Use of internal and external variables and extremum principle in limit equilibrium formulations with application to bearing capacity and slope stability problems

Y.M. Cheng\textsuperscript{a,}\textsuperscript{*}, T. Lansivaara\textsuperscript{b}, R. Baker\textsuperscript{c}, N. Li\textsuperscript{a}

\textsuperscript{a}Department of Civil and Structural Engineering, Hong Kong Polytechnic University, Hong Kong
\textsuperscript{b}Department of Civil Engineering, Tampere University of Technology, Finland
\textsuperscript{c}Department of Structural Engineering & Construction Management, Technion—Israel Institute of Technology, Haifa, Israel

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Abstract

Limit equilibrium methods, satisfying both force and moment equilibrium can be formulated using assumptions on the internal variables or the external variables. Even though most stability methods are based on force and moment equilibrium, as well as the Mohr–Coulomb yield criterion, there are great differences between the results of the different formulations due to variations in the assumptions. The authors believe that the use of the interslice force function $f(x)$, the thrust line or the base normal forces should provide an equivalent concept at the ultimate/failure state. In the present study, the authors have used the well-known bearing capacity solutions to determine $f(x)$, the thrust line and the base normal forces for a “horizontal slope”. The equivalence between the different formulations under the ultimate condition is demonstrated. It is shown that it is not important which forces are used in the stability formulation, external boundary forces or internal forces, if only that the ultimate state is considered. It is also demonstrated in the present paper that the maximum extremum from the limit equilibrium analysis is equivalent to the slip line solution using a classical bearing capacity problem.

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1. Introduction

Slope stability analyses, using the limit equilibrium method (LEM), are well known for their applications to statically indeterminate problems; therefore, slope stability methods require assumptions on the internal forces or the base forces (stress) before the problems can be solved. Broadly speaking, there are two major groups of “rigorous” methods in the LEM analysis: (1) internal variables in the form of the direction or the location of the interslice forces and (2) external variables (boundary stress) in the form of base normal forces (stress) acting on a potential slip surface.

For the first group of methods, Morgenstern–Price’s (MP) method (1965) and Janbu’s rigorous method (1957, 1973) are the most common formulations. In the MP method (1965), a method popular among many engineers, the inclination of the total internal force is usually expressed as $\dot{f}(x)$, where $\dot{z}$ is a...
mobilization factor, while \( f(x) \) is a function specifying the normalized inclination of the interslice forces. This function takes a value between 0 and 1, and \( x \) is a normalized distance in the range 0–1.0 (using the horizontal length of the failure surface in the normalization). Since only the global moment equilibrium is used in the MP formulation (1965), the back-calculated thrust line may lie outside of the soil mass, which is physically not possible. This situation is equivalent to the violation of the local moment equilibrium, and the classical MP cannot enforce the local moment equilibrium automatically.

In Janbu’s rigorous method (1957, 1973), the distance between the thrust line and the base of the slip surface is assumed to be known, while the local moment equilibrium is used in the formulation. It should be noted that there are some differences between the international adaption of Janbu’s method and the way Janbu himself intended (1957, 1973) it to be adapted. In the original computer implementation by Janbu and others in the Nordic countries, the moment equilibrium is taken on the interfaces of the slices, but not on the actual slice itself. Janbu (1973) has discussed “the moment equilibrium for a slice of infinite small width”, but this does not mean that the slices themselves would be very narrow. The method was developed during the time of hand calculations (Janbu, 1957), so relatively few slices had to be used. The trick was to take the moment equilibrium for an infinite extra small slice in the intersection of the normal slices. This averaging method resulted in quite a good convergence for normal problems, but convergence problems began to appear as the slices got thinner.

In the international implementation of Janbu’s rigorous method (Abramson et al., 2002), the moment equilibrium is considered about the center of the base of each slice; and hence, the local equilibrium and the overall moment equilibrium are implicitly satisfied. As the problem is actually over-specified by one unknown, the moment equilibrium of the last slice is not enforced in Janbu’s rigorous method (1973). Thus, the true moment equilibrium is still not maintained in this method. Besides these two methods, there are many other variants of slope stability methods which are usually based on these two important slope stability formulations. As long as a statically admissible stress field is defined over a domain, the solution will be a lower bound of the ultimate limit state (or equivalent failure state for which no more external or internal loads can be added). In this respect, LEM is an approximate, but not exact, lower bound solution (Chen, 1975), as force (lumping the stress over a finite length), instead of stress, is considered in the classical LEM.

In the second group of methods, the variational principle by Baker and Garber (1978) (BG) is the representative method. The BG method minimizes the safety functional with respect to both the potential slip surface \( y(x) \) and the potential normal stress \( \sigma(x) \) acting on this surface, using equilibrium requirements as the constraints. It should be noted that in the BG formulation (1978), the failure mass bounded by the potential slip surface and the ground surface is not divided into slices; complete equilibrium can be achieved using this group of methods, which is not possible with the first group of methods. It is important to realize that the variational technique by BG (1978) is just one of the many different minimization procedures available. The variational technique is an analytical procedure which is convenient for the solution of simple slope stability problems, but it is difficult to adopt in cases when the layered geometry or the ground/loading conditions are complicated. Cheng et al. have demonstrated the equivalence between the variational principle and a global optimization analysis; the simpler global optimization analysis can be applied to general complicated cases without any problems.

