dots, b_n \in \mathcal{B}$ be given. The right-hand side of (4.18) is $\widetilde{\kappa}_n(b_{\Gamma(1)}, \dots, b_{\Gamma(n)})$, which is by definition equal to

$$\sum_{\pi \in NC(n)} \mathrm{Möb}^{(A)}(\pi, 1_n) \cdot \widetilde{\varphi}_\pi(b_{\Gamma(1)}, \dots, b_{\Gamma(n)}). \tag{4.19}$$

By taking into account the traciality of $\widetilde{\varphi}$ on $\mathcal{B}$ it is easily verified that $\widetilde{\varphi}_\pi(b_{\Gamma(1)}, \dots, b_{\Gamma(n)}) = \widetilde{\varphi}_{\Gamma \cdot \pi}(b_1, \dots, b_n)$, $\forall \pi \in NC(n)$. Since $\mathrm{Möb}^{(A)}(\Gamma \cdot \pi, 1_n) = \mathrm{Möb}^{(A)}(\Gamma \cdot \pi, \Gamma \cdot 1_n) =$

$\text{Möb}^{(A)}(\pi, 1_n)$, $\forall \pi \in NC(n)$, it becomes clear that the change of variable $\Gamma \cdot \pi =: \rho$ will convert the sum from (4.19) into the one which defines $\widetilde{\kappa}_n(b_1, \ldots, b_n)$. $\square$

Proposition 4.11. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ that are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. If $\varphi \mid \mathcal{A}_i$ and $\varphi' \mid \mathcal{A}_i$ are traces for every $1 \leqslant i \leqslant k$, then φ and φ' are traces on $Alg(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$.*

Proof. The given hypothesis and the required conclusion can be rephrased by saying that $\widetilde{\varphi}$ is a trace on every $\mathcal{A}_i$, and respectively that $\widetilde{\varphi}$ is a trace on $\text{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$. Clearly, the rephrased conclusion will follow if we prove that

$$\widetilde{\varphi}(x_1 \cdots x_{n-1} x_n) = \widetilde{\varphi}(x_n x_1 \cdots x_{n-1}) \tag{4.20}$$

for $x_1 \in \mathcal{A}_{i_1}, \ \ldots, \ x_n \in \mathcal{A}_{i_n}$, with $n \geqslant 2$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$. Let us fix such $n, i_1, \ldots, i_n$ and $x_1, \ldots, x_n$. It is moreover convenient to denote $y_1 := x_n$, $y_2 := x_1, \ldots, y_n := x_{n-1}$, so that (4.20) takes the form $\widetilde{\varphi}(x_1 \cdots x_n) = \widetilde{\varphi}(y_1 \cdots y_n)$.

Let π_o be the partition of $\{1, \ldots, n\}$ defined by the requirement that for $1 \leqslant p < q \leqslant n$ we have $(p, q \text{ in the same block of } \pi_o) \Leftrightarrow i_p = i_q$. The hypothesis that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free and Proposition 4.7 imply that

$$\widetilde{\varphi}(x_1 \cdots x_n) = \sum_{\substack{\pi \in NC(n) \text{ such} \\ \text{that } \pi \leqslant \pi_o}} \widetilde{\kappa}_\pi(x_1, \ldots, x_n). \tag{4.21}$$

(Note that π_o may not belong to $NC(n)$, but the inequality $\pi \leqslant \pi_o$ still makes sense, in reverse refinement order.) Now, by using Lemma 4.10 it is easily checked that for every $\pi \in NC(n)$ such that $\pi \leqslant \pi_o$ one has

$$\widetilde{\kappa}_\pi(x_1, \ldots, x_n) = \widetilde{\kappa}_{\Gamma \cdot \pi}(y_1, \ldots, y_n), \tag{4.22}$$

where "$\Gamma \cdot \pi$" has the same significance as in the proof of Lemma 4.10. If we combine (4.21) with (4.22) and then make the change of variable $\Gamma \cdot \pi =: \rho$, we arrive to

$$\widetilde{\varphi}(x_1 \cdots x_n) = \sum_{\substack{\rho \in NC(n) \text{ such} \\ \text{that } \rho \leqslant \Gamma \cdot \pi_o}} \widetilde{\kappa}_\rho(y_1, \ldots, y_n). \tag{4.23}$$

Finally, we invoke once more the infinitesimal freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ and Proposition 4.7, to conclude that the right-hand side of (4.23) is precisely the moment-cumulant expansion for $\widetilde{\varphi}(y_1 \cdots y_n)$. $\square$

5. Alternating products of infinitesimally free random variables

In Proposition 4.7 we saw that infinitesimal freeness can be described as a vanishing condition for mixed $\mathbb{G}$-valued cumulants. Because of this fact and because $\mathbb{G}$ is commutative, (which makes practically all calculations with non-crossing cumulants go without any change from $\mathbb{C}$-valued to $\mathbb{G}$-valued framework) we get a "generic method" for proving infinitesimal versions of various results presented in the monograph [5] – replace $\mathbb{C}$ by $\mathbb{G}$ in the proof of the original result, then take the soul part of what comes out. Note that the infinitesimal results so obtained do not have $\mathbb{G}$ in their statement, hence could also be attacked by using other approaches to infinitesimal freeness (in which case, however, proving them may be more than a straightforward routine).

In this section we show how the generic method suggested above works when applied to the topic of alternating products of infinitesimally free random variables. In particular, we will obtain the infinitesimal versions for two important facts related to this topic, that were originally found in [4] – one of them is about compressions by free projections, the other concerns a method of constructing free families of free Poisson elements. Since the proofs of the $\mathbb{G}$-valued formulas that we need are identical to those of their $\mathbb{C}$-valued counterparts, we will not give them here, but we will merely indicate where in [5] can the $\mathbb{C}$-valued proofs be exactly found. The starting point is provided by the following formulas, obtained by doing the $\mathbb{C}$-to-$\mathbb{G}$ change in Theorem 14.4 of [5].

Proposition 5.1. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps and let $\mathcal{A}_1, \mathcal{A}_2$ be unital subalgebras of $\mathcal{A}$ which are infinitesimally free. Consider the functional $\widetilde{\varphi} = \varphi + \varepsilon\varphi' : \mathcal{A} \to \mathbb{G}$ and the associated cumulant functionals $(\widetilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G})_{n\geqslant 1}$. Recall that for every $n \geqslant 1$ and $\pi \in NC(n)$ we also have functionals $\widetilde{\varphi}_\pi, \widetilde{\kappa}_\pi : \mathcal{A}^n \to \mathbb{G}$, as defined in Notation* 3.12.

1^o *For every $a_1, \ldots, a_n \in \mathcal{A}_1$ and $b_1, \ldots, b_n \in \mathcal{A}_2$ one has that*

$$\widetilde{\varphi}(a_1b_1 \cdots a_nb_n) = \sum_{\pi \in NC(n)} \widetilde{\kappa}_\pi(a_1, \ldots, a_n) \cdot \widetilde{\varphi}_{Kr(\pi)}(b_1, \ldots, b_n). \tag{5.1}$$

2^o *For every $a_1, \ldots, a_n \in \mathcal{A}_1$ and $b_1, \ldots, b_n \in \mathcal{A}_2$ one has that*

$$\widetilde{\kappa}_n(a_1b_1, \ldots, a_nb_n) = \sum_{\pi \in NC(n)} \widetilde{\kappa}_\pi(a_1, \ldots, a_n) \cdot \widetilde{\kappa}_{Kr(\pi)}(b_1, \ldots, b_n). \tag{5.2}$$

We now start on the application to free compressions.

Definition 5.2. Let $(\mathcal{A}, \varphi, \varphi')$ be an incps, and let $p \in \mathcal{A}$ be an idempotent element such that $\varphi(p) \neq 0$. We denote $\varphi(p) =: \alpha$ and $\varphi'(p) = \alpha'$. The *compression* of $(\mathcal{A}, \varphi, \varphi')$ by p is then defined to be the incps $(\mathcal{B}, \psi, \psi')$ where

$$\mathcal{B} := p\mathcal{A}p = \{b \in \mathcal{A} \mid pb = b = bp\} \tag{5.3}$$

and where $\psi, \psi' : \mathcal{B} \to \mathbb{C}$ are defined by

$$\psi(b) = \frac{1}{\alpha}\varphi(b), \qquad \psi'(b) = \frac{1}{\alpha}\varphi'(b) - \frac{\alpha'}{\alpha^2}\varphi(b), \quad b \in \mathcal{B}. \tag{5.4}$$

Remark 5.3. 1^o In the preceding definition, note that the Grassman number $\widetilde{\alpha} := \alpha + \varepsilon\alpha'$ is invertible in $\mathbb{G}$, with inverse $1/\widetilde{\alpha} = (1/\alpha) - \varepsilon(\alpha'/\alpha^2)$. As a consequence, the two formulas given in (5.4) are equivalent to the consolidated functional $\widetilde{\psi} = \psi + \varepsilon\psi' : \mathcal{B} \to \mathbb{G}$ satisfying

$$\widetilde{\psi}(b) = \frac{1}{\widetilde{\alpha}}\widetilde{\varphi}(b), \quad \forall b \in \mathcal{B}. \tag{5.5}$$

2^o If in the preceding definition $(\mathcal{A}, \varphi, \varphi')$ is a $*$-incps and p is a projection, then by using the relations $p = p^* = p^2$ we immediately infer that $0 < \alpha \leqslant 1$ and $\alpha' \in \mathbb{R}$. As a consequence, $(\mathcal{B}, \psi, \psi')$ defined there is a $*$-incps as well.

