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Finite p-groups all of whose non-abelian proper subgroups are generated by two elements ☆

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Abstract

In this paper, we classify the finite p-groups all of whose non-abelian proper subgroups are generated by two elements.

Keywords: Metabelian groups; Metacyclic groups; Minimal non-abelian groups; Minimal non-metacyclic groups; $\mathcal{A}_t$-groups; p-groups of maximal class

1. Introduction

To determine a finite group G by using its subgroup structure is an important theme in the group theory. Let G be a finite p-group. If every proper subgroup of G is abelian then G is either abelian or minimal non-abelian determined by Rédei [8]. If every proper subgroup of G is generated by two elements then Blackburn [5] proved that G is either metacyclic or a 3-group of maximal class with a few exceptions. Moreover, every subgroup of a p-group of maximal

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class with an abelian maximal subgroup is either abelian or generated by two elements (see Theorem 2.5). Motivated by the above results, Berkovich [1] posed the following

Problem 46. Classify the p-groups all of whose proper subgroups are either abelian or generated by two elements.

The present paper is devoted to this problem and all such groups are classified.

Following Berkovich and Janko [2], for a positive integer t, a finite p-group G is called an $\mathcal{A}_t$*-group* if every subgroup of index p^t is abelian, but there is at least one subgroup of index p^{t-1} which is not abelian. So $\mathcal{A}_1$-groups are nothing but the minimal non-abelian p-groups. All $\mathcal{A}_1$- and $\mathcal{A}_2$-groups are known (see [12]).

We use $\mathcal{B}_p$ to denote the class of p-groups satisfying the condition in this problem. It is obvious that abelian, metacyclic, $\mathcal{A}_1$- and $\mathcal{A}_2$-groups are in $\mathcal{B}_p$. Also, as mentioned above, p-groups of maximal class having an abelian maximal subgroup and 3-groups of maximal class are in $\mathcal{B}_p$. So, the class $\mathcal{B}_p$ is rather wide.

We use $\mathcal{B}'_p$ to denote the class consisting of groups in $\mathcal{B}_p$ which are neither abelian nor minimal non-abelian and let $\mathcal{D}_p = \{G \in \mathcal{B}'_p \mid G$ has an abelian maximal subgroup$\}$, $\mathcal{M}_p = \{G \in \mathcal{B}'_p \mid G$ has no abelian maximal subgroup$\}$. Then we need only classify $\mathcal{D}_p$-groups and $\mathcal{M}_p$-groups. For convenience, we also let $\mathcal{D}_p(2) = \{G \in \mathcal{D}_p \mid d(G) = 2\}$ and $\mathcal{D}_p(3) = \{G \in \mathcal{D}_p \mid d(G) = 3\}$.

We also say G is an $\mathcal{X}$-group if G is in the class $\mathcal{X}$.

Let G be a finite p-group. We use $c(G)$, $d(G)$, and $p^{e(G)}$ to denote the nilpotency class, the minimal number of generators, and the exponent of G, respectively. For any integer s with $0 \leqslant s \leqslant e = e(G)$, we define

$$\Omega_s(G) = \langle a \in G \mid a^{p^s} = 1 \rangle, \qquad \mho_s(G) = \langle a^{p^s} \mid a \in G \rangle.$$

In this paper we use

$$G = G_1 > G_2 > \cdots > G_{c+1} = 1$$

to denote the lower central series of G, where $c = c(G)$ is the nilpotency class of G. Following P. Hall, we say that G has *lower central complexion* $(f_1, f_2, \ldots, f_c)$ if $|G : G_2| = p^{f_1}$ and $|G_i : G_{i+1}| = p^{f_i}$ for $2 \leqslant i \leqslant c$.

We use C_{p^m} to denote the cyclic group of order p^m, $C_{p^m}^n$ the direct product of n cyclic groups of order p^m and $M \lessdot G$ the maximal subgroup of a group G. For undefined notation and terminology the reader is referred to Huppert [6, Kap. III].

2. Preliminaries

In this preliminary section we list some known results about minimal non-metacyclic groups, $\mathcal{A}_1$-groups, p-groups of maximal class and some other results due to Blackburn, and some commutator formulae. These results will be used later.

Recall that a finite group G is said to be *metacyclic* if there is a cyclic normal subgroup N such that G/N is also cyclic. For a classification of metacyclic p-groups, the reader is referred to [7,11].

A non-metacyclic group G is said to be *minimal non-metacyclic* if all proper subgroups of G are metacyclic.

Lemma 2.1. *A minimal non-metacyclic p-group has an abelian maximal subgroup.*

Proof. It is an immediate consequence of [5, Theorem 3.2]. □

For minimal non-abelian p-groups, i.e., $\mathcal{A}_1$-groups, the following lemma is well known.

Lemma 2.2. *Let G be a finite p-group. Then the following statements are equivalent.*

(1) *G is minimal non-abelian*;
(2) *$d(G) = 2$ and $|G'| = p$*;
(3) *$d(G) = 2$ and $\mathrm{Z}(G) = \Phi(G)$.*

Theorem 2.3. *(See [8].) Let $G \in \mathcal{A}_1$. Then G is one of the following groups*:

(1) Q_8;
(2) $\langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}}\rangle$ (*metacyclic*);
(3) $\langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1,\ [a, b] = c,\ [c, a] = [c, b] = 1\rangle$, *where $m + n \geqslant 3$ if $p = 2$*, (*non-metacyclic*).

A non-abelian p-group G of order p^n is said to be *of maximal class* if $n \geqslant 3$ and $c(G) = n - 1$. It is known that 2-groups of maximal class are dihedral, semi-dihedral and generalized quaternion, which have a cyclic maximal subgroup.