Under the lower bound theorem in a limit analysis (Chen, 1975), the loads determined from the stress distribution alone, that satisfies: (a) the equilibrium equations, (b) the stress boundary conditions and (c) nowhere violates the yield criterion, are not greater than the actual collapse load. Under the upper bound theorem, the loads determined by equating the external rate of work to the internal rate of dissipation, associated with a prescribed deformation mode (or velocity field) that satisfies: (a) the velocity boundary conditions and (b) the strain and velocity compatibility conditions, are not less than the actual collapse load. The lower bound theorem, which does not involve energy dissipation, is also applicable to the limit equilibrium formulation. The major difference between the limit equilibrium and the limit analysis is the upper bound approach. In the limit analysis, the energy balance is considered in determining the critical solution; in the limit equilibrium formulation, the minimum resistance (force/moment) against failure is considered. Cheng et al. (2010) have demonstrated the equivalence of the ultimate limit and the maximum extremum of the system (whereby the maximum strength of a prescribed failure surface is utilized) by a simple footing on clay based on the slip line solution. The slip line solution corresponds to the instant of impending plastic flow, where both the equilibrium and the yield conditions are satisfied. Combining the Mohr–Coulomb criterion with the equations of equilibrium will provide a set of differential equations for plastic equilibrium which can be solved with appropriate boundary conditions (Sokolovskii, 1965). Together with the stress boundary conditions, this set of differential equations can be used to investigate the stress at the ultimate condition. In the formulation by Cheng et al. (2010), which treats \( f(x) \) as a variable to be determined, the overall moment equilibrium is used, while the local moment equilibrium of an individual slice is not directly enforced. The local moment equilibrium (or acceptability of the thrust line location by Cheng et al., 2010) is indirectly enforced by rejecting those \( f(x) \) which are associated with the thrust line outside of the soil mass. Cheng et al. (2010) have pointed out that as long as a \( f(x) \) is prescribed, the solution will be a lower bound to the ultimate limit state, which is the lower bound theorem. Under the ultimate limit state, where the strength of a system is fully mobilized, \( f(x) \) is actually determined by this requirement, a boundary condition which has not been used in the past. Cheng et al. (2010) have applied a modern heuristic optimization algorithm to determine \( f(x) \)
for arbitrary problems, and have pointed out that every kinematically acceptable failure surface should have a factor of safety. Failure to converge in the classical stability analysis is caused by the use of an inappropriate \( f(x) \) in the analysis.

Cheng’s approach (2010) can be classified as a hybrid formulation of the first and second groups of methods. The adoption of the maximum extremum of the system is conceptually similar to the second group of methods, but there are two major differences between the formulations by Cheng et al. (2010) and Baker and Garber (1978). Baker and Garber (1978) minimizes the factor of safety simultaneously with respect to the base normal forces as well as the locations of the failure surfaces, while Cheng et al. (2010) determine the maximum extremum of the system for any prescribed failure surface. The acceptability of the internal forces is not enforced in the BG approach (1978); however, a reasonable distribution of the internal stress is obtained by this approach (e.g., Baker, 1981, 2005). Cheng et al. (2010) have enforced the acceptability of the internal forces during the extremum computation, which is different from the BG approach (1978).

In this paper, the authors will firstly use the well-known slip line solutions for a bearing capacity problem to determine \( f(x) \) and the thrust line for a “horizontal slope”. Based on the ultimate load, the failure surfaces and \( f(x) \) or the thrust line from the slip line solutions, the factors of safety will then be back-computed from Morgentern and Price’s method (1965) and Janbu’s rigorous method (1957, 1973), and the equivalence between the two stability methods under the ultimate condition will be illustrated. It is well known that difficulty is encountered in the convergence of the thin slices with Janbu’s rigorous method (1973). It is found from the present study that the international adaption of Janbu’s rigorous method (1973) can be very sensitive to the location of the thrust line, and a method to improve the convergence has been proposed in this paper.

The authors will demonstrate that at the maximum extremum condition, there is no difference between the use of external and internal variables in specifying a problem. The authors believe that the use of the internal variables is preferable over the use of the boundary variables as the imposition of the acceptability of the internal forces can be easily enforced. Furthermore, the authors will demonstrate clearly that the classical limit equilibrium methods with a prescribed internal/external force assumption will be a lower bound to the ultimate condition. The maximum extremum of the system from LEM is also shown to be equivalent to the slip line solution in the present study.

2. Interslice force function \( f(x) \) and thrust line for horizontal slope problem

In the present study, a “horizontal slope” is considered under the action of an applied load. This case is actual, as the plasticity solutions (slip line solutions) are available for this ultimate “horizontal slope”/bearing capacity problem. The slip line method is based on the theory of plasticity, and it considers the yield and the equilibrium of a soil mass controlled by the Mohr–Coulomb criterion, which is a typical lower bound method. Combining the Mohr–Coulomb criterion with the equations of equilibrium gives a set of differential equations for the plastic equilibrium. Together with the stress boundary conditions, a set of differential equations given by Eqs. (1) and (2) can be used to investigate the stresses at the ultimate condition. Sokolovskii (1965), Booker and Zheng (2000), Cheng (2003), Cheng and Au (2005), Cheng et al. (2007a) and many others have provided solutions to slip line equations. In the present study, the slip line program, SLIP, developed by Cheng and Au (2005) (which has been compared and verified with program ABC by Martin, 2004 as well as many published results), is used for the slip line analysis of a bearing capacity problem.