Theorem 5.4. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps. Let $p \in \mathcal{A}$ be an idempotent element such that $\varphi(p) \neq 0$. Denote $\varphi(p) =: \alpha$, $\varphi'(p) =: \alpha'$, and consider the compressed incps $(\mathcal{B}, \psi, \psi')$ from Definition 5.2. For every $n \geqslant 1$ let $\kappa_n, \kappa_n' : \mathcal{A}^n \to \mathbb{C}$ and $\underline{\kappa}_n, \underline{\kappa}_n' : \mathcal{B}^n \to \mathbb{C}$ be the nth non-crossing cumulant and infinitesimal cumulant functional associated to $(\mathcal{A}, \varphi, \varphi')$ and to $(\mathcal{B}, \psi, \psi')$, respectively. Let $\mathcal{X}$ be a subset of $\mathcal{A}$ which is infinitesimally free from $\{p\}$. Then we have*

$$\underline{\kappa}_n(px_1p, \ldots, px_np) = \frac{1}{\alpha}\kappa_n(\alpha x_1, \ldots, \alpha x_n), \quad \forall n \geqslant 1,\ x_1, \ldots, x_n \in \mathcal{X} \tag{5.6}$$

and

$$\begin{cases} \underline{\kappa}_1'(px_1p) = \kappa_1'(x_1), \quad \forall x_1 \in \mathcal{X}, \\ \underline{\kappa}_n'(px_1p, \ldots, px_np) = \dfrac{(n-1)\alpha'}{\alpha^2}\kappa_n'(\alpha x_1, \ldots, \alpha x_n), \quad \forall n \geqslant 2,\ x_1, \ldots, x_n \in \mathcal{X}. \end{cases} \tag{5.7}$$

Proof. It is easily verified that Eqs. (5.6) and (5.7) are the body part and respectively the soul part for the formula

$$\widetilde{\underline{\kappa}}_n(px_1p, \ldots, px_np) = \widetilde{\alpha}^{n-1} \cdot \widetilde{\kappa}_n(x_1, \ldots, x_n) \in \mathbb{G}, \quad \forall n \geqslant 1,\ x_1, \ldots, x_n \in \mathcal{X}, \tag{5.8}$$

where the "tilde" notations have their usual meaning ($\widetilde{\underline{\kappa}}_n = \underline{\kappa}_n + \varepsilon \cdot \underline{\kappa}_n'$, $\widetilde{\alpha} = \alpha + \varepsilon \cdot \alpha'$). But the latter formula is just the $\mathbb{G}$-valued counterpart for Theorem 14.10 in [5]; its proof is obtained by faithfully doing the $\mathbb{C}$-to-$\mathbb{G}$ transcription of the proof of that theorem from [5], with the minor change that the powers of $\widetilde{\alpha}$ must be kept outside the cumulant functionals (one cannot write "$\widetilde{\kappa}_n(\widetilde{\alpha}x_1, \ldots, \widetilde{\alpha}x_n)$", since $\mathcal{A}$ is only a $\mathbb{C}$-algebra). Note that the argument obtained in this way is indeed an application of Proposition 5.1, in the same way as Theorem 14.10 is an application of Theorem 14.4 in [5]. □

Corollary 5.5. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps. Let $p \in \mathcal{A}$ be an idempotent element with $\varphi(p) \neq 0$, and consider the compressed incps $(\mathcal{B}, \psi, \psi')$ defined as above. Let $\mathcal{X}_1, \ldots, \mathcal{X}_k$ be subsets of $\mathcal{A}$ such that $\{p\}, \mathcal{X}_1, \ldots, \mathcal{X}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. Put $\mathcal{Y}_i = p\mathcal{X}_ip \subseteq \mathcal{B}$, $1 \leqslant i \leqslant k$. Then $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$ are infinitesimally free in $(\mathcal{B}, \psi, \psi')$.*

Proof. This is an immediate consequence of Corollary 4.8, where the needed vanishing of mixed cumulants follows from the explicit formulas found in Theorem 5.4. □

We now go to the construction of families of infinitesimally free Poisson elements. We will use the infinitesimal (a.k.a "type B") versions of semicircular and of free Poisson elements that appeared in [7] in connection to limit theorems of type B, and are discussed in detail in Sections 4 and 5 of [1]. For the present paper it is most convenient to introduce these elements in terms of their infinitesimal cumulants, as stated in Definitions 5.6 and 5.8 below.

Definition 5.6. Let $(\mathcal{A}, \varphi, \varphi')$ be a $*$-incps. A selfadjoint element $x \in \mathcal{A}$ will be called *infinitesimally semicircular* when it satisfies

$$\kappa_n(x, \dots, x) = \kappa'_n(x, \dots, x) = 0, \quad \forall n \geqslant 3. \tag{5.9}$$

If in addition to that we also have

$$\kappa_1(x) = 0, \qquad \kappa_2(x, x) = 1, \tag{5.10}$$

then we will say that x is a *standard* infinitesimally semicircular element.

Remark 5.7. 1^o By using the multilinearity of κ_n, κ'_n and Proposition 3.14, it is immediately seen that if x is infinitesimally semicircular then so is $\alpha(x - \beta 1_{\mathcal{A}})$ for any $\alpha > 0$ and $\beta \in \mathbb{R}$. Moreover, leaving aside the trivial case when $\kappa_2(x, x) = 0$, one can always pick α and β so that $\alpha(x - \beta 1_{\mathcal{A}})$ is standard.

2^o Let x be standard infinitesimally semicircular in $(\mathcal{A}, \varphi, \varphi')$. Then all moments $\varphi(x^n)$ and $\varphi'(x^n)$ for $n \geqslant 1$ are completely determined by the real parameters[6] α'_1, α'_2 defined by

$$\alpha'_1 := \kappa'_1(x) = \varphi'(x), \quad \text{and} \quad \alpha'_2 := \kappa'_2(x, x) = \varphi'\big(x^2\big). \tag{5.11}$$

It is in fact very easy to calculate what these moments are. Indeed, one can calculate the $\mathbb{G}$-valued moments $\widetilde{\varphi}(x^n) = \varphi(x^n) + \varepsilon\varphi'(x^n)$ by using the moment-cumulant formula (3.16), where one takes into account that

$$\widetilde{\kappa}_1(x) = \varepsilon\alpha'_1, \qquad \widetilde{\kappa}_2(x, x) = 1 + \varepsilon\alpha'_2, \quad \text{and} \quad \widetilde{\kappa}_n(x, \dots, x) = 0 \quad \text{for all } n \geqslant 3.$$

The expansion of $\widetilde{\varphi}(x^n)$ in terms of $\{\widetilde{\kappa}_\pi(x, \dots, x) \mid \pi \in NC(n)\}$ can get non-zero contributions only from such partitions π where every block V of π has $|V| \leqslant 2$ and where there is at most one block of π of cardinality 1 (the latter condition coming from the fact that $(\widetilde{\kappa}_1(x))^2 = 0$). We distinguish two cases, depending on the parity of n.

Case 1. n is even, $n = 2m$. We get a sum extending over non-crossing pairings in $NC(n)$, which gives us

$$\widetilde{\varphi}\big(x^{2m}\big) = C_m \cdot \big(1 + \varepsilon\alpha'_2\big)^m = C_m \cdot \big(1 + \varepsilon m\alpha'_2\big),$$

or in other words

$$\varphi\big(x^{2m}\big) = C_m, \qquad \varphi'\big(x^{2m}\big) = \alpha'_2 \cdot (mC_m), \tag{5.12}$$

where C_m stands for the mth Catalan number.

[6] Any two numbers $\alpha'_1, \alpha'_2 \in \mathbb{R}$ can appear here. Indeed, Example 8.7 shows situations where one has $\alpha'_1 = 1, \alpha'_2 = 0$ and respectively $\alpha'_1 = 0, \alpha'_2 = 2$. One can rescale the functionals φ' of these two special cases to get standard infinitesimal semicirculars x_1, x_2 having any pairs of parameters $\alpha'_1, 0$ and respectively $0, \alpha'_2$; then due to Proposition 2.4 one may assume that x_1, x_2 are infinitesimally free, and form the average $(x_1 + x_2)/\sqrt{2}$, which is standard infinitesimally semicircular with generic parameters in (5.11).