Theorem 2.4. *(See [4] also see [1, Section 9, Exercise 1].) Let G be a p-group of maximal class and of order p^n. Then*

(1) *$|G : G'| = p^2$, $G' = \Phi(G)$, $\mathrm{d}(G) = 2$.*
(2) *$|G_i/G_{i+1}| = p$, $i = 2, 3, \dots, n - 1$.*
(3) *If $i \geqslant 2$, then G_i is the unique normal subgroup of G of order p^{n-i}.*
(4) *If $N \trianglelefteq G$ and $|G/N| \geqslant p^3$, then G/N is also of maximal class.*
(5) *If $0 \leqslant i \leqslant n - 1$, then $\mathrm{Z}_i(G) = G_{n-i}$, where $1 = Z_0(G) < \cdots < Z_{n-1}(G) = G$ is the upper central series of G.*
(6) *If $p > 2$ and $n > 3$, then G has no cyclic normal subgroup of order p^2.*

Theorem 2.5. *(See [1, Section 1, Exercise 4].) Suppose that a non-abelian p-group G has an abelian maximal subgroup A. Then*

(1) *G is of maximal class if and only if $|\mathrm{Z}(G)| = p$ or $|G : G'| = p^2$.*
(2) *If G is of maximal class, then every non-abelian subgroup of G is of maximal class. In particular, $G \in \mathcal{D}_p(2)$.*

Theorem 2.6. *(See [4].) Let G be a group of maximal class and order p^m, $m \leqslant p + 1$. Then $\Phi(G)$ and $G/\mathrm{Z}(G)$ have exponent p.*

Theorem 2.7. *(See [5, Lemma 1.2].) Let G be a group of order p^n with $n \geqslant 4$. Assume that G has a maximal subgroup which is of maximal class. Then either G is of maximal class or $d(G) = 3$, $G' = \Phi(G)$ and $c(G) = n - 2$.*

Theorem 2.8. *(See [5, Theorem 3.1].) Suppose that G is a finite p-group (p odd), $d(G) > 2$, and $d(H) \leqslant 2$ for all $H \lessdot G$. Then G has an abelian maximal subgroup.*

Theorem 2.9. *(See [5, Theorem 4.2].) Suppose that G is a group of order p^n, where p is odd and $n \geqslant 6$, and that all maximal subgroups of G have two generators. Then either G is metacyclic or G/G_3 is of order p^3 and $\mho_1(G) = G_3$.*

Theorem 2.10. *(See [5, Theorem 5.1].) Let G be a group of order 2^n, where $n \geqslant 5$. Suppose, for some integer r with $5 \leqslant r \leqslant n$, that all subgroups of order 2^{r-1} and 2^r have two generators. Then G is metacyclic.*

Theorem 2.11. *(See [3, Theorem 4].) Let G be a p-group. If both G and G' can be generated by two elements, then G' is abelian.*

A non-abelian group G is said to be *metabelian* if G' is abelian. The following commutator formulae are useful in this paper, and we will use it freely.

Proposition 2.12. *(See [10].) Let G be a metabelian group and $a, b \in G$. For any positive integers i and j, let*

$$[ia, jb] = [a, b, \underbrace{a, \dots, a}_{i-1}, \underbrace{b, \dots, b}_{j-1}].$$

Then

(1) *For any positive integers m and n,*

$$\left[a^m, b^n\right] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^{\binom{m}{i}\binom{n}{j}}.$$

(2) *Let n be a positive integer. Then*

$$\left(ab^{-1}\right)^n = a^n \prod_{i+j \leqslant n} [ia, jb]^{\binom{n}{i+j}} b^{-n}.$$

Lemma 2.13. *(See [9], see also [6, Aufgabe 2, p. 259].) Suppose that a finite non-abelian p-group G has an abelian normal subgroup A, and $G/A = \langle bA \rangle$ is cyclic. Then the map $a \mapsto [a, b]$, $a \in A$, is an epimorphism from A to G', and $G' \cong A/A \cap Z(G)$. In particular, if a non-abelian p-group G has an abelian maximal subgroup, then $|G| = p|G'||Z(G)|$.*

Finally, we prove the following lemma.

Lemma 2.14. *Assume that a p-group G has an abelian maximal subgroup A. Then*:

(1) *Let $a_1, a_2 \in A$ and $b \in G$. Then $[a_1 a_2, b] = [a_1, b][a_2, b]$.*
(2) *Let $a \in A$, $b \in G$ and n a positive integer. Then*:

(2a) $[a^n, b] = [a, b]^n$;
(2b) $(ba)^n = b^n a^n \prod_{i=1}^{n-1} [a, ib]^{\binom{n}{i+1}}$.
(3) *Let $d \in G'$ and $b \in G \backslash A$. Then $(bd)^p = b^p$.*

Proof. For (1) and (2), see [6, III, Hilfsatz 10.9]. By Lemma 2.13, there is an $a \in A$ such that $d = [b, a]$. Thus $(bd)^p = (b[b, a])^p = (b^a)^p = (b^p)^a = b^p$. □

3. The classification of $\mathcal{D}_p(2)$-groups

Let $\mathcal{D}'_p(2) = \{G \in \mathcal{D}_p(2) \mid G$ is not of maximal class$\}$. According to Theorem 2.5, we only need to classify $\mathcal{D}'_p(2)$-groups.