\[
\alpha \text{ characteristics: } -\frac{\partial p}{\partial S_y} \sin 2\mu + 2R \frac{\partial \theta}{\partial S_y} + \gamma \left( \sin(\varepsilon + 2\mu) \frac{\partial y}{\partial S_y} + \cos(\varepsilon + 2\mu) \frac{\partial x}{\partial S_y} \right) = 0 \tag{1}
\]

\[
\beta \text{ characteristics: } \frac{\partial p}{\partial S_y} \sin 2\mu + 2R \frac{\partial \theta}{\partial S_y} + \gamma \left( \sin(\varepsilon - 2\mu) \frac{\partial y}{\partial S_y} + \cos(\varepsilon - 2\mu) \frac{\partial x}{\partial S_y} \right) = 0 \tag{2}
\]

where \( p = (\sigma_1 + \sigma_3)/2 \), \( R = (\sigma_1 - \sigma_3)/2 = \rho \sin \phi + c \cos \phi \), \( \sigma_1 \) and \( \sigma_3 \) are the major and minor principle stresses, respectively, \( S_y \) and \( S_y \) are the characteristic lines, as shown in Fig. 1(a), \( \varepsilon \) and \( \phi \) are the cohesive strength and the friction angle of the soil, \( \mu = (\pi/4 - \phi/2) \), \( \gamma \) is the unit weight of the soil, \( \theta \) is the direction of the principal stress to the \( y \) axis and \( \varepsilon \) is the angle between the body force and the \( y \)-axis. For simplicity, effective soil parameters \( \varepsilon \) and \( \phi \) are represented by \( \varepsilon \) and \( \phi \) in the present paper.

Classically, bearing capacity factors \( N_c \), \( N_q \) and \( N_y \) are determined by a simple super position principle. Michalowski (1997) and Cheng (2002) have demonstrated that this simple super position is a good approximation and is only slightly conservative even for high friction angle conditions. A bearing capacity problem can be considered as a “horizontal slope” where the failure is induced by the bearing pressure from the foundation. This reasoning is physically obvious, but is practically not adopted for engineering use (due to poor results) because of its inability to specify a correct \( f(x) \), as pointed out by Cheng et al. (2010). In the present study, these three factors, which correspond to the ultimate condition, together with the corresponding \( f(x) \), the thrust line and the base normal forces will be determined from slip line solutions. Once the stress field \( (p, R, \theta) \) for the slip line field, shown in Fig. 1(b), has been determined by Eqs. (1)
and (2) using SLIP, the interslice force function and the thrust line location can be determined as follows:

**STEP 1:** Calculate the normal stress \((\sigma_x, \sigma_y)\) and the shear stress \((t_{xy})\) at any grid point by SLIP using the following relations:

\[
\begin{align*}
\sigma_1 &= p + R \\
\sigma_3 &= p - R \\
\end{align*}
\]

Hence, by the Mohr–Coulomb relation

\[
\begin{align*}
\sigma_x &= p + R \cos \theta \\
\sigma_y &= p - R \cos \theta \\
t_{xy} &= R \sin \theta
\end{align*}
\]

**STEP 2:** For any specified section (with a given \(x\)-ordinate), as shown in Fig. 1(c), determine the normal stress and the shear stress at equal vertical intervals by interpolation from the 4 grid points in the slip line field enclosing any given \(x\) and \(y\) coordinates using a bilinear equation similar to the 4-node quadrilateral element used in the finite element analysis.

**STEP 3:** Calculate interslice normal force \(E\) and shear force \(X\) in each specified section by integrating the normal stress and the shear stress at a vertical interval \(\Delta y\) in a vertical direction (as shown in Fig. 1(a)) by

\[
\begin{align*}
E &= \sum \sigma_x \Delta y \\
X &= \sum t_{xy} \Delta y
\end{align*}
\]

**STEP 4:** Determine the maximum ratio of \(X/E\) across all sections from STEP 3, denoted as mobilization factor \(\lambda\).

**STEP 5:** Obtain \(f(x)\) across the slip surface by

\[
f(x) = \frac{X/E \text{ (at each location } x)}{\lambda}
\]

**STEP 6:** Determine average normal stress \(\sigma_{ar}\) at each element along the vertical direction by stress \(\sigma_{x1}\) at the top and stress \(\sigma_{x2}\) at bottom of the element. Determine the lever arm \(h\) of the normal stresses above the base of the slip surface from

\[
\sigma_{ar} = \frac{\sigma_{x1} + \sigma_{x2}}{2}
\]
where \( h_i \) and \( y_j \) are the average \( y \)-ordinate of the element and the \( y \)-ordinate of the slip surface at Section 1, respectively. In the present analysis, a very fine grid is used (1 mm for \( N_c \) and \( N_q \) and 0.1 mm for \( N_g \)). With such fine grids, simple interpolation within a subdomain, as shown in Fig. 1(c), and a simple trapezoidal rule, as used in Eqs. (6) and (7), are good enough for the analysis.

For the assessment of the three bearing capacity factors, a direct super-position approach is assumed, which is also the basis for the determination of these three factors. For example, in determining \( N_g \), the surcharge and the cohesive strength are assumed to be zero in the slip line or the limit equilibrium analysis. From the results of SLIP, \( f(x) \) is determined for different \( \phi \) for cases associated with \( N_c \), \( N_q \) and \( N_g \). A typical slip line field for the case of \( N_c \) is shown in Fig. 2, where the pressure on the ground surface at the left-hand side is determined from a slip line analysis.

The slip line field from SLIP is in accordance with the classical solution where active and passive wedges exist at the left- and right-hand sides of the problem, and the two wedges are connected by a log-spiral zone in-between, as shown in Fig. 2. On the other hand, for the slip line field for \( N_g \), as shown in Fig. 3 with \( \phi = 30^\circ \), the active zone is actually curved, while the intermediate radial shear zone is not a true log-spiral zone with an inscribed angle less than 90\(^\circ\). When \( \phi \) is further reduced to 10\(^\circ\), the radial shear zone becomes very small and the active zone will dominate the problem. As given in Figs. 3 and 4, the bearing capacity factors from SLIP are very close to the slip line solutions from Sokolovskii (1965).