Case 2. n is odd, $n = 2m + 1$. Here we get a sum extending over the partitions $\pi \in NC(n)$ which have one block of 1 element and m blocks of 2 elements. There are $(2m+1)C_m$ such partitions; so we obtain

$$\widetilde{\varphi}\big(x^{2m+1}\big) = (2m+1)C_m \cdot \big(\big(\varepsilon\alpha_1'\big)\big(1+\varepsilon\alpha_2'\big)^m\big),$$

leading to

$$\varphi\big(x^{2m+1}\big) = 0, \qquad \varphi'\big(x^{2m+1}\big) = \alpha_1' \cdot \big((2m+1)C_m\big). \tag{5.13}$$

Definition 5.8. Let $(\mathcal{A}, \varphi, \varphi')$ be a $*$-incps, and let λ, β', γ' be real parameters, where $\lambda > 0$. A selfadjoint element $y \in \mathcal{A}$ will be called *infinitesimally free Poisson* of parameter λ and[7] infinitesimal parameters β', γ' when it has non-crossing cumulants given by

$$\begin{cases} \kappa_n(y, \ldots, y) = \lambda, \\ \kappa_n'(y, \ldots, y) = \beta' + n\gamma', \quad \forall n \geqslant 1. \end{cases} \tag{5.14}$$

Theorem 5.9. *Let $(\mathcal{A}, \varphi, \varphi')$ be a $*$-incps. Let $x \in \mathcal{A}$ be a standard infinitesimally semicircular element, and let $\mathcal{S}$ be a subset of $\mathcal{A}$ which is infinitesimally free from $\{x\}$. Then for every $n \geqslant 1$ and $a_1, \ldots, a_n \in \mathcal{S}$ we have*

$$\kappa_n(xa_1x, \ldots, xa_nx) = \varphi(a_1 \cdots a_n) \tag{5.15}$$

and

$$\kappa_n'(xa_1x, \ldots, xa_nx) = \varphi'(a_1 \cdots a_n) + n\varphi'\big(x^2\big) \cdot \varphi(a_1 \cdots a_n). \tag{5.16}$$

Proof. Eqs. (5.15) and (5.16) are the body part and respectively the soul part for the formula

$$\widetilde{\kappa}_n(xa_1x, \ldots, xa_nx) = \big(\widetilde{\kappa}_2(x, x)\big)^n \cdot \widetilde{\varphi}(a_1 \cdots a_n) \in \mathbb{G}. \tag{5.17}$$

The proof of the latter formula is obtained by doing the $\mathbb{C}$-to-$\mathbb{G}$ transcription either for the arguments used in Proposition 12.18 and Example 12.19 on pp. 207–208 of [5], or for the arguments in Propositions 17.20 and 17.21 on pp. 283–284 of [5]. □

The ensuing construction of families of infinitesimally free Poisson elements is stated in the next corollary. Part 2^o of the corollary has also appeared as Corollary 36 of [1].

Corollary 5.10. *Let $(\mathcal{A}, \varphi, \varphi')$ be a $*$-incps, and let $x \in \mathcal{A}$ be a standard infinitesimally semicircular element. Let $e_1, \ldots, e_k \in \mathcal{A}$ be projections such that $e_i \perp e_j$ for $1 \leqslant i < j \leqslant k$ and such that $\{e_1, \ldots, e_k\}$ is infinitesimally free from $\{x\}$. Then*

- 1^o *The elements $xe_1x, \ldots, xe_kx$ form an infinitesimally free family in $(\mathcal{A}, \varphi, \varphi')$.*
- 2^o *For every $1 \leqslant i \leqslant k$, xe_ix is infinitesimally free Poisson with parameter λ_i and infinitesimal parameters β_i', γ_i' given by $\lambda_i = \varphi(e_i)$, $\beta_i' = \varphi'(e_i)$, $\gamma_i' = \varphi'(x^2) \cdot \varphi(e_i)$.*

[7] A more complete definition of these elements would also use a 4th parameter $r > 0$, and have each of λ, β', γ' multiplied by r^n in Eqs. (5.14). For the sake of simplicity, here we have set this additional parameter to $r = 1$.

Proof. 1^o This is an immediate consequence of Corollary 4.8, where the needed vanishing of mixed cumulants follows from the explicit formulas found in Theorem 5.9.

2^o By putting $a_1 = \cdots = a_n := e_i$ in (5.15) and (5.16) we see that the cumulants of xe_ix have the form required in Definition 5.8, with parameters $\lambda_i, \beta'_i, \gamma'_i$ as stated. □

6. Relations with the lattices $NC^{(B)}(n)$

In this section we remember that the concept of incps has its origins in the considerations "of type B" from [2], and we look at how the essence of these considerations persists in the framework of the present paper.

The strategy of [2] was to study the type B analogue for an operation with power series called *boxed convolution* and denoted by $\boxed{\star}$. The focus on $\boxed{\star}$ was motivated by the fact that it provides in some sense a "middle ground" between alternating products of free random variables and the structure of intervals in the lattices $NC(n)$ (see discussion on pp. 2282-2283 of [2]). The key point discovered in [2] (stated in the form of the equation $\boxed{\star}^{(B)} = \boxed{\star}^{(A)}_{\mathbb{G}}$ in the introduction of that paper) was that boxed convolution of type B can still be defined by the formulas from type A, provided that one uses scalars from $\mathbb{G}$.

For a detailed discussion on $\boxed{\star}$ we refer the reader to Lecture 17 of [5]. What is important for us here is that the formula used to define $\boxed{\star}$ (cf. Eq. (17.1) on p. 273 of [5]) has already made an appearance, in $\mathbb{G}$-valued context, in Eqs. (5.1), (5.2) of the preceding section. So then, the present incarnation of the "$\boxed{\star}^{(B)} = \boxed{\star}^{(A)}_{\mathbb{G}}$" principle from [2] should just amount to the following fact: if one takes the soul parts of Eqs. (5.1) and (5.2), then summations over $NC^{(B)}(n)$ must arise. This is stated precisely in Theorem 6.4 below, which is actually an easy application of the fact that the absolute value map $\mathrm{Abs} : NC^{(B)}(n) \to NC(n)$ is an $(n+1)$-to-1 cover.

We start by introducing some notations that will be used in Theorem 6.4, namely the type B analogues for the functionals $\varphi_\pi^{(A)}$ and $\kappa_\pi^{(A)}$ from Section 3.2.

Notation 6.1. Let $(\mathcal{A}, \varphi, \varphi')$ be an incps and let $(\kappa_n)_{n\geqslant 1}$, $(\kappa'_n)_{n\geqslant 1}$ be the families of non-crossing and respectively of infinitesimal non-crossing cumulant functionals associated to this incps. For every $n \geqslant 1$ and $\tau \in NC^{(B)}(n)$ we define a multilinear functional $\kappa_\tau^{(B)} : \mathcal{A}^n \to \mathbb{C}$, as follows.

Case 1. If $\tau \in NCZ^{(B)}(n)$, $\tau = \{Z, V_1, -V_1, \ldots, V_p, -V_p\}$, then we put

$$\kappa_\tau^{(B)}(a_1, \ldots, a_n) := \kappa'_{|Z|/2}\big(a_1, \ldots, a_n) \mid \mathrm{Abs}(Z)\big) \times \prod_{j=1}^{p} \kappa_{|V_j|}\big((a_1, \ldots, a_n) \mid \mathrm{Abs}(V_j)\big), \quad \text{for } a_1, \ldots, a_n \in \mathcal{A}. \tag{6.1}$$

Case 2. If $\tau \in NC^{(B)}(n) \setminus NCZ^{(B)}(n)$, $\tau = \{V_1, -V_1, \ldots, V_p, -V_p\}$, then we put

$$\kappa_\tau^{(B)}(a_1, \ldots, a_n) := \prod_{j=1}^{p} \kappa_{|V_j|}\big((a_1, \ldots, a_n) \mid \mathrm{Abs}(V_j)\big), \quad a_1, \ldots, a_n \in \mathcal{A}. \tag{6.2}$$

Notation 6.2. Let $(\mathcal{A}, \varphi, \varphi')$ be an incps. Consider the families of multilinear functionals $(\varphi_n, \varphi'_n : \mathcal{A}^n \to \mathbb{C})_{n\geqslant 1}$ defined by $\varphi_n = \varphi \circ \mathrm{Mult}_n$, $\varphi'_n = \varphi' \circ \mathrm{Mult}_n$, where $\mathrm{Mult}_n : \mathcal{A}^n \to \mathcal{A}$ is the multiplication map, $n \geqslant 1$ (same as used in Remark 3.10). Then for every $n \geqslant 1$ and every

$\tau \in NC^{(B)}(n)$ we define a multilinear functional $\varphi_\tau^{(B)} : \mathcal{A}^n \to \mathbb{C}$ by the same recipe as in Notation 6.1 (with discussion separated in 2 cases), where every occurrence of κ_m (respectively κ'_m) is replaced by φ_m (respectively φ'_m). For example, the analogue of Case 1 is like this: for $n \geqslant 1$ and for $\tau = \{Z, V_1, -V_1, \ldots, V_p, -V_p\}$ in $NCZ^{(B)}(n)$ we define $\varphi_\tau^{(B)} : \mathcal{A}^n \to \mathbb{C}$ by putting

$$\varphi_\tau^{(B)}(a_1, \ldots, a_n) := \varphi'_{|Z|/2}\big(a_1, \ldots, a_n) \mid \mathrm{Abs}(Z)\big) \cdot \prod_{j=1}^{p} \varphi_{|V_j|}\big((a_1, \ldots, a_n) \mid \mathrm{Abs}(V_j)\big), \tag{6.3}$$

for $a_1, \ldots, a_n \in \mathcal{A}$.