Lemma 3.1. *Let G be a non-abelian two-generator p-group having an abelian maximal subgroup A. Assume that $|G/G'| = p^{m+1}$ and $c(G) = c$. Then $m \geqslant 1$, $c \geqslant 2$ and*

(1) *G has the lower central complexion $(m+1, \underbrace{1, \ldots, 1}_{c-1})$ and hence $|G'| = p^{c-1}$, $|G| = p^{m+c}$.*
(2) *$|\mathrm{Z}(G)| = p^m$ and $G/\mathrm{Z}(G)$ is of maximal class.*
(3) *$\mathrm{Z}(G) \leqslant \Phi(G)$, $\Phi(G) = G'\mathrm{Z}(G)$ and $G' \cap \mathrm{Z}(G) = G_c$.*
(4) *Let M be a non-abelian maximal subgroup of G. Then $\mathrm{Z}(M) = \mathrm{Z}(G)$ and*

$$M' = G_3, M_3 = G_4, \ldots, M_{c-1} = G_c.$$

Proof. (1) Let $b \in G \backslash A$, and $a_1 \in A \backslash \Phi(G)$. Then $G = \langle b, a_1 \rangle$, $b^p \in A$ and $G_i = \langle [a_1, (i-1)b], G_{i+1} \rangle$ for $2 \leqslant i \leqslant c$. In particular, $G_c = \langle [a_1, (c-1)b] \rangle$. Since $[a_1, (c-1)b] \in \mathrm{Z}(G)$, $[a_1, (c-1)b]^p = [a_1, (c-2)b, b^p] = 1$ and hence $|G_c| = p$.

To prove that G has the lower central complexion (l.c.c. for short) $(m+1, \underbrace{1, \ldots, 1}_{c-1})$, we use induction on c. If $c = 2$, then $|G'| = p$ and G has the l.c.c. $(m+1, 1)$. Now assume $c > 2$. Since G/G_c is not abelian and has an abelian maximal subgroup A/G_c, the induction hypothesis gives that G/G_c has the l.c.c. $(m+1, \underbrace{1, \ldots, 1}_{c-2})$. Since $|G_c| = p$, (1) holds.

(2) By Lemma 2.13, $|G| = p|G'||\mathrm{Z}(G)|$. Since $|G| = p^{m+c}$ and $|G'| = p^{c-1}$, $|\mathrm{Z}(G)| = p^m$ and $|G/\mathrm{Z}(G)| = p^c$. Since $c(G/\mathrm{Z}(G)) = c - 1$, $G/\mathrm{Z}(G)$ is of maximal class.

(3) Suppose that $\mathrm{Z}(G) \nleqslant \Phi(G)$. Then there exists an element $x \in \mathrm{Z}(G) \setminus \Phi(G)$. Since $d(G) = 2$, there is a $y \in G$ such that $\langle x, y \rangle = G$. It follows that G is abelian, a contradiction. Thus $\mathrm{Z}(G) \leqslant \Phi(G)$.

Since $G/\mathrm{Z}(G)$ is of maximal class, $G'\mathrm{Z}(G)/\mathrm{Z}(G) = \Phi(G/\mathrm{Z}(G))$. Since $\mathrm{Z}(G) \leqslant \Phi(G)$, we have $G'\mathrm{Z}(G) = \Phi(G)$.

Finally, since $|G' \cap \mathrm{Z}(G)| = \frac{|G'||\mathrm{Z}(G)|}{|G'\mathrm{Z}(G)|} = \frac{|G|}{p|\Phi(G)|} = p$ and $G_c \leqslant G' \cap \mathrm{Z}(G)$, we have $G' \cap \mathrm{Z}(G) = G_c$.

(4) Since $\mathrm{Z}(G) \leqslant \Phi(G) < M$, $\mathrm{Z}(G) \leqslant \mathrm{Z}(M)$. By Lemma 2.13, $|G| = p|G'||\mathrm{Z}(G)|$ and $|M| = p|M'||\mathrm{Z}(M)|$. It follows that $|G'/M'| = p|\mathrm{Z}(M)/\mathrm{Z}(G)|$.

Let $\overline{G} = G/M'$. Then $\overline{G}$ is not abelian since $G'/M' \neq 1$. Since $\overline{G}$ has two distinct abelian maximal subgroup A/M' and M/M', we know that $\overline{G}$ is minimal non-abelian. Then $|G'/M'| = p$ and $G_3 \leqslant M'$. It follows that $\mathrm{Z}(M) = \mathrm{Z}(G)$ and $M' = G_3$.

By induction on c, we may assume that for $3 \leqslant i \leqslant c-1$, $G_i = M_{i-1}$. Then we have $G_{i+1} = [G_i, G] = [M_{i-1}, AM] = [M_{i-1}, M] = M_i$. $\square$

Note that a non-abelian proper subgroup K of a $\mathcal{D}_p(2)$-group G is in $\mathcal{D}_p(2)$ or $\mathcal{A}_1$.

Lemma 3.2. *Let G, A, m, c be the same as in Lemma* 3.1. *Assume that $G \in \mathcal{D}'_p(2)$. Then $c \geqslant 3$, $m \geqslant 2$ and*

(1) *If M is a non-abelian subgroup of G with $|G : M| = p^t$. Then $t \leqslant c-2$, $\mathrm{Z}(M) = \mathrm{Z}(G)$ and*

$$M' = G_{t+2}, M_3 = G_{t+3}, \ldots, M_{c-t} = G_c.$$

In particular, $c(M) = c - t$;

(2) *M is minimal non-abelian if and only if $|G : M| = p^{c-2}$*;

(3) *All subgroups of G of index p^{c-1} are abelian*;

(4) *For any $b \in G \backslash A$, $\mathrm{Z}(G) = \langle b^p, G_c\rangle$ and $o(bG') = o(bG_c) = p^m$. There is some $a_1 \in A \backslash \Phi(G)$ such that $o(a_1G') = p$. Moreover, G/G' has type invariants (p^m, p) and A/G' has type invariants (p^{m-1}, p).*

Proof. Since $\mathcal{D}'_p(2)$-groups are neither minimal non-abelian nor of maximal class, we have $c(G) \geqslant 3$ and $m \geqslant 2$.