\[ f(x) \text{, for the cases associated with the determination of } N_c, N_q \text{ and } N_g, \text{ is given in Figs. 5, 6 and 7. In Figs. 5 and 6, } f(x) \text{ is symmetrical about } x = 0.5 \text{ when } \phi = 0, \text{ which is consistent with the classical plasticity solution. } f(x) \text{ is zero at the left- and right-hand sides of the wedge, as shown in Figs. 5 and 6. These results are consistent with the slip line results in which the principal stresses are the vertical and horizontal stresses in the active and passive wedges. When } \phi > 0, \text{ } f(x) \text{ moves towards the left-hand side of the figure with an increasing } \phi. \text{ These results are also obvious because the failure zone will become longer with an increasing } \phi. \text{ It is also interesting to note that } f(x) \text{ is the same for factors } N_c \text{ and } N_q. \text{ These results are not surprising because the failure mechanisms for } N_c \text{ and } N_q \text{ are the same based on the classical plasticity solution. On the other hand, the principal stresses for } N_g \text{ are the vertical and horizontal stresses only in the passive wedges, which can be observed from the results in Figs. 3 and 4. Hence, } f(x) \text{ is zero only for the right-hand side in Fig. 7.}

The results of the thrust line for \( N_c \) (and \( N_q \)) at \( \phi = 0, 10^\circ, 20^\circ, 30^\circ \) and 40\(^\circ\) are given in Fig. 8. The horizontal axis is dimensionless distance \( x \) in the range of 0–1. In the passive wedge region on the right-hand side, the thrust line
ratio (LOT) is always 0.5. In the very beginning of the active wedge, the thrust line ratios are very close to 0.5, but deviate slightly from 0.5 due to the minor error arising from the iteration analysis in the slip line analysis. Outside the foundation, LOT will go below 0.5 and then gradually rebound to 0.5. The fluctuation in LOT outside the foundation is mainly caused by the radial shear zone. It should be noted that while the ultimate bearing capacity factors from SLIP are relatively insensitive to the grid sizes used in the analysis, the thrust line is more sensitive to the size of the grid. This situation is particularly important for the case of $N_g$, and thus, a very fine grid is adopted for the case of $N_g$ or else there will be a larger fluctuation in the location of the thrust line.

The results of the thrust line ratio for $N_q$ are the same as those for $N_c$ and are also given in Fig. 8. These results are in line with those for which $f(x)$ is the same for the cases of $N_q$ and $N_c$.

For the case of $N_q$, the results are different from those of the previous two cases. Just beneath the foundation, the stresses are mainly controlled by the ground pressure so that the thrust line is slightly less than 0.5, which is obtained in Fig. 9. At the passive zone, where there is no imposed pressure, the vertical pressure is totally controlled by the weight of the soil. Therefore, the thrust line ratio is 1/3, which implies a triangular pressure distribution; this is consistent with the recommendation by Janbu (1973). It should be noted that the linear distribution of the ground pressure, as determined from the slip line analysis in Fig. 3, applies only when $c$ is taken as zero, which is also the way $N_q$ is defined. For simplicity, the coupling effect between the unit weight and $c$ is not considered in the present study, but the present study is not limited to the case of $c=0$.

The results for the thrust line are in line with the suggestion from Janbu (1973). Janbu (1973) suggested that LOT could be determined based on the earth pressure theory. For a general slope from the frictional material, the lateral earth pressure distribution is largely controlled by the unit weight of the soil and will be close to a triangular shape; hence, a generally referred value of 1/3 for LOT is suggested. In the present study, for both $N_c$ and $N_q$, where the unit weight of the soil is zero, the horizontal and vertical pressure under half of the footing will be constant, and thus, LOT should be exactly 0.5. The later part of the slip surface represents a passive earth pressure state in which the earth pressure distribution is similarly constant, and again LOT = 0.5. For $N_g$, there is a triangular-shaped earth pressure distribution in the passive zone, and hence, LOT = 1/3.

When $f(x)$ or the thrust line is defined, the problem can be back-analyzed in the following ways. For the case of $N_c$, a uniform pressure corresponding to $cN_c$ is applied on ground surface without any surcharge outside the foundation. The unit weight of the soil is set to zero in the slope stability analysis. For the case of $N_q$, a surcharge of 1 unit is applied outside the foundation, and the uniform foundation pressure is given by unit $N_q$, while the unit weight of the soil and the cohesive strength are set to zero in the analysis. For the
case of \( N_g \), a triangular pressure (see Fig. 3) with a maximum equal to \( \gamma BN_g \), while the cohesive strength and the surcharge outside the foundation are set to zero. Based on the \( f(x) \) shown in Figs. 5, 6 and 7, and the failure surfaces given by the slip line solutions, the authors have back-computed the factors of safety to be nearly 1.0 (only 0.001 to 0.002 less than 1.0) for all the cases. These results are obvious as the solutions from the slip line equations are the ultimate solutions of the system. On the other hand, when the thrust line ratios shown in Figs. 8 and 9 are used in Janbu’s rigorous method (1973), there are major difficulties in the convergence with the international adaption of Janbu’s rigorous method (1973) (equations by Janbu are approximations only). The majority of the analysis using the Janbu’s rigorous method (1973) cannot converge using the exact thrust line location from the slip line solution. After a series of investigations, the authors found that the solutions can be very sensitive to the thrust line location, and they have finally proposed another procedure which can truly satisfy the moment equilibrium (instead of the approximations by Janbu, 1973).