Remark 6.3. 1^o It is immediate that for $\tau \in NC^{(B)}(n) \setminus NCZ^{(B)}(n)$ one has

$$\kappa_\tau^{(B)} = \kappa_{\mathrm{Abs}(\tau)}^{(A)}, \qquad \varphi_\tau^{(B)} = \varphi_{\mathrm{Abs}(\tau)}^{(A)}. \tag{6.4}$$

2^o The functionals introduced in Notation 6.1 extend both families κ_n and κ'_n. Indeed, we have that $\kappa'_n = \kappa_{1_{\pm n}}^{(B)}$ and that $\kappa_n = \kappa_{1_n}^{(A)} = \kappa_\tau^{(B)}$ for every $n \geqslant 1$ and any $\tau \in NC^{(B)}(n)$ such that $\mathrm{Abs}(\tau) = 1_n$ (e.g. $\tau = \{\{1, \ldots, n\}, \{-1, \ldots, -n\}\}$). A similar remark holds in connection to the functionals $\varphi_\tau^{(B)}$ – they extend both families φ_n and φ'_n.

Theorem 6.4. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps, and consider multilinear functionals on $\mathcal{A}$ as in Notations* 6.1, 6.2. *Let $\mathcal{A}_1, \mathcal{A}_2$ be unital subalgebras of $\mathcal{A}$ which are infinitesimally free. Then for every $a_1, \ldots, a_n \in \mathcal{A}_1$ and $b_1, \ldots, b_n \in \mathcal{A}_2$ one has*

$$\varphi'(a_1 b_1 \cdots a_n b_n) = \sum_{\tau \in NC^{(B)}(n)} \kappa_\tau^{(B)}(a_1, \ldots, a_n) \cdot \varphi_{Kr(\tau)}^{(B)}(b_1, \ldots, b_n) \tag{6.5}$$

and

$$\kappa'_n(a_1 b_1, \ldots, a_n b_n) = \sum_{\tau \in NC^{(B)}(n)} \kappa_\tau^{(B)}(a_1, \ldots, a_n) \cdot \kappa_{Kr(\tau)}^{(B)}(b_1, \ldots, b_n). \tag{6.6}$$

Proof. Consider the "tilde" notations from Proposition 5.1. Let π be a partition in $NC(n)$, and let us look at the expression

$$\begin{aligned}
&\mathrm{So}\big(\widetilde{\kappa}_\pi(a_1, \ldots, a_n)\widetilde{\kappa}_{Kr(\pi)}(b_1, \ldots, b_n)\big) \\
&\quad = \mathrm{So}\bigg(\prod_{V \in \pi} \widetilde{\kappa}_{|V|}\big((a_1, \ldots, a_n) \mid V\big) \cdot \prod_{W \in Kr(\pi)} \widetilde{\kappa}_{|W|}\big((b_1, \ldots, b_n) \mid W\big)\bigg).
\end{aligned}$$

In view of the formula (3.12) describing the soul part of a product, the latter expression is equal to a sum of $n+1$ terms, some of them indexed by the blocks $V \in \pi$, and the others indexed by the blocks $W \in \mathrm{Kr}(\pi)$. We leave it as a straightforward exercise to the reader to write these $n+1$ terms explicitly, and verify that the natural correspondence to the $n+1$ partitions in $\{\tau \in NC^{(B)}(n) \mid \mathrm{Abs}(\tau) = \pi\}$ leads to the formula

$$\begin{aligned}
&\mathrm{So}\big(\widetilde{\kappa}_\pi(a_1,\ldots,a_n)\widetilde{\kappa}_{Kr(\pi)}(b_1,\ldots,b_n)\big)\\
&\quad=\sum_{\substack{\tau\in NC^{(B)}(n) \text{ such}\\ \text{that } \mathrm{Abs}(\tau)=\pi}} \kappa_\tau^{(B)}(a_1,\ldots,a_n)\cdot\varphi_{Kr(\tau)}^{(B)}(b_1,\ldots,b_n).
\end{aligned}\tag{6.7}$$

(Note: the Kreweras complement $\mathrm{Kr}(\tau)$ from (6.7) is taken in the lattice $NC^{(B)}(n)$; we use here the fact that $\mathrm{Abs}(\tau)=\pi \Rightarrow \mathrm{Abs}(\mathrm{Kr}(\tau))=\mathrm{Kr}(\pi)$ – cf. Lemma 1.4 in [2].)

By summing over $\pi\in NC(n)$ on both sides of (6.7), we obtain that

$$\mathrm{So}\big(\text{right-hand side of Eq. (5.1)}\big)=\big(\text{right-hand side of Eq. (6.5)}\big).$$

Since the soul part of the left-hand side of Eq. (5.1) is $\varphi'(a_1b_1\cdots a_nb_n)$, this proves that (6.5) holds. The verification of (6.6) is done in exactly the same way, by starting from Eq. (5.2) of Proposition 5.1. □

Remark 6.5. If in the preceding theorem we make $\mathcal{A}_1=\mathcal{A}$ and $\mathcal{A}_2=\mathbb{C}1_{\mathcal{A}}$, and if in Eq. (6.5) we take $b_1=\cdots=b_n=1_{\mathcal{A}}$, then we obtain the formula

$$\varphi'(a_1\cdots a_n)=\sum_{\tau\in NCZ^{(B)}(n)}\kappa_\tau^{(B)}(a_1,\ldots,a_n),\quad \forall a_1,\ldots,a_n\in\mathcal{A}.\tag{6.8}$$

The terms indexed by $\sigma\in NC^{(B)}(n)\setminus NCZ^{(B)}(n)$ have disappeared in (6.8), due to the fact that $\varphi'(1_{\mathcal{A}})=0$. This formula was also noticed (via a direct argument from the definition of the $\mathbb{G}$-valued functionals $\widetilde{\kappa}_n$) in Proposition 7.4.4 of [6].

7. Dual derivation systems

Notation 7.1. Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$, and for every $n\geqslant 1$ let $\mathfrak{M}_n$ denote the vector space of multilinear functionals from $\mathcal{A}^n$ to $\mathbb{C}$. If $\pi=\{V_1,\ldots,V_p\}$ is a partition in $NC(n)$ where the blocks $V_1,\ldots,V_p$ are listed in increasing order of their minimal elements, then we define a multilinear map

$$J_\pi:\mathfrak{M}_{|V_1|}\times\cdots\times\mathfrak{M}_{|V_p|}\ni(f_1,\ldots,f_p)\to f\in\mathfrak{M}_n,\tag{7.1}$$

where

$$f(a_1,\ldots,a_n):=\prod_{j=1}^p f_j\big((a_1,\ldots,a_n)\,\big|\,V_j\big),\quad \forall a_1,\ldots,a_n\in\mathcal{A}.\tag{7.2}$$

Remark 7.2. 1^o For a concrete example of how Notation 7.1 works, say for instance that $n=6$ and that $\pi=\{\{1,3,6\},\{2\},\{4,5\}\}\in NC(6)$. Then we have $J_\pi:\mathfrak{M}_3\times\mathfrak{M}_1\times\mathfrak{M}_2\to\mathfrak{M}_6$, and the fact that $J_\pi(f_1,f_2,f_3)=f$ means that

$$f(a_1,\ldots,a_6)=f_1(a_1,a_3,a_6)\cdot f_2(a_2)\cdot f_3(a_4,a_5),\quad \forall a_1,\ldots,a_6\in\mathcal{A}.$$

2^o The formula (7.2) from the preceding notation is the same as those used to define the families of functionals $\{\varphi_\pi^{(A)}\mid\pi\in NC(n)\}$ and $\{\kappa_\pi^{(A)}\mid\pi\in NC(n)\}$ in Remark 3.10. Hence if

$(\mathcal{A}, \varphi)$ is a noncommutative probability space and if $(\kappa_n)_{n\geqslant 1}$ are the non-crossing cumulant functionals associated to φ, then for $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ as in Notation 7.1 we get that

$$J_\pi(\kappa_{|V_1|}, \ldots, \kappa_{|V_p|}) = \kappa_\pi^{(A)}. \tag{7.3}$$

Likewise, for the same $(\mathcal{A}, \varphi)$ and π we get

$$J_\pi(\varphi_{|V_1|}, \ldots, \varphi_{|V_p|}) = \varphi_\pi^{(A)}, \tag{7.4}$$

where $\varphi_m = \varphi \circ \mathrm{Mult}_m : \mathcal{A}^m \to \mathbb{C}$, $m \geqslant 1$ (same as in Remark 3.10).