(1) The case $t = 0$ is trivial. So we assume below that M is a proper subgroup of G. Take $K \lessdot G$ with $M \leqslant K$. Lemma 3.1(4) gives that $c(K) = c - 1$, $\mathrm{Z}(K) = \mathrm{Z}(G)$ and

$$K' = G_3, K_3 = G_4, \ldots, K_{c-1} = G_c.$$

If $c = 3$, then $K \in \mathcal{A}_1$. It follows that $M = K$ and $t = 1$. Then we are done. So we may assume that $c \geqslant 4$ below. Hence $K \in \mathcal{D}'_p(2)$. Since $|K : M| = p^{t-1}$, by induction on $c(G)$ we have $t - 1 \leqslant c - 3$, $\mathrm{Z}(K) = \mathrm{Z}(M)$ and

$$M' = K_{t+1} = G_{t+2}, \qquad M_3 = K_{t+2} = G_{t+3}, \qquad \ldots, \qquad M_{-t} = K_{c-1} = G_c.$$

It follows that $t \leqslant c - 2$ and $\mathrm{Z}(M) = \mathrm{Z}(G)$.

(2)–(3) are immediate consequences of (1).

(4) Since $G = \langle b, A\rangle$, $\mathrm{Z}(G) = C_A(b)$. Let $M = \langle b, G_{c-1}\rangle$. Since $[G_{c-1}, b] = G_c$, $M' = G_c$. By Lemma 2.2, $M \in \mathcal{A}_1$. It follows that $\mathrm{Z}(M) = \Phi(M) = \langle b^p, G_c\rangle$, and hence $\mathrm{Z}(G) = \mathrm{Z}(M) = \langle b^p, G_c\rangle$ by (1). Since $|\mathrm{Z}(G)| = p^m$, $o(bG_c) = p^m$. Since $b^p \in C_A(b) = \mathrm{Z}(G)$ and $G' \cap \mathrm{Z}(G) = G_c$, we have $o(bG') = o(bG_c) = p^m$. So G/G' has type invariants (p^m, p) and there is an element a_1 such that $o(a_1G') = p$ and $G/G' = \langle bG', a_1G'\rangle$. We claim that $a_1 \in A$. For if $a_1 \notin A$, the above argument would give $o(a_1G') = p^m > p$, which is a contradiction. Hence $A/G' = \langle b^pG', a_1G'\rangle$ and A/G' has type invariants (p^{m-1}, p). $\square$

Theorem 3.3. *$G \in \mathcal{D}'_p(2)$ if and only if G/G' has type invariants (p^m, p) where $m \geqslant 2$, $c(G) \geqslant 3$ and G has an abelian maximal subgroup A such that A/G' has type invariants (p^{m-1}, p).*

Proof. We only need to prove the "if" part. Since $c = c(G) \geqslant 3$, $G \notin \mathcal{A}_1$. Since $m \geqslant 2$, G is not of maximal class. Note that if all non-abelian maximal subgroups of G are $\mathcal{D}'_p(2)$- or $\mathcal{A}_1$-groups then $G \in \mathcal{D}'_p(2)$.

Let M be a non-abelian maximal subgroup of G. Then, by Lemma 3.1, $\mathrm{Z}(M) = \mathrm{Z}(G)$, $M' = G_3, M_3 = G_4, \ldots$, and $c(M) = c - 1$. Since G/G' has type invariants (p^m, p) and A/G' has (p^{m-1}, p), M/G' has type invariant (p^m). It follows that $M/M'(= M/G_3)$ has type invariants (p^m, p). If $c = 3$, then $M \in \mathcal{A}_1$ and we are done. So we may assume that $c > 3$ below. Hence $c(M) \geqslant 3$.

We claim that $\Phi(G)/G_3$ is not cyclic. Otherwise, since $|\mathrm{Z}(G)| = p^m \geqslant p^2$, we would have $\mathrm{Z}(G)G_3/G_3 \cong \mathrm{Z}(G)/(G_3 \cap \mathrm{Z}(G)) = \mathrm{Z}(G)/G_c \neq \bar{1}$ and $\mathrm{Z}(G)G_3/G_3 \leqslant \Phi(G)/G_3$ is cyclic. Since $|G'/G_3| = p$, $G'/G_3 \leqslant \mathrm{Z}(G)G_3/G_3$, and hence $G' \leqslant \mathrm{Z}(G)G_3$. Thus $G' = G' \cap \mathrm{Z}(G)G_3 = (G' \cap \mathrm{Z}(G))G_3 = G_c G_3 = G_3$, a contradiction. Then $A \cap M/M'(= \Phi(G)/G_3)$ has type invariant (p^{m-1}, p). By induction, $M \in \mathcal{D}'_p(2)$, and finally $G \in \mathcal{D}'_p(2)$ by the arbitrariness of M. $\square$

The following corollary give a characterization of $\mathcal{D}'_p(2)$-groups.

Corollary 3.4. *$G \in \mathcal{D}'_p(2)$ if and only if $d(G) = 2$, $c(G) \geqslant 3$, $\Phi(G)/G'$ is cyclic and G has an abelian maximal subgroup A such that A/G' is not cyclic.*

The next examples show that the conditions for G/G' and A/G' in Theorem 3.3 are indispensable.