For the slice shown in Fig. 10, \( P \) and \( T \) are the base normal and the shear forces, respectively, \( l \) is the base length of the slice, \( E_L \) and \( X_L \) are the interslice normal and shear forces at the left, respectively, while \( E_r \) and \( X_r \) are the interslice normal and shear forces at the right, respectively, \( h_t \) is the height of the thrust line above the base of the slice at the right, \( W \) is the weight of the soil mass and \( \alpha \) is the base angle of the slice base. Based on the Coulomb relation applied to force, which is the common approach for slope stability analyses, we obtain

\[
T = \frac{1}{F} (cl + (P-ul)\tan \phi) \tag{11}
\]

For the vertical force equilibrium,

\[
P \cos \alpha - T \sin \alpha = W + (X_r - X_L) \tag{12}
\]
Rearranging and substituting for $T$ gives

$$P = \left[ W + (X_r - X_L) + \frac{1}{F} (c \sin \alpha - u \tan \phi \sin \alpha) \right] / m_x$$  \hspace{1cm} (13)

where

$$m_x = \cos \alpha - \sin \alpha \frac{\tan \phi}{F}$$

For the horizontal force equilibrium,

$$-P \sin \alpha - T \cos \alpha = E_r - E_L$$  \hspace{1cm} (14)

Rearranging and substituting for $T$ gives

$$P = \left[ -(E_r - E_L) - \frac{1}{F} (c \sin \alpha - u \tan \phi \cos \alpha) \right] / i_x$$  \hspace{1cm} (15)

where

$$i_x = \sin \alpha + \frac{1}{F} \tan \phi \cos \alpha$$

For the force equilibrium, resolving the forces parallel to the base of the slice along the base shear force direction yields

$$-T - (E_r - E_L) \cos \alpha = (W + (X_r - X_L)) \sin \alpha$$  \hspace{1cm} (16)

Rearranging Eq. (16) gives

$$X_r - X_L = -W - \frac{1}{\sin \alpha} \left( T + (E_r - E_L) \cos \alpha \right)$$  \hspace{1cm} (16a)

The moment equilibrium about the center of the base of the slice gives

$$X_L \frac{b}{2} + E_L \left( h_j - \frac{\tan \alpha \frac{b}{2}}{2} \right) + X_r \frac{b}{2} = E_r \left( h_j + 1 + \frac{\tan \alpha \frac{b}{2}}{2} \right)$$

Rearranging Eq. (17) yields

$$X_L + X_r = -E_L \left( 2 \frac{h_j}{b} - \tan \alpha \right) + E_r \left( 2 \frac{h_j + 1}{b} + \tan \alpha \right)$$  \hspace{1cm} (17a)

For the overall force equilibrium in the horizontal and vertical directions, in the absence of surface loading, the internal forces will balance out and produce

$$\sum (E_r - E_L) = 0 \quad \sum (X_r - X_L) = 0$$  \hspace{1cm} (18)

From Eqs. (16) and (18)

$$\sum (X_r - X_L) = \sum \left[ -W - \frac{1}{\sin \alpha} \left( T + (E_r - E_L) \cos \alpha \right) \right] = 0$$  \hspace{1cm} (19)

Hence, the factor of safety is given by

$$F = \frac{cl + (P - ul) \tan \phi}{\sum (-W \sin \alpha - (E_r - E_L) \cos \alpha)}$$  \hspace{1cm} (20)

The iteration solution starts from a good estimate of the initial factor of safety and the first interslice normal force $E_L$. From Eqs. (13) and (15), $P$ and $X_L$ for the first slice are then computed. Once $X_L$ is known, Eqs. (16) and (17) can be used to compute $X_r$ and $E_r$. The process is continued until all the internal forces and the factor of safety have been computed. This solution procedure is advantageous in that no finite difference scheme is required. Here are some notes about this new modified method:

1) This solution procedure is still sensitive to the location of the thrust line, but is better than Janbu’s rigorous method (1973) for thin slices.

2) A good initial choice for the factor of safety has to be defined. In general, the factor of safety from Janbu’s simplified method can be used as the initial solution.

3) A good initial guess for the first interslice normal force should be supplied in the beginning, and this value can be estimated from Morgenstern–Price’s solution.

Based on these procedures, all the problems can now converge nicely with the factors of safety close to 1.0 for all cases (only 0.001–0.002 less than 1.0). It should be noted that for normal slopes, the convergence of Janbu’s rigorous method (1973) is actually not too bad, although it is not very good either. However, for a horizontal slope, as in the present problem, Janbu’s rigorous method (1973) for thin slices is very poor in terms of the convergence. Again, it should be emphasized that Janbu’s method (1973) has been used with great success with and with few convergence problems in Nordic countries (without the rigorous moment equilibrium). For example, when $\phi = 30^\circ$ for a 1-m-wide footing, using thrust line ratios of 1/3, 0.4 and 0.5, and that from the slip line solution, the factors of safety are 0.96, 0.953, 0.943 and 0.964, respectively. Some of the results are shown in Fig. 11. When the thrust line ratio based on the slip line solution is used, the factor of safety from the original Janbu’s rigorous method (1973) is 0.964 instead of 1.0, which indicates that true moment equilibrium has not been achieved in the original Janbu’s rigorous method (1973). From Fig. 11(b) and (c), it is seen that the interslice shear force is not zero when $x$ is less than 0.5 m from Janbu’s rigorous method in Nordic countries, and that some of the $f(x)$ are actually less than zero, which are different from the results from the slip line analysis. When the thrust line ratio is 0.5 or based on that from the slip line solution, the corresponding $f(x)$ obtained is a close approximation of that from the slip line analysis (except for the initial part). On the other hand, if a more rigorous consideration of the moment equilibrium is given, using the approach suggested above, the factor of safety is 1.0, while the $f(x)$ and internal forces obtained are virtually the same as those from the slip line analysis. That means, a correct thrust line will correspond to a correct $f(x)$, and the choice of internal or external variables is not important under the ultimate condition.

It is interesting to note that all the factors of safety for the three bearing capacity factors are very close to 1.0 (0.001–0.002 different from 1.0) using either $f(x)$ or the thrust line from the ultimate condition. That means, as long as the ultimate condition is given consideration, there is no difference between the uses of $f(x)$ or the thrust line in defining a problem. In this respect, Morgenstern–Price’s
method (1965) and Janbu’s method (1973) are judged to be equivalent methods when specifying a problem under the ultimate condition. Other than the ultimate condition, the choice of \( f(x) \) or the thrust line will give different factors of safety (well known in limit equilibrium analyses) as the solutions are only typical lower bound solutions. Since iteration analyses are sensitive to the thrust line location, the use of \( f(x) \) for normal routine engineering analyses and designs is advantageous in that it is easier to achieve convergence for normal cases.