3^o Let $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ be as in Notation 7.1, and let $1 \leqslant j \leqslant p$ be such that V_j is an interval-block of π. Denote $|V_j| =: m$ and let $\overset{\vee}{\pi} \in NC(n-m)$ be the partition obtained by removing the block V_j out of π and by redenoting the elements of $\{1, \ldots, n\} \setminus V_j$ as $1, \ldots, n-m$, in increasing order. On the other hand, let us denote by $\gamma \in NC(n)$ the partition of $\{1, \ldots, n\}$ into the two blocks V_j and $\{1, \ldots, n\} \setminus V_j$. It is then immediate that for every $f_1 \in \mathfrak{M}_{|V_1|}, \ldots, f_p \in \mathfrak{M}_{|V_p|}$ we can write

$$J_\pi(f_1, \ldots, f_p) = J_\gamma(g, f_j) \quad \text{where } g := J_{\overset{\vee}{\pi}}(f_1, \ldots f_{j-1}, f_{j+1}, \ldots, f_p). \tag{7.5}$$

Due to this observation and to the fact that every non-crossing partitions has interval-blocks, considerations about the multilinear functions J_π from Notation 7.1 can sometimes be reduced (via an induction argument on $|\pi|$) to discussing the case when $|\pi| = 2$.

Definition 7.3. Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let the spaces $(\mathfrak{M}_n)_{n\geqslant 1}$ and the multilinear functions $\{J_\pi \mid \pi \in \bigcup_{n=1}^\infty NC(n)\}$ be as in Notation 7.1. We will call *dual derivation system* a family of linear maps $(d_n : \mathfrak{D}_n \to \mathfrak{M}_n)_{n\geqslant 1}$ where, for every $n \geqslant 1$, $\mathfrak{D}_n$ is a linear subspace of $\mathfrak{M}_n$, and where the following two conditions are satisfied.

(i) Let $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ be as in Notation 7.1. Then for every $f_1 \in \mathfrak{D}_{|V_1|}, \ldots, f_p \in \mathfrak{D}_{|V_p|}$ one has that $J_\pi(f_1, \ldots, f_p) \in \mathfrak{D}_n$ and that

$$d_n\big(J_\pi(f_1, \ldots, f_p)\big) = \sum_{j=1}^p J_\pi\big(f_1, \ldots, f_{j-1}, d_{|V_j|}(f_j), f_{j+1}, \ldots, f_p\big). \tag{7.6}$$

(ii) For every $f \in \mathfrak{D}_1$ and every $n \geqslant 1$ one has that $f \circ \mathrm{Mult}_n \in \mathfrak{D}_n$ and that

$$d_n(f \circ \mathrm{Mult}_n) = (d_1 f) \circ \mathrm{Mult}_n, \tag{7.7}$$

where $\mathrm{Mult}_n : \mathcal{A}^n \to \mathcal{A}$ is the multiplication map.

Remark 7.4. 1^o When verifying condition (i) in Definition 7.3, it suffices to check the particular case when $|\pi| = 2$. Indeed, the general case of Eq. (7.6) can then be obtained by induction on $|\pi|$, where one invokes the argument from (7.5).

2^o In the setting of Definition 7.3, let us use the notation $f \times g$ for the functional obtained by "concatenating" $f \in \mathfrak{M}_m$ and $g \in \mathfrak{M}_n$. So $f \times g \in \mathfrak{M}_{m+n}$ acts simply by

$$(f \times g)(a_1, \ldots, a_m, b_1, \ldots, b_n) = f(a_1, \ldots, a_m)g(b_1, \ldots, b_n), \quad \forall a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathcal{A}.$$

Clearly one can write $f \times g = J_\gamma(f, g)$ where $\gamma \in NC(m+n)$ is the partition with two blocks $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$. By using Eq. (7.6) we thus obtain that

$$d_{m+n}(f \times g) = \big(d_m(f) \times g\big) + \big(f \times d_n(g)\big), \quad \forall m, n \geqslant 1,\ f \in \mathfrak{M}_m,\ g \in \mathfrak{M}_n. \tag{7.8}$$

So a dual derivation system gives in particular a derivation on the algebra structure defined by using concatenation on $\bigoplus_{n=1}^{\infty} \mathfrak{M}_n$. Note however that Eq. (7.8) alone is not sufficient to ensure condition (i) from Definition 7.3 (since it cannot control J_π for partitions such as $\pi = \{\{1,3\},\{2\}\} \in NC(3)$).

Proposition 7.5. *Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let $(d_n : \mathfrak{D}_n \to \mathfrak{M}_n)_{n \geqslant 1}$ be a dual derivation system on $\mathcal{A}$. Let φ be a linear functional in $\mathfrak{D}_1$, and denote $d_1(\varphi) =: \varphi'$. Consider the incps $(\mathcal{A}, \varphi, \varphi')$, and let $(\kappa_n)_{n \geqslant 1}$, $(\kappa'_n)_{n \geqslant 1}$ be the families of non-crossing and respectively of infinitesimal non-crossing cumulant functionals associated to this incps. Then for every $n \geqslant 1$ we have that*

$$\kappa_n \in \mathfrak{D}_n \quad \text{and} \quad d_n(\kappa_n) = \kappa'_n. \tag{7.9}$$

Proof. Denote as usual $\varphi_n := \varphi \circ \mathrm{Mult}_n$, $\varphi'_n := \varphi' \circ \mathrm{Mult}_n$, $n \geqslant 1$. Since $\varphi \in \mathfrak{D}_1$, condition (ii) from Definition 7.3 implies that $\varphi_n \in \mathfrak{D}_n$ and $d_n(\varphi_n) = \varphi'_n$ for every $n \geqslant 1$.

Now let $\pi = \{V_1, \ldots, V_p\}$ be a partition in $NC(n)$, with $V_1, \ldots, V_p$ written in increasing order of their minimal elements. By using Eq. (7.4) from Remark 7.2 and condition (i) in Definition 7.3 we find that

$$d_n\big(\varphi_\pi^{(A)}\big) = \sum_{j=1}^{p} J_\pi\big(\varphi_{|V_1|}, \ldots, \varphi_{|V_{j-1}|}, \varphi'_{|V_j|}, \varphi_{|V_{j+1}|}, \ldots, \varphi_{|V_p|}\big) \tag{7.10}$$

(where the latter formula incorporates the fact that $d_{|V_j|}(\varphi_{|V_j|}) = \varphi'_{|V_j|}$).

We next consider the formula (3.9) which expresses a cumulant functional κ_n in terms of the functionals $\{\varphi_\pi^{(A)} \mid \pi \in NC(n)\}$. From this formula it follows that $\kappa_n \in \mathfrak{D}_n$ and that

$$d_n(\kappa_n) = \sum_{\substack{\pi \in NC(n),\\ \pi = \{V_1, \ldots, V_p\}}} \mathrm{M\ddot{o}b}(\pi, 1_n)\left(\sum_{j=1}^{p} J_\pi\big(\varphi_{|V_1|}, \ldots, \varphi_{|V_{j-1}|}, \varphi'_{|V_j|}, \varphi_{|V_{j+1}|}, \ldots, \varphi_{|V_p|}\big)\right). \tag{7.11}$$

It is immediate that on the right-hand side of (7.11) we have obtained precisely the sum over $\{(\pi, V) \mid \pi \in NC(n),\ V \text{ block of } \pi\}$ which was used to introduce κ'_n in Definition 4.2. $\square$

Proposition 7.6. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps, and consider the multilinear functionals $\varphi_\pi^{(A)}$ ($\pi \in NC(n)$, $n \geqslant 1$) which were introduced in Remark 3.10. Suppose that for every $n \geqslant 1$ the set $\{\varphi_\pi^{(A)} \mid \pi \in NC(n)\}$ is linearly independent in $\mathfrak{M}_n$; let $\mathfrak{D}_n$ denote its span, and let $d_n : \mathfrak{D}_n \to \mathfrak{M}_n$ be the linear map defined by the requirement that*

$$d_n(\varphi_\pi^{(A)}) = \sum_{\substack{\tau \in NCZ^{(B)}(n) \text{ such}\\ \text{that } Abs(\tau) = \pi}} \varphi_\tau^{(B)}, \quad \forall \pi \in NC(n), \tag{7.12}$$

with $\varphi_\tau^{(B)}$ as in Notation 6.2. Then $(d_n)_{n \geqslant 1}$ is a dual derivation system, and $d_1(\varphi) = \varphi'$.