Example 3.5. (1) $G = \langle a_1, b \mid b^{p^2} = a_1^{p^2} = a_2^p = a_3^p = 1,\ [a_1, b] = a_2,\ [a_2, b] = a_3,\ [a_1, a_2] = [a_1, a_3] = [a_2, a_3] = [a_3, b] = 1\rangle$, where p is odd.

Then G has a unique abelian maximal subgroup $A = \langle b^p, a_1, a_2, a_3\rangle$ such that A/G' has type invariants (p^2, p) and G/G' has type invariants (p^2, p^2). We have $G \notin \mathcal{D}'_p(2)$ since $\langle b, a_1^p, a_2, a_3\rangle$ has three generators.

(2) $G = \langle a_1, b \mid b^p = a_1^{p^2} = a_2^p = a_3^p = 1,\ [a_1, b] = a_2,\ [a_2, b] = a_3,\ [a_1, a_2] = [a_1, a_3] = [a_2, a_3] = [a_3, b] = 1\rangle$, where p is odd.

Then G/G' has type invariants (p^2, p) and G has a unique abelian maximal subgroup $A = \langle a_1, a_2, a_3\rangle$ such that A/G' has type invariants (p^2). We have $G \notin \mathcal{D}'_p(2)$ since $\langle b, a_1^p, a_2, a_3\rangle$ has three generators.

Theorem 3.6. *Let $G \in \mathcal{D}'_p(2), c(G) = c$ and $|G| = p^{m+c}$. Let A, b, a_1 be the same as in Lemma 3.2. Then $m \geqslant 2$, $c \geqslant 3$ and $G = \langle a_1, b\rangle$. Let*

(a) *$a_i = [a_{i-1}, b]$, where $i = 2, 3, \ldots$.*

Then $a_i \neq 1$ for $i \leqslant c$, $G' = \langle a_2, a_3, \ldots, a_c\rangle$, $G_c = \langle a_c\rangle$, and the following relations hold:

(b) $[a_c, b] = 1$;
(c) $[a_i, a_j] = 1$, $i, j = 1, 2, \ldots, c$;
(d) *$b^{p^m} = a_c^\delta$ for some integer δ with $0 \leqslant \delta \leqslant p - 1$;*
(e) *$a_1^{\binom{p}{1}} a_2^{\binom{p}{2}} \ldots a_p = a_c^\gamma$ for some integer γ with $0 \leqslant \gamma \leqslant p - 1$;*
(f) $a_i^{\binom{p}{1}} a_{i+1}^{\binom{p}{2}} \cdots a_{i+p-1} = 1$, $i = 2, 3, \ldots, c$;
(g) *$(b^s a_1^{s_1} a_2^{s_2} \ldots a_c^{s_c})^p = b^{sp} a_c^{s_1\gamma}$, where $s, s_1, \ldots, s_c$ are integers and $(s, p) = 1$.*

Furthermore, the relations (a)–(f) *are defining relations of* G. *Conversely, a* p*-group having defining relations* (a)–(f) *is a* $\mathcal{D}'_p(2)$*-group.*

Proof. Since $c(G)=c$, we have (b). Then for $i>c$, $a_i=1$. Since $a_i\in A$ for any i, (c) holds and $a_i=1$ for $c+1\leqslant i\leqslant c+p-1$. Since $b^p\in A$, we have $[a_{i-1},b^p]=1$ for $2\leqslant i\leqslant c$. By Proposition 2.12(1), we have

$$\left[a_{i-1},b^p\right]=a_i^{\binom{p}{1}}a_{i+1}^{\binom{p}{2}}\dots a_{i+p-1}=1.$$

Then (f) holds. Since

$$\left[a_1^{\binom{p}{1}}a_2^{\binom{p}{2}}\dots a_p^{\binom{p}{p}},b\right]=a_2^{\binom{p}{1}}a_3^{\binom{p}{2}}\dots a_p^{\binom{p}{p-1}}a_{p+1}=1$$

by Lemma 2.14, we have $a_1^{\binom{p}{1}}a_2^{\binom{p}{2}}\dots a_p^{\binom{p}{p}}\in Z(G)$. Since $o(bG')=o(bG_c)=p^m$, (d) holds. Since $o(a_1G')=p$, we have $a_1^p\in G'$ and hence $a_1^{\binom{p}{1}}a_2^{\binom{p}{2}}\dots a_p^{\binom{p}{p}}\in G'\cap Z(G)=G_c$. Then (e) holds.

By Lemma 2.14 and (e), $(ba_1^{s_1})^p=b^pa_1^{s_1p}\prod_{i=1}^{p-1}[a^{s_1p},ib]^{\binom{p}{i+1}}=b^p(a_1^pa_2^{\binom{p}{2}}\dots a_p^{\binom{p}{p}})^{s_1}=b^pa_c^{s_1\gamma}$. If $(s,p)=1$, then there is some $d\in G'$ such that $b^sa_1^{s_1}=(ba_1^{s's_1})^sd$ where $ss'\equiv 1 \pmod p$. By Lemma 2.14(3), $(b^sa_1^{s_1}a_2^{s_2}\dots a_c^{s_c})^p=(b^sa_1^{s_1})^p=(ba_1^{s's_1})^{sp}=b^{sp}a_c^{s_1\gamma}$. Then (g) holds.

To prove that (a)–(f) are defining relations of G. (Note that (f) can be derived from (a)–(c) and (e), so the relations (a)–(e) are already the defining relations of G.) We first use (f), taking $i=c,c-1,\dots,$ to get $o(a_c)=p,\dots,o(a_{c-p+2})=p,\dots,$ $A_i:=\langle a_i,a_{i+1},\dots,a_c\rangle$ has order p^{c-i+1} for $i=2,3,\dots$. Hence $G'=\langle a_2,a_3,\dots,a_c\rangle$ has order p^{c-1}. By (c) G' is abelian. Now by (e), $A_1:=\langle G',a_1\rangle$ is abelian and of order p^c. Finally, by (d), G is a cyclic extension of A_1 by C_{p^m}. By Proposition 2.12(1) and (a), $[a_1,b^p]=a_2^{\binom{p}{1}}a_3^{\binom{p}{2}}\cdots a_{p+1}=1$, implying $b^p\in Z(G)$.