3. Boundary forces in limit equilibrium analysis

Baker and Garber (1978) have proposed the use of base normal forces as the variables in the variational principle formulation of slope stability problems. For \( N_q \) where \( \phi = 30^\circ \), the base normal stresses under an external surcharge of 1 kPa outside the foundation are determined by the slip line method which is shown in Fig. 12.

Based on the stresses determined from the slip line analysis, the base normal stresses, and hence, the forces for the slices, can be determined correspondingly. Once \( P \) is known, based on \( \sum (E_r-E_L) = 0 \), and using Eq. (14), the factor of safety can be computed by the force equilibrium as

\[
F = \frac{\sum (cl + P\tan \phi ) \cos \alpha}{\sum - P \sin \alpha}
\]  
(21)

Based on the base normal stress in Fig. 12, which have been tested against different grid sizes used for the slip line analysis, the factor of safety from Eq. (21) is exactly equal to 1.0, which is as expected. However, using \( P \) or the thrust line as the control variables is less satisfactory compared to using the interslice force function in the optimization analysis. When the thrust line is defined, the moment equilibrium of the last slice is not used in the analysis, so the true moment equilibrium cannot be satisfied (the well-known problem of Janbu’s rigorous method (1973)). It should be pointed out that for the original Janbu’s moment equilibrium (1973), the moment equilibrium of the slice interface, instead of the slice, is considered so that there is no problem for the last slice, but the moment equilibrium for each slice is not strictly enforced. The use of \( P \) also suffers from this limitation. Once \( P \) is prescribed, \( F \) will be known from Eq. (21). Based on the force equilibrium in the horizontal and vertical directions, as well as the moment equilibrium, the interslice normal and

![Fig. 11. (a) E for thrust line ratio=1/3 for \( N_q \) when \( \phi = 30^\circ \), (b) X for thrust line ratio=1/3 for \( N_q \) when \( \phi = 30^\circ \), and (c) \( f(x) \) based on different thrust line ratios.](image)

![Fig. 12. Base normal stress distribution along slip surface for \( N_q \) when \( \phi = 30^\circ \) and \( q=1 \) kPa.](image)
shear forces, as well as the thrust line, will be defined for the first slice. These results can then be used to compute the internal forces between slices 2 and 3. The computation progresses until the last slice for which both the force and the moment equilibrium cannot be enforced automatically. To apply the base normal force as the control variables in the extremum evaluation, the base normal forces for N-1 slices (N = total number of slices) are taken as the variables, while the base normal forces for the last slice will be determined from a trial and error process when the equilibrium of the last slice is satisfied. The same principle can also be applied to the thrust line, where the thrust line for only N-2 interfaces are prescribed and the thrust line for the last interface is obtained by a trial and error process until the moment equilibrium has been achieved. The use of \( f(x) \) is simpler in that the majority of the back-computed thrust lines are acceptable, so that the solution for the prescribed \( f(x) \) can be adopted directly without the trial and error process. If the thrust line is not acceptable, then the solution is simply rejected and another trial \( f(x) \) can be considered.

### 4. Lower bound solution and maximum extremum from limit equilibrium analysis

For the previous problems where \( f(x) \) or the thrust line from the ultimate limit state is used, the factors of safety of the system will be very close to 1.0. As discussed by Cheng et al. (2010), the authors of the present paper, whenever \( f(x) \) is prescribed, the solution will always be the lower bound, which can be illustrated by the results in Table 1. The maximum extremum corresponds to the state for which a system will exercise its maximum resistance before failure, and this condition is simply the ultimate condition of the system (Cheng et al., 2010). It is interesting to note that as \( \phi \) increases, the rate of decrease in the factor of safety increases, and the factors of safety corresponding to \( N_c \) and \( N_q \) can be considered to be the same. The maximum difference for the factors of safety corresponding to \( N_c \) and \( N_q \) with the ultimate limit state solution (1.0) is about 10%. On the other hand, the factor of safety corresponding to \( N_c \) is close to 1.0 (bearing in mind that the failure surface is not the classical wedge/log-spiral mechanism).

Based on the stresses at yield at the two ends of a failure surface, Chen and Morgenstern (1983) have established a requirement which states that the inclination of the internal forces at the two ends of a failure surface must be parallel to the ground slope. For the previous problems, \( f(x) \) is zero at the two ends and it satisfies this requirement. Consider a bearing capacity problem (equivalently a slope stability problem) for a soil with \( c = 0 \) and \( \phi = 30^\circ \), and the ground is sloping at an angle of 15°. The solution to this problem is given by Sokolovskii (1965) as

\[
N_c = \left[ 1 + \sin \phi \frac{x}{1 - \sin \phi} e^{165/\phi} \cot \phi - 1 \right] \cot \phi
\]

where \( \alpha = 165^\circ \) in the present problem. If Spencer’s method (1967) is used, a factor of safety of 0.951 with \( \lambda = 0.281 \) is obtained for this slip surface, which should bear a factor of safety of 1.0 by the classical plasticity solution. Based on the slip line solution by SLIP, \( f(x) \) for this problem is given in Fig. 13. \( f(x) \), at the end of the failure surface, is 0.268, which is exactly \( \tan 15^\circ \); the results clearly satisfy the requirement by Chen and Morgenstern (1983). If Spencer’s method (1967) is used in the global minimum analysis, the minimum factor of safety is 0.825 with \( \lambda = 0.219 \). The critical failure surface based on Spencer’s method (1967), shown in Fig. 14(b), is slightly deeper than the classical slip line solution, shown in Fig. 14(a). More importantly, the critical solution based on Spencer’s method (1967) appears to be a wedge type of failure which is different from the classical solution. The low factor of safety from Spencer’s method (1967) has illustrated the importance of \( f(x) \) in the analysis.