Proof. It is obvious that the unique partition $\tau \in NCZ^{(B)}(n)$ such that $\mathrm{Abs}(\tau) = 1_n$ is $\tau = 1_{\pm n}$. Thus if we put $\pi = 1_n$ in Eq. (7.12) we obtain that $d_n(\varphi^{(A)}_{1_n}) = \varphi^{(B)}_{1_{\pm n}}$; in other words, this means that

$$d_n(\varphi \circ \mathrm{Mult}_n) = \varphi' \circ \mathrm{Mult}_n, \quad \forall n \geqslant 1. \tag{7.13}$$

The particular case $n = 1$ of (7.13) gives us that $d_1(\varphi) = \varphi'$. Moreover, it becomes clear that

$$d_n(f \circ \mathrm{Mult}_n) = (d_1 f) \circ \mathrm{Mult}_n, \quad \forall n \geqslant 1 \quad \text{and} \quad f \in \mathbb{C}\varphi;$$

since in this proposition we have $\mathfrak{D}_1 = \mathbb{C}\varphi$, we thus see that condition (ii) from Definition 7.3 is verified.

The rest of the proof is devoted to verifying (i) from Definition 7.3. We fix a partition $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ for which we will prove that Eq. (7.6) holds. Both sides of (7.6) behave multilinearly in the arguments $f_1 \in \mathfrak{D}_{|V_1|}, \ \ldots, \ f_p \in \mathfrak{D}_{|V_p|}$; hence, due to how $\mathfrak{D}_{|V_1|}, \ldots, \mathfrak{D}_{|V_p|}$ are defined, it suffices to prove the following statement: for every $\pi_1 \in NC(|V_1|), \ \ldots, \ \pi_p \in NC(|V_p|)$ we have that $J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_p}) \in \mathfrak{D}_n$ and that

$$\begin{aligned} &d_n\big(J_\pi\big(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_p}\big)\big) \\ &\quad = \sum_{j=1}^{p} J_\pi\big(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_{j-1}}, d_{|V_j|}\big(\varphi^{(A)}_{\pi_j}\big), \varphi^{(A)}_{\pi_{j+1}}, \ldots, \varphi^{(A)}_{\pi_p}\big). \end{aligned} \tag{7.14}$$

In what follows we fix some partitions $\pi_1 \in NC(|V_1|), \ \ldots, \ \pi_p \in NC(|V_p|)$, for which we will prove that this statement holds.

Observe that, in view of how the maps $d_{|V_j|}$ are defined, on the right-hand side of (7.14) we have

$$\sum_{j=1}^{p} \sum_{\substack{\tau \in NCZ^{(B)}(n) \text{ such} \\ \text{that } \mathrm{Abs}(\tau) = \pi_j}} J_\pi\big(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_{j-1}}, \varphi^{(B)}_{\tau}, \varphi^{(A)}_{\pi_{j+1}}, \ldots, \varphi^{(A)}_{\pi_p}\big).$$

But let us recall from Remark 3.5 that the partitions in $\{\tau \in NCZ^{(B)}(n) \mid \mathrm{Abs}(\tau) = \pi_j\}$ are indexed by the set of blocks of π_j. More precisely, for every $1 \leqslant j \leqslant p$ and $V \in \pi_j$ let us denote by $\tau(j, V)$ the unique partition $\tau \in NCZ^{(B)}(n)$ such that $\mathrm{Abs}(\tau) = \pi_j$ and such that the zero-block Z of τ has $\mathrm{Abs}(Z) = V$; then the double sum written above for the right-hand side of Eq. (7.14) becomes

$$\sum_{j=1}^{p} \sum_{V \in \pi_j} J_\pi\big(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_{j-1}}, \varphi^{(B)}_{\tau(j,V)}, \varphi^{(A)}_{\pi_{j+1}}, \ldots, \varphi^{(A)}_{\pi_p}\big). \tag{7.15}$$

Now to the left-hand side of (7.14). For every $1 \leqslant j \leqslant p$ let $\widehat{\pi}_j$ be the partition of V_j obtained by transporting the blocks of π_j via the unique order preserving bijection from $\{1, \ldots, |V_j|\}$ onto V_j. Then $\widehat{\pi}_1, \ldots, \widehat{\pi}_p$ form together a partition $\rho \in NC(n)$ which refines π, and it is immediate that $J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_p}) = \varphi^{(A)}_\rho$. In particular this shows of course that

$J_\pi(\varphi_{\pi_1}^{(A)}, \ldots, \varphi_{\pi_p}^{(A)}) \in \mathfrak{D}_n$. Moreover, by using how $d_n(\varphi_\rho^{(A)})$ is defined, we obtain that the left-hand side of (7.14) is equal to $\sum_{W \in \rho} \varphi_{\sigma(W)}^{(B)}$, where for every $W \in \rho$ we denote by $\sigma(W)$ the unique partition in $NCZ^{(B)}(n)$ such that $\mathrm{Abs}(\sigma(W)) = \rho$ and such that the zero-block Z of $\sigma(W)$ has $\mathrm{Abs}(Z) = W$.

Finally, we observe that the set of blocks of ρ is the disjoint union of the sets of blocks of the partitions $\widehat{\pi}_1, \ldots, \widehat{\pi}_p$, and is hence in natural bijection with $\{(j, V) \mid 1 \leqslant j \leqslant p \text{ and } V \in \pi_j\}$. We leave it as a straightforward (though somewhat notationally tedious) exercise to the reader to verify that when $W \in \rho$ corresponds to (j, V) via this bijection, then the term indexed by (j, V) in (7.15) is precisely equal to $\varphi_{\sigma(W)}^{(B)}$. Hence the double sum from (7.15) is identified term by term to $\sum_{W \in \rho} \varphi_{\sigma(W)}^{(B)}$ via the bijection $W \leftrightarrow (j, V)$, and the required formula (7.14) follows. □

Remark 7.7. The linear independence hypothesis in Proposition 7.6 is necessary, otherwise we need some relations to be satisfied by φ and φ'. Indeed, suppose for example that the set $\{\varphi_\pi^{(A)} \mid \pi \in NC(2)\}$ is linearly dependent in $\mathfrak{M}_2$. It is immediately verified that this is equivalent to the fact that φ is a character of $\mathcal{A}$ ($\varphi(ab) = \varphi(a)\varphi(b)$, $\forall a, b \in \mathcal{A}$). Hence $\kappa_2 = 0$, so if Proposition 7.6 is to work then we should have $\kappa_2' = d_2(\kappa_2) = 0$ as well, implying that φ' satisfies the condition $\varphi'(ab) = \varphi(a)\varphi'(b) + \varphi'(a)\varphi(b)$, $\forall a, b \in \mathcal{A}$.

8. Soul companions for a given φ

In this section we elaborate on the facts announced in the Section 1.3 of the introduction. We start by recording some basic properties of the set of functionals φ' which can appear as soul-companions for φ, when $(\mathcal{A}, \varphi)$ and $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are given.

Proposition 8.1. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are freely independent in $(\mathcal{A}, \varphi)$.*

1^o *The set of linear functionals*

$$\mathcal{F}' := \left\{ \varphi' : \mathcal{A} \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi' \text{ linear, } \varphi'(1_\mathcal{A}) = 0, \text{ and } \mathcal{A}_1, \ldots, \mathcal{A}_k \\ \text{are infinitesimally free in } (\mathcal{A}, \varphi, \varphi') \end{array} \right\} \tag{8.1}$$

is a linear subspace of the dual of $\mathcal{A}$.

2^o *Suppose that $Alg(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k) = \mathcal{A}$, and consider the linear map*

$$\mathcal{F}' \ni \varphi' \mapsto \left(\varphi' \middle| \mathcal{A}_1, \ldots, \varphi' \middle| \mathcal{A}_k\right) \in \mathcal{F}_1' \times \cdots \times \mathcal{F}_k', \tag{8.2}$$

where $\mathcal{F}'$ is as in (8.1) *and where for $1 \leqslant i \leqslant k$ we denote $\mathcal{F}_i' = \{\varphi' : \mathcal{A}_i \to \mathbb{C} \mid \varphi'$ linear, $\varphi'(1_\mathcal{A}) = 0\}$. The map from* (8.2) *is one-to-one.*

Proof. 1^o This is immediate from Definition 1.1, and specifically from the fact that φ' makes a linear appearance on the right-hand side of Eq. (1.5).