Finally, let G be a group defined by (a)–(f). Then $\langle A_1,b^p\rangle$ is an abelian maximal subgroup of G. Now it is easy to see that G/G' and A/G' has type invariants (p^m,p) and (p^{m-1},p), respectively. Hence $G\in\mathcal{D}'_p(2)$ by Theorem 3.3. □

Remark 3.7. The proofs of Lemma 3.2, Theorems 3.3 and 3.6 also work for the case $m=1$. This gives a description of the structures of p-groups of maximal class having an abelian maximal subgroup and coincides with Blackburn's results; see [4, §4].

Corollary 3.8. *Suppose that* $G\in\mathcal{D}_p(2)$. *Then*

(1) $G\in\mathcal{A}_2$ *if and only if* $c=3$.
(2) *If* $p=2$, *then* G *is metacyclic.*
(3) *If* $c\leqslant p$, *then* $d(G')=c-1$. *If* $c\geqslant p+1$, *then* $d(G')=p-1$.

Theorem 3.9. *Let* G *and* $\tilde G$ *be two groups having defining relations in Theorem* 3.6 *with generators* $\{b,a_1\}$ *and* $\{\tilde b,\tilde a_1\}$ *and parameters* (δ,γ) *and* $(\tilde\delta,\tilde\gamma)$, *respectively. Then* G *and* $\tilde G$ *are isomorphic if and only if there exist integers* s,t_1,k *with* $p\nmid st_1$ *such that*

(1) $s^{c-2}t_1\tilde\delta\equiv\delta \pmod p$; *and*
(2) $\tilde\gamma t_1s^{c-1}\equiv t_1\gamma+k\delta \pmod p$.

Proof. Assume θ is an isomorphism from $\tilde{G}$ onto G. We have $\tilde{A}^\theta = A$ and $(\tilde{G}')^\theta = G'$ since these four subgroups are characteristic in G or $\tilde{G}$, respectively. So, we may let

$$\tilde{b}^\theta = b^s a_1^{s_1} a_2^{s_2} \dots a_c^{s_c}, \qquad \tilde{a}_1^\theta = b^{kp^{m-1}} a_1^{t_1} a_2^{t_2} \dots a_c^{t_c}$$

where $p \nmid st_1$. By calculations, using Lemma 2.14, we get

$$\tilde{a}_i^\theta \equiv a_i^{s^{i-1}t_1} \pmod{G_{i+1}} \quad (2 \leqslant i \leqslant c).$$

By Theorem 3.6(g), we have

$$\big(\tilde{b}^p\big)^\theta = \big(b^s a_1^{s_1} a_2^{s_2} \dots a_c^{s_c}\big)^p = b^{sp} a_c^{s_1\gamma},$$

and hence

$$\big(\tilde{b}^{p^m}\big)^\theta = b^{sp^m} = a_c^{s\delta}.$$

Since

$$\big(\tilde{b}^{p^m}\big)^\theta = \big(\tilde{a}_c^{\tilde{\delta}}\big)^\theta = a_c^{s^{c-1}t_1\tilde{\delta}},$$

we get

$$s^{c-2}t_1\tilde{\delta} \equiv \delta \pmod{p}.$$

By Theorem 3.6(g), we have

$$(ba_1)^p = b^p a_c^\gamma \quad \text{and} \quad (\tilde{b}\tilde{a}_1)^p = \tilde{b}^p \tilde{a}_c^{\tilde{\gamma}}.$$

Transforming the latter equation by θ, the left-hand side becomes

$$\big(b^s a_1^{s_1} a_2^{s_2} \dots a_c^{s_c} b^{kp^{m-1}} a_1^{t_1} a_2^{t_2} \dots a_c^{t_c}\big)^p,$$

and by Theorem 3.6(g) again it is equal to $b^{sp} a_c^{(s_1+t_1)\gamma} a_c^{k\delta}$; while the right-hand side of that equation is equal to $b^{sp} a_c^{s_1\gamma} a_c^{s^{c-1}t_1\tilde{\gamma}}$. Hence we have

$$\tilde{\gamma} t_1 s^{c-1} \equiv t_1\gamma + k\delta \pmod{p}.$$

Conversely, if parameters (δ, γ) and $(\tilde{\delta}, \tilde{\gamma})$ satisfy congruent equations in theorem, then, by using the above argument, it is easy to check that the map $\theta : \tilde{b} \mapsto b^s$, $\tilde{a}_1 \mapsto b^{kp^{m-1}} a_1^{t_1}$ is an isomorphism from $\tilde{G}$ onto G. $\square$

Theorem 3.10. *The number of non-isomorphic $\mathcal{D}'_p(2)$-groups of order p^{m+c} with $c(G) = c$ is $2 + \gcd(c-1, p-1)$.*

Proof. If δ is relatively prime to p, then δ can be chosen to be 1 and γ can be chosen to be zero. If p divides δ, then γ can be assigned either 0 or one of $\gcd(c-1, p-1)$ values. $\square$

Lemma 3.11. *Let G, A, $a_1, \ldots, a_c, b$ and γ be the same as in Theorem* 3.6. *Let $A_i = \langle a_i, a_{i+1}, \ldots, a_c\rangle$, $c-1 = (p-1)q + r$ where $q \geqslant 0$ and $0 \leqslant r \leqslant p-2$. Then*:

(1) *If $\gamma = 0$, then $A_1 = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{p-1}\rangle = C_{p^{q+1}}^{r+1} \times C_{p^q}^{p-r-2}$ and $a_c = a_{r+1}^{(-p)^q}$.*

(2) *If $\gamma \neq 0$, then*
   - (i) *for $q = 0$ (namely $c < p$), $A_1 = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{c-1}\rangle = C_{p^2} \times C_p^{c-2}$ and $a_1^p = a_c^\gamma$;*
   - (ii) *for $q = 1$, $r = 0$ (namely $c = p$) and $\gamma = 1$, $A_1 = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_p\rangle = C_p^p$;*
   - (iii) *for $q = 1$, $r = 0$ (namely $c = p$) and $\gamma \neq 1$, $A_1 = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{p-1}\rangle = C_{p^2} \times C_p^{p-2}$ and $a_1^p = a_p^{\gamma-1}$;*
   - (iv) *for $c \geqslant p+1$, $A_1 = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{p-1}\rangle = C_{p^{q+1}}^{r+1} \times C_{p^q}^{p-r-2}$ and $a_c = a_{r+1}^{(-p)^q}$.*

Proof. For the case $c = c(G) \leqslant p-1$, Theorem 3.6(e)–(f) gives $a_c^p = a_{c-1}^p = \cdots = a_2^p = 1$ and $\exp(A_2) = p$. Since $a_1^p = a_c^\gamma$, $\exp(A_1) = p$ or p^2, depending on $\gamma = 0$ or $\neq 0$, respectively. So, we get (1) and (2)(i).

Now we assume that $c = p$. By Theorem 3.6(e), $a_1^p = a_c^{\gamma-1}$. Similar argument gives (1) and (2)(ii)–(iii).

Next, assume $c > p$. By Theorem 3.6(e)–(f) again, we have that for any $i \geqslant 2$, $o(a_i) = po(a_{i+p-1})$, and that for $i \geqslant p$, a_i is a pth power of some element in A_1 and hence $a_i \in \Phi(A_1)$. Thus $A_1 = \langle a_1, \ldots, a_{p-1}\rangle$. Since $|A_1| = p^c$, and it is easy to check that $o(a_1) = \cdots = o(a_{r+1}) = p^{q+1}$ and $o(a_{r+2}) = \cdots = o(a_{p-1}) = p^q$, we have $o(a_1)o(a_2)\cdots o(a_{p-1}) = |A_1|$. So we have $A_1 = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{p-1}\rangle$.

Finally, to prove $a_c = a_{r+1}^{(-p)^q}$, we use induction on c. If $p+1 \leqslant c \leqslant 2p-1$, Theorem 3.6(e) gives the result. If $c \geqslant 2p$, $\Omega_1(A_1) = \langle a_c, \ldots, a_{c-p+2}\rangle = G_{c-p+2}$. Consider $\bar{G} = G/\Omega_1(A_1)$. Then $c(\bar{G}) = c-p+1$. So, the induction hypothesis gives $a_{c-p+1} = a_{r+1}^{(-p)^{q-1}}$ modulo $\Omega_1(A_1)$, i.e., $a_{c-p+1}^{-p} = a_{r+1}^{(-p)^q}$. Applying Theorem 3.6(e) again, we get $a_c = a_{c-p+1}^{-p}$. Hence $a_c = a_{r+1}^{(-p)^q}$. $\square$

Theorem 3.12. *Suppose $G \in \mathcal{D}_2'(2)$, $c(G) = c$ and $|G| = p^{m+c}$. Then $m \geqslant 2$, $c \geqslant 3$ and G is one of following non-isomorphic groups*:

(1) $\langle a, b \mid a^{2^c} = b^{2^m} = 1,\ a^b = a^{-1}\rangle$;
(2) $\langle a, b \mid a^{2^c} = 1,\ b^{2^m} = a^{2^{c-1}},\ a^b = a^{-1}\rangle$;
(3) $\langle a, b \mid a^{2^c} = b^{2^m} = 1,\ a^b = a^{-1+2^{c-1}}\rangle$.

Theorem 3.13. *For p odd suppose that $G \in \mathcal{D}_p'(2)$, $c(G) = c$ and $|G| = p^{m+c}$. Then $m \geqslant 2$, $c \geqslant 3$ and*

(I) *If $c \leqslant p$, then G is one of following non-isomorphic groups*:
   - (1) *an extension of an abelian p-group $\langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_c\rangle (= C_p^c)$ by a cyclic group of order p^m generated by b, i.e. $\langle a_1, b \mid a_i^p = b^{p^m} = 1,\ [a_j, b] = a_{j+1},\ [a_c, b] = 1,\ [a_i, a_j] = 1\rangle$, where $1 \leqslant i \leqslant c \leqslant p$, $1 \leqslant j \leqslant c-1$;*
   - (2) *an extension of an abelian p-group $\langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_c\rangle (= C_p^c)$ by a cyclic group of order p^m generated by b, i.e. $\langle a_1, b \mid a_i^p = b^{p^{m+1}} = 1,\ [a_j, b] = a_{j+1},\ [a_{c-1}, b] = b^{p^m},\ [a_i, a_j] = 1\rangle$, where $1 \leqslant i \leqslant c-1 \leqslant p-1$, $1 \leqslant j \leqslant c-2$;*