Based on the previous problems and the present problem, it is established that the maximum extremum of a system is very close to the ultimate limit state of the system. For a prescribed failure mechanism, the system will exercise its maximum strength before failure, and this is conceptually the lower bound theorem. Either \( f(x) \) or the thrust line corresponding to the ultimate limit state will be sufficient to define the system, and \( f(x) \) and the thrust line can be determined if the method by Cheng et al. (2010) is adopted.

The results in Table 1 and Fig. 14 have clearly illustrated the concept of the lower bound analysis, and every prescribed \( f(x) \) with acceptable internal forces will give a lower bound solution to the problem. On the other hand,
if unacceptable internal forces are accepted for any prescribed set of $f(x)$, thrust line or base normal forces, it is actually possible to obtain a low factor of safety or a value higher than 1.0 which is in conflict with the assumption of the lower bound analysis. In this respect, the acceptability of the internal forces, which is not explicitly imposed in the methods by Morgentern and Price (1965), Janbu (1973) or Baker and Garber (1978), is actually important if arbitrary internal or external variables are imposed in a stability analysis. The lower bound concept can be visualized clearly by Fig. 15, which shows all the temporary factors of safety during the simulated annealing analysis in searching for the maximum factor of safety using $f(x)$ as the variables in the extremum determination for factor $N_q$ when $\phi = 30^\circ$. $f(x)$ is set to 1.0 for the initial trial in the optimization analysis, and the factors of safety are far from 1.0 initially. As the global optimization analysis proceeds, the extremum of the system will tend towards the theoretical value of 1.0, and no factor of safety exceeding 1.0 can be found. The results, shown in Fig. 15, comply well with the assumption of the lower bound analysis, and they further support the adoption of the maximum extremum as the lower bound solution. A further demonstration of the extremum principle is shown for a very thin slice for a $30^\circ$ slope with $\phi = 30^\circ$. According to classical soil mechanics, the factor of safety for this thin slice should be 1.0. From the maximum extremum principle, a factor of 1.0014 is obtained for the slip surface (a minor difference from 1.0 as a circular arc is actually used). From the results in Figs. 15 and 16, it is clear that the maximum resistance of the system has been mobilized in the maximum extremum analysis, and that the extremum principle will not over-predict the factor of safety of the system. Besides the use of $f(x)$, the thrust line and the base normal forces from the slip line analysis, the authors have also adopted the maximum extremum principle and have obtained factors of safety close to 1.0.

Fig. 14. (a) Failure surface based on classical plasticity solution using $f(x)$ from Fig. 13 ($\lambda = 0.79$ and FOS = 1.0) and (b) critical failure surface based on the Spencer’s method ($\lambda = 0.219$, FOS = 0.825).

Fig. 15. Distribution of acceptable factor of safety during simulated annealing analysis for $N_q$ with $\phi = 30^\circ$ using $f(x)$ as variable.

Fig. 16. Factor of safety from extremum principle for cohesionless soil with $\phi = 30^\circ$. 

for the three bearing capacity factors. Using the maximum extremum principle, all the factors of safety corresponding to the three bearing capacity factors are very close to 1.0 by using either \( f(x) \), the thrust line or the base normal forces. These results further demonstrate that the results from the maximum extremum principle practically correspond to the ultimate condition of a system from a limit equilibrium viewpoint.

5. Discussions

In this paper, the authors have demonstrated that even though a limit equilibrium problem can be formulated in various ways, giving different results for the analyses, there will be no difference between formulations if the ultimate condition is considered. For demonstration, the classical bearing capacity problem, where the slip line solution is available, has been used for illustration. Since the stresses are determinate for this problem, all the internal or external variables at the ultimate condition could be evaluated. The problem is then re-considered by taking \( f(x) \) as the control variables in the limit equilibrium and determining the maximum extremum. Based on this study, the following have been found.

1. The authors have proved that a classical bearing capacity problem is equivalent to a horizontal slope stability problem if \( f(x) \), the thrust line or the external boundary forces \( P \) are known at the ultimate condition. The use of \( f(x) \), the thrust line or the external boundary forces are simply different ways to specify a limit equilibrium slope stability problem, and the use of \( f(x) \), the thrust line or the base normal forces as the control variables are actually equivalent at the ultimate condition.

2. It has been demonstrated in the present study that \( f(x) \) are the same for \( N_c \) and \( N_q \) factors from both the slip line methods and the extremum principle; these results are not surprising as the failure mechanisms for \( N_c \) and \( N_q \) are actually the same. It is also demonstrated that \( f(x) \) for \( N_c \) and \( N_q \) are zero at the two ends and take the maximum value at an \( x \) ratio less than 0.5. On the other hand, \( f(x) \) for \( N_p \) takes the maximum value at a different \( x \) ratio. For \( N_c \) and \( N_p \), the thrust line ratio cluster is around 0.4–0.5, while for \( N_q \), the thrust line ratio starts at a value between 0.4 and 0.5 and decreases gradually to 1/3 outside the foundation.

3. Based on the \( f(x) \), thrust line or the base normal forces, as obtained from the slip line analysis, the factors of safety using the MP (1965), Janbu’s rigorous method (1973) or the BG method (1978) are very close to 1.0. That means, as long as the ultimate condition is considered, there is no practical difference between the uses of the internal or the external variables in defining a problem. Different formulations should give the same results under the ultimate condition, which has been clearly illustrated in the present paper.