2^o Let $\varphi' \in \mathcal{F}'$ be such that $\varphi' \mid \mathcal{A}_i = 0$, $\forall 1 \leqslant i \leqslant k$. Then from Eq. (1.5) it is immediate that $\varphi'(a_1 \cdots a_n) = 0$ for all choices of $a_1, \ldots, a_n \in \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$. The linear span of the products $a_1 \cdots a_n$ formed in this way is the algebra generated by $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$, hence is all of $\mathcal{A}$, and the conclusion that $\varphi' = 0$ follows. □

Remark 8.2. In the framework of Proposition 8.1, the linear map (8.2) may not be surjective. For an example, consider the full Fock space over $\mathbb{C}^2$,

$$\mathcal{T} = \mathbb{C}\Omega \oplus \mathbb{C}^2 \oplus (\mathbb{C}^2 \otimes \mathbb{C}^2) \oplus \cdots \oplus (\mathbb{C}^2)^{\otimes n} \oplus \cdots,$$

and let $L_1, L_2 \in B(\mathcal{T})$ be the left-creation operators associated to the two vectors in the canonical orthonormal basis of $\mathbb{C}^2$. Then L_1, L_2 are isometries with mutually orthogonal ranges; this is recorded in algebraic form by the relations

$$L_1^* L_1 = L_2^* L_2 = 1 \quad \text{(identity operator on } \mathcal{T}\text{)}, \qquad L_1^* L_2 = 0.$$

For $i = 1, 2$ let $\mathcal{A}_i$ denote the unital $*$-subalgebra of $B(\mathcal{T})$ generated by L_i, and let $\mathcal{A} = \mathrm{Alg}(\mathcal{A}_1 \cup \mathcal{A}_2)$, the unital $*$-algebra generated by L_1 and L_2 together. It is well known (see e.g. [5, Lecture 7]) that $\mathcal{A}_1$ and $\mathcal{A}_2$ are free in $(\mathcal{A}, \varphi)$ where φ is the vacuum-state on $\mathcal{A}$. Let $\varphi_2' : \mathcal{A}_2 \to \mathbb{C}$ be any linear functional such that $\varphi_2'(1_{\mathcal{A}}) = 0$ and $\varphi_2'(L_2) = 1$. Then there exists no linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi' \mid \mathcal{A}_2 = \varphi_2'$ and such that $\mathcal{A}_1, \mathcal{A}_2$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. Indeed, if such φ' would exist then from Eq. (2.3) of Remark 2.2 it would follow that

$$\varphi'(L_1^* L_2 L_1) = \varphi(L_1^* L_1)\varphi'(L_2) + \varphi'(L_1^* L_1)\varphi(L_2) = 1 \cdot 1 + 0 \cdot 0 = 1,$$

which is not possible, since $L_1^* L_2 L_1 = 0$.

Remark 8.3. The example from the above remark shows that we can't always extend a given system of functionals φ_i' in order to get a soul companion φ' for φ. But Proposition 2.4 gives us an important case when we are sure this is possible, namely the one when $(\mathcal{A}, \varphi)$ is the free product $(\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$.

In the remaining part of this section we will look at the two recipes for obtaining a soul companion that were stated in Corollary 1.4 and Proposition 1.5. For the first of them, we start by verifying that a derivation on $\mathcal{A}$ does indeed define a dual derivation system as indicated in Eq. (1.15).

Proposition 8.4. *Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let $D : \mathcal{A} \to \mathcal{A}$ be a derivation. For every $n \geqslant 1$ let $\mathfrak{M}_n$ denote the space of multilinear functionals from $\mathcal{A}^n$ to $\mathbb{C}$, and define $d_n : \mathfrak{M}_n \to \mathfrak{M}_n$ by putting*

$$(d_n f)(a_1, \ldots, a_n) := \sum_{m=1}^{n} f(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n), \tag{8.3}$$

for $f \in \mathfrak{M}_n$ and $a_1, \ldots, a_n \in \mathcal{A}$. Then $(d_n)_{n \geqslant 1}$ is a dual derivation system on $\mathcal{A}$.

Proof. We first do the immediate verification of condition (ii) from Definition 7.3. Let f be a functional in $\mathfrak{M}_1$, let n be a positive integer, and denote $g = f \circ \mathrm{Mult}_n \in \mathfrak{M}_n$. Then for every $a_1, \ldots, a_n \in \mathcal{A}$ we have

$$(d_n g)(a_1, \ldots, a_n) = \sum_{m=1}^{n} f(a_1 \cdots a_{m-1} \cdot D(a_m) \cdot a_{m+1} \cdots a_n) = f(D(a_1 \cdots a_n))$$

(where at the first equality sign we used the definitions of d_n and of g, and at the second equality sign we used the derivation property of D). Since $d_1 f$ is just $f \circ D$, it is clear that we have obtained $d_n g = (d_1 f) \circ M_n$, as required.

For the remaining part of the proof we fix $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ and $f_1 \in \mathfrak{M}_{|V_1|}, \ldots, f_p \in \mathfrak{M}_{|V_p|}$ as in (i) of Definition 7.3, and we verify that the formula (7.6) holds. Denote $f := J_\pi(f_1, \ldots, f_p) \in \mathfrak{M}_n$. In the summation which defines $d_n f$ in Eq. (8.3) we group the terms by writing

$$\sum_{j=1}^{p}\biggl(\sum_{m \in V_j} f\bigl(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\bigr)\biggr). \tag{8.4}$$

It will clearly suffice to prove that, for every $1 \leqslant j \leqslant p$, the term indexed by j in the sum (8.4) is equal to the term indexed by j on the right-hand side of (7.6).

So then let us also fix a j, $1 \leqslant j \leqslant p$. We write explicitly the block V_j of π as $\{v_1, \ldots, v_s\}$ with $v_1 < \cdots < v_s$. From the definition of f as $J_\pi(f_1, \ldots, f_p)$ it is then immediate that for $m = v_r \in V_j$ we have

$$\begin{aligned} &f\bigl(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\bigr) \\ &\quad = \biggl(\prod_{\substack{1 \leqslant i \leqslant p, \\ i \neq j}} f_i\bigl((a_1, \ldots, a_n) \bigm| V_i\bigr)\biggr) \cdot f_j\bigl(a_{v_1}, \ldots, a_{v_{r-1}}, D(a_{v_r}), a_{v_{r+1}}, \ldots, a_{v_s}\bigr). \end{aligned} \tag{8.5}$$

When summing over $1 \leqslant r \leqslant s$ in (8.5), the sum on the right-hand side only affects the last factor of the product, which gets summed to $(d_s f_j)(a_{v_1}, \ldots, a_{v_s})$. The result of this summation is hence that

$$\sum_{m \in V_j} f\bigl(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\bigr) = J_\pi\bigl(f_1, \ldots, f_{j-1}, d_{|V_j|}(f_j), f_{j+1}, \ldots, f_p\bigr),$$

as required. $\square$

Corollary 8.5. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $D : \mathcal{A} \to \mathcal{A}$ be a derivation. Define $\varphi' := \varphi \circ D$. Let the non-crossing and the infinitesimal non-crossing cumulant functionals associated to $(\mathcal{A}, \varphi, \varphi')$ be denoted by $(\kappa_n)_{n \geqslant 1}$ and $(\kappa'_n)_{n \geqslant 1}$, in the usual way. Then for every $n \geqslant 1$ and $a_1, \ldots, a_n \in \mathcal{A}$ one has*

$$\kappa'_n(a_1, \ldots, a_n) = \sum_{m=1}^{n} \kappa_n\bigl(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\bigr). \tag{8.6}$$

Proof. This follows from Proposition 7.5, where we use the specific dual derivation system put into evidence in Proposition 8.4. $\square$

Corollary 8.6. *In the notations of Corollary* 8.5*, let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are freely independent with respect to φ, and such that $D(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for $1 \leqslant i \leqslant k$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.*

Proof. We verify that condition (2) from Theorem 1.2 is satisfied. The vanishing of mixed cumulants κ_n follows from the hypothesis that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \varphi)$. But then the specific formula obtained for the infinitesimal cumulants κ'_n in Corollary 8.5, together with the hypothesis that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are invariant under D, imply that the mixed infinitesimal cumulants κ'_n vanish as well. □

Example 8.7. Consider the situation where $\mathcal{A}$ is the algebra $\mathbb{C}\langle X_1, \ldots, X_k\rangle$ of noncommutative polynomials in k indeterminates. We will view $\mathcal{A}$ as a $*$-algebra, with $*$-operation uniquely determined by the requirement that each of $X_1, \ldots, X_k$ is selfadjoint. Consider the unital $*$-subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$ where $\mathcal{A}_i = \operatorname{span}\{X_i^n \mid n \geqslant 0\}$, $1 \leqslant i \leqslant k$. We will look at two natural derivations on $\mathcal{A}$ that leave $\mathcal{A}_1, \ldots, \mathcal{A}_k$ invariant, and we will examine some examples of infinitesimal freeness given by these derivations.

(a) Let $D : \mathcal{A} \to \mathcal{A}$ be the linear operator defined by putting $D(1) = 0$, $D(X_i) = 1$ $\forall 1 \leqslant i \leqslant k$, and

$$D(X_{i_1} \cdots X_{i_n}) = \sum_{m=1}^{n} X_{i_1} \cdots X_{i_{m-1}} X_{i_{m+1}} \cdots X_{i_n}, \quad \forall n \geqslant 2,\ \forall 1 \leqslant i_1, \ldots, i_n \leqslant k. \tag{8.7}$$

It is immediate that D is a derivation on $\mathcal{A}$, which is selfadjoint (in the sense that $D(P^*) = D(P)^*$, $\forall P \in \mathcal{A}$). For every $1 \leqslant i \leqslant k$ we have that $D(\mathcal{A}_i) \subseteq \mathcal{A}_i$ and that D acts on $\mathcal{A}_i$ as the usual derivative (in the sense that $D(P(X_i)) = P'(X_i)$, $\forall P \in \mathbb{C}[X]$).