4. The interslice force functions obtained from the slip line method in this study are not simple functions; the functions have fully complied with the requirements by Chen and Morgenstern (1983), and the inclination of the internal forces at the two ends of a failure surface are always parallel to the ground slope in the present study. On the other hand, \( \lambda f(x) \) is set to 1.0 in the popular Spencer’s method (1967), which can be considered as an approximate lower bound of the true failure mechanism. Poor results are obtained for the bearing capacity problem if Spencer’s method (1967) is used, which implies that the precise values for the internal or external variables can be very important in some problems. For critical and highly complicated problems, the classical approach of using a simple \( f(x) \) may not be adequate.

5. The authors have also clearly illustrated that as long as \( f(x) \) is prescribed, the solution will be a lower bound of the ultimate condition; hence, the classical limit equilibrium methods are practically lower bounds to the ultimate condition.

It should be pointed out that the extremum principle by Cheng et al. (2010) can be viewed as a form of the variational principle (Cheng et al., 2011; Sieniutycz and Farkas, 2005). However, the maximum factor of safety as determined may deviate slightly from the true ultimate limit state, as the Coulomb relation is applied as a constraint along the vertical interface instead of any arbitrarily small domain. For a true ultimate limit state, the Mohr–Coulomb relation should be applicable throughout the whole medium instead of applying it to lumped global interslice normal and shear forces. In this respect, the maximum extremum principle, as proposed by Cheng et al. (2010), is only a good approximation of the ultimate condition, as the yield condition is checked globally at each interface instead of being enforced at each infinitesimal domain. Nevertheless, the method by Cheng et al. (2010) provides a good solution with minimum effort in providing a practical solution to a problem without a complete discretization of the solution domain. In the strength reduction analysis of a slope, the factor of safety is varied until the system cannot maintain stability. Stress will redistribute during the nonlinear elasto-plastic analysis, and as long as the stress can redistribute without violating yield and equilibrium, the trial factor of safety can be further increased. It should be noted that the concept of the extremum is actually in line with the concept of the strength reduction method in this respect. Hence, it is not surprising that the factors of safety from the strength reduction analysis (ultimate condition) are always greater than those from the lower bound Spencer’s analysis (Cheng et al., 2007b), provided that the global minima from Spencer’s analysis are used for comparisons.

The use of the classical slope stability methods (limit equilibrium-based methods) will not yield good results for
bearing capacity problems (plasticity-based solutions), which is well known among many engineers and has also been illustrated by Cheng et al. (2010). Thus, it is not surprising that very few engineers adopt slope stability analysis methods for determining the bearing capacity (actually this approach is not recommended for use in Hong Kong). In this paper, the authors have demonstrated that the maximum extremum of the limit equilibrium method is practically equivalent to the plasticity solution so that the limitation in applying the slope stability method to a bearing capacity determination will be removed. There is a special application of the present approach to the bearing capacity problem of a buried foundation adjacent to a slope with a horizontal set-back which has been considered by Cheng and Au (2005) and Graham et al. (1988). An approximate slip line solution has been adopted by Cheng and Au (2005) and Graham et al. (1988) (see Fig. 17) as part of the solution domain has not yielded and is not controlled by the plasticity slip line equations. As shown in Fig. 17, Point C1 has to be obtained on from trial and error analysis. The weight of the triangular zone of the soil and the external surcharge is assumed to be known external pressure acting on A1C1 in order to solve the plasticity slip line equation. The zone A1B1C1 is totally considered as an external loading without any consideration given to the strength contribution which appears not to be reasonable, but without this assumption, there is no way to solve the slip line equation. Using the present approach, a fully yielded solution domain is not required, and the slope stability solution based on the maximum extremum can be used to assess the bearing capacity of the footing. The results shown in Table 2 have also demonstrated that the present approach provides a good bearing capacity factor which is not possible for the classical Spencer’s method using $f(x)=1.0$ (which always underestimates $N_g$).

6. Conclusions

Although some of the case studies in this paper are based on the use of horizontal slopes, as their analytical solutions are available for comparisons, the results from the present study are generally valid. This has been demonstrated by the problems shown in Figs. 14 and 17. Based on the present study, Morgenstern–Price’s method (1965) and Janbu’s rigorous method (1973) can be considered as the same under the ultimate condition. These two methods have been demonstrated to be equivalent at the ultimate condition, as they are controlled by both the yield and the equilibrium equations, except for the ease of mathematical manipulation. Besides that, it has also been demonstrated that the use of the base normal forces can be an alternative to Morgenstern–Price’s (1965) and Janbu’s rigorous methods (1973). Hence, the choice of the assumption is physically not important at the ultimate condition. On the other hand, if the ultimate condition is

![Fig. 17. Approximate modeling of footing with embedment and set back by the slip line method.](image)

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not considered, there are practical differences between the use of \( f(x) \), the thrust line or the base normal forces in LEM, and every prescribed assumption giving acceptable internal forces will be a lower bound to the ultimate limit state of the problem, which is clearly illustrated in Fig. 16. It is also interesting that the present study has demonstrated the equivalence between the LEM and the slip line solution for a medium which is fully in the plastic condition, provided that the maximum extremum from the LEM is used in the comparison. The hyperbolic partial equations governing the \( \alpha \) and \( \beta \) characteristic lines, as given by Eqs. (1) and (2), can be well approximated by tuning \( f(x) \) in the slope stability analysis until the critical solution has been obtained with the simple force and the moment equilibrium. This is an interesting and useful application of the lower bound concept to more general problems where the classical slip line method fails to work. For a homogeneous problem with a continuous stress field, the conclusions from the present study will be valid. It should be kept in mind, however, that for nonhomogeneous problems with discontinuous stress fields, the present conclusion will not be valid, but the difference between the use of \( f(x) \), the thrust line and the base normal force should still be small at the ultimate condition.

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