Now let $\mu : \mathcal{A} \to \mathbb{C}$ be a positive definite functional with $\mu(1) = 1$ and such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \mu)$. Then Corollary 8.6 implies that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in the $*$-incps $(\mathcal{A}, \mu, \mu')$, where $\mu' := \mu \circ D$.

Note that in this special example we actually have

$$\kappa'_n(X_{i_1}, \ldots, X_{i_n}) = 0, \quad \forall n \geqslant 2,\ \forall 1 \leqslant i_1, \ldots, i_n \leqslant k; \tag{8.8}$$

this is an immediate consequence of formula (8.6), combined with the fact that a non-crossing cumulant vanishes when one of its arguments is a scalar.

Eq. (8.8) gives in particular that

$$\kappa'_n(X_i, \ldots, X_i) = 0, \quad \forall n \geqslant 2 \text{ and } 1 \leqslant i \leqslant k.$$

So if μ is defined such that every X_i has a standard semicircular distribution in $(\mathcal{A}, \mu)$, then every X_i will become a standard infinitesimal semicircular element in $(\mathcal{A}, \mu, \mu')$, where in Eq. (5.11) of Remark 5.7 we take $\alpha'_1 = 1$, $\alpha'_2 = 0$.

(b) Let $D_\# : \mathcal{A} \to \mathcal{A}$ be the linear operator defined by putting $D_\#(1) = 0$ and

$$D_\#(X_{i_1} \cdots X_{i_n}) = n\, X_{i_1} \cdots X_{i_n}, \quad \forall n \geqslant 1,\ \forall 1 \leqslant i_1, \ldots, i_n \leqslant k. \tag{8.9}$$

Then $D_\#$ is a selfadjoint derivation, sometimes called "the number operator" on $\mathcal{A}$. It is clear that $D_\#$ leaves every $\mathcal{A}_i$ invariant, $1 \leqslant i \leqslant k$. Hence if $\mu : \mathcal{A} \to \mathbb{C}$ is as in part (a) above (such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \mu)$), then Corollary 8.6 implies that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in the $*$-incps $(\mathcal{A}, \mu, \mu'_\#)$, where $\mu'_\# := \mu \circ D_\#$.

Since $D_{\#}(X_i) = X_i$ for $1 \leqslant i \leqslant k$, the formula (8.6) for infinitesimal non-crossing cumulants now gives

$$\kappa_n'(X_{i_1}, \ldots, X_{i_n}) = n \cdot \kappa_n(X_{i_1}, \ldots, X_{i_n}), \quad \forall n \geqslant 1, \ \forall 1 \leqslant i_1, \ldots, i_n \leqslant k. \tag{8.10}$$

In the particular case when μ is such that every X_i is standard semicircular in $(\mathcal{A}, \mu)$, it thus follows that every X_i becomes a standard infinitesimal semicircular element in $(\mathcal{A}, \mu, \mu_{\#}')$, where we set the parameters from Eq. (5.11) to be $\alpha_1' = 0$ and $\alpha_2' = 2$. On the other hand, if μ is defined such that every X_i has a standard free Poisson distribution in $(\mathcal{A}, \mu)$ (with $\kappa_n(X_i, \ldots, X_i) = 1$ for all $n \geqslant 1$), then the X_i will become infinitesimal free Poisson elements in $(\mathcal{A}, \mu, \mu_{\#}')$, in the sense of Definition 5.8 and where we take $\beta' = 0$, $\gamma' = 1$ in Eq. (5.14).

We now move to the situation described in Proposition 1.5. Clearly, this is just an immediate consequence of Proposition 4.3.

Corollary 8.8. *In the notations of Proposition* 4.3*, suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are unital subalgebras of $\mathcal{A}$ which are freely independent with respect to φ_t for every $t \in T$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.*

Proof. Consider elements $a_1 \in \mathcal{A}_{i_1}, \ \ldots, \ a_n \in \mathcal{A}_{i_n}$ where the indices $i_1, \ldots, i_n$ are not all equal to each other. The freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ in $(\mathcal{A}, \varphi_t)$ implies that $\kappa_n^{(t)}(a_1, \ldots, a_n) = 0$ for every $t \in T$. The limit and derivative at 0 of the function $t \mapsto \kappa_n^{(t)}(a_1, \ldots, a_n)$ must therefore vanish, which means (by Proposition 4.3) that $\kappa_n(a_1, \ldots, a_n) = \kappa_n'(a_1, \ldots, a_n) = 0$. Hence condition (2) from Theorem 1.2 is satisfied, and the conclusion follows. □

Example 8.9. Consider again the situation where $\mathcal{A}$ is the $*$-algebra $\mathbb{C}\langle X_1, \ldots, X_k\rangle$, as in Example 8.7, and where $\mu : \mathcal{A} \to \mathbb{C}$ is a positive definite functional with $\mu(1) = 1$. Let $(\kappa_n)_{n \geqslant 1}$ be the non-crossing cumulant functionals of μ, and let $\{\kappa_\pi^{(A)} \mid \pi \in \bigcup_{n=1}^{\infty} NC(n)\}$ be the extended family of multilinear functionals from Remark 3.10.

For every $t > 0$, let $\mu_t : \mathcal{A} \to \mathbb{C}$ be the linear functional defined by putting $\mu_t(1) = 1$ and

$$\mu_t(X_{i_1}, \ldots, X_{i_n}) = \sum_{\pi \in NC(n)} (t+1)^{|\pi|} \cdot \kappa_\pi(X_{i_1}, \ldots, X_{i_n}), \tag{8.11}$$

for all $n \geqslant 1$ and $1 \leqslant i_1, \ldots, i_n \leqslant k$. As is easily seen, μ_t is uniquely determined by the fact that its non-crossing cumulant functionals $(\kappa_n^{(t)})_{n \geqslant 1}$ satisfy

$$\kappa_n^{(t)}(X_{i_1}, \ldots, X_{i_n}) = (t+1) \cdot \kappa_n(X_{i_1}, \ldots, X_{i_n}), \quad \forall n \geqslant 1, \ 1 \leqslant i_1, \ldots, i_n \leqslant k. \tag{8.12}$$

Due to this fact, μ_t is called the "$(t+1)$th convolution power of μ" with respect to the operation $\boxplus$ of free additive convolution – see pp. 231–233 of [5] for details.

From (8.11) it is clear that the family $\{\mu_t \mid t > 0\}$ has infinitesimal limit (μ, μ') at $t = 0$, where μ is the functional we started with, while μ' is defined by putting $\mu'(1) = 0$ and

$$\mu'(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in NC(n)} |\pi| \cdot \kappa_\pi(X_{i_1}, \ldots, X_{i_n}), \quad \forall n \geqslant 1, \ 1 \leqslant i_1, \ldots, i_n \leqslant k. \tag{8.13}$$

Note also that by using Eq. (8.12) and by invoking Proposition 4.3 we get

$$\kappa_n'(X_{i_1},\dots,X_{i_n}) = \kappa_n(X_{i_1},\dots,X_{i_n}), \quad \forall n \geqslant 1,\ 1 \leqslant i_1,\dots,i_n \leqslant k. \tag{8.14}$$

Now let $\mathcal{A}_1,\dots,\mathcal{A}_k$ be the unital $*$-subalgebras of $\mathcal{A}$ that were also considered in Example 8.7, $\mathcal{A}_i = \operatorname{span}\{X_i^n \mid n \geqslant 0\}$ for $1 \leqslant i \leqslant k$. Suppose that $\mathcal{A}_1,\dots,\mathcal{A}_k$ are free in $(\mathcal{A},\mu)$. Then they are free in $(\mathcal{A},\mu_t)$ for every $t > 0$; this follows from Eq. (8.12) and the description of freeness in terms of non-crossing cumulants (cf. [5, Theorem 11.20]), where we take into account that $\mathcal{A}_i$ is the unital algebra generated by X_i. Hence this is a situation where Corollary 8.8 applies, and we conclude that $\mathcal{A}_1,\dots,\mathcal{A}_k$ are infinitesimally free in $(\mathcal{A},\mu,\mu')$.

Note also that if X_i has a standard semicircular distribution in $(\mathcal{A},\mu)$, then Eq. (8.14) implies that X_i becomes an infinitesimal semicircular element in $(\mathcal{A},\mu,\mu')$, where the parameters α_1',α_2' from Remark 5.7 are taken to be $\alpha_1'=0$, $\alpha_2'=1$. Likewise, if X_i is a standard free Poisson in $(\mathcal{A},\mu)$, then Eq. (8.14) implies that X_i becomes an infinitesimal free Poisson element in $(\mathcal{A},\mu,\mu')$, where the parameters β',γ' from Definition 5.8 are taken to be $\beta'=1$, $\gamma'=0$.

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