(3) *an extension of an abelian p-group $\langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{c-1}\rangle (= C_{p^2} \times C_p^{c-2})$ by a cyclic group of order p^m generated by b, i.e. $\langle a_1, b \mid a_1^{p^2} = a_i^p = b^{p^m} = 1,\ [a_j, b] = a_{j+1},\ [a_{c-1}, b] = a_1^{tp},\ [a_i, a_j] = 1\rangle$, where $2 \leqslant i \leqslant c-1 \leqslant p-1$, $1 \leqslant j \leqslant c-2$, $t = t_1, t_2, \ldots, t_{(c-1,p-1)}$, where $t_1, t_2, \ldots, t_{(c-1,p-1)}$ are the coset representatives of the subgroup $\mathbb{F}$ consisting of $c-1$ powers in $\mathbb{Z}_p^*$ (there are* $\gcd(c-1, p-1)$ *groups).*

(II) *If $c \geqslant p+1$, we let $c-1 = (p-1)q + r$ where $q \geqslant 1$ and $0 \leqslant r \leqslant p-2$. Then G is one of following non-isomorphic groups*:

(4) *an extension of an abelian p-group $\langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{p-1}\rangle (= C_{p^{q+1}}^{r+1} \times C_{p^q}^{p-r-2})$ by a cyclic group of order p^m generated by b, i.e. $\langle a_1, b \mid a_i^{p^{q+1}} = a_j^{p^q} = b^{p^m} = 1,\ [a_k, b] = a_{k+1},\ [a_{p-1}, b] = a_1^{-\binom{p}{1}} a_2^{-\binom{p}{2}} \ldots a_{p-1}^{-p} a_{r+1}^{t(-p)^q},\ [a_u, a_v] = 1\rangle$, where $1 \leqslant i \leqslant r+1$, $r+2 \leqslant j \leqslant p-1$, $1 \leqslant k \leqslant p-2$, $1 \leqslant u, v \leqslant p-1$, $t = p, t_1, t_2, \ldots, t_{(r,p-1)}$, where $t_1, t_2, \ldots, t_{(r,p-1)}$ are the coset representatives of the subgroup $\mathbb{F}$ consisting of r powers in $\mathbb{Z}_p^*$ (there are* $\gcd(r, p-1) + 1$ *groups*);

(5) *an extension of an abelian p-group $\langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{p-1}\rangle (= C_{p^{q+1}}^{r+1} \times C_{p^q}^{p-r-2})$ by a cyclic group of order p^m generated by b, i.e. $\langle a_1, b \mid a_i^{p^{q+1}} = a_j^{p^q} = b^{p^{m+1}} = 1,\ b^{p^m} = a_{r+1}^{(-p)^q},\ [a_k, b] = a_{k+1},\ [a_{p-1}, b] = a_1^{-\binom{p}{1}} a_2^{-\binom{p}{2}} \ldots a_{p-1}^{-p},\ [a_u, a_v] = 1\rangle$, where $1 \leqslant i \leqslant r+1$, $r+2 \leqslant j \leqslant p-1$, $1 \leqslant k \leqslant p-2$, $1 \leqslant u, v \leqslant p-1$.*

4. The classification of $\mathcal{D}_p(3)$-groups

Note that a $\mathcal{D}_p(3)$-group G has a non-abelian maximal subgroup since G is not minimal non-abelian.

Theorem 4.1. *If $G \in \mathcal{D}_p(3)$, then $G \in \mathcal{A}_2$. Conversely, let $G \in \mathcal{A}_2$, $d(G) = 3$ and G has an abelian maximal subgroup, then $G \in \mathcal{D}_p(3)$.*

Proof. Assume that $G \in \mathcal{D}_p(3)$ but $G \notin \mathcal{A}_2$. Let A be an abelian maximal subgroup and M a non-abelian maximal subgroup of G. Then $d(M) = 2$ and M has an abelian maximal subgroup $A \cap M$. It follows from Lemma 3.1(3) that $\mathrm{Z}(M) \leqslant \Phi(M)$. Since $d(G) = 3$, $\Phi(M) = \Phi(G)$. Since $\mathrm{Z}(M) \leqslant \Phi(M) = \Phi(G) \leqslant A$ and $G = AM$, $\mathrm{Z}(M) \leqslant \mathrm{Z}(G)$. Applying Lemma 2.13, we have $|G| = p|G'||\mathrm{Z}(G)|$ and $|M| = p|M'||\mathrm{Z}(M)|$. It follows that $|G'/M'||\mathrm{Z}(G)/\mathrm{Z}(M)| = p$.

Since $G \notin \mathcal{A}_2$, G has a non-abelian maximal subgroup M which is not minimal non-abelian and hence $M \in \mathcal{D}_p(2)$, $c(M) \geqslant 3$ and $|M'| \geqslant p^2$. By using the above argument, we consider two cases: (1) $G' = M'$ and $|\mathrm{Z}(G)/\mathrm{Z}(M)| = p$, and (2) $\mathrm{Z}(G) = \mathrm{Z}(M)$ and $|G'/M'| = p$.

Case 1: Let $K \leqslant M$ and $K \in \mathcal{A}_1$. Take $d \in \mathrm{Z}(G) \backslash M$ and let $N = \langle K, d\rangle$. Then $|N'| = |K'| = p$. It follows that $N < G$ since $|G'| = |M'| \geqslant p^2$. Hence $d(N) = 2$. Since $|N'| = p$, $N \in \mathcal{A}_1$ and hence $N = K$. Furthermore, $d \in K \leqslant M$, a contradiction.

Case 2: By Lemma 3.1(3), $\Phi(M) = M'\mathrm{Z}(M)$. Then $G' \leqslant \Phi(G) = \Phi(M) = M'\mathrm{Z}(M)$. Let $e \in G' \backslash M'$ and $e = mz$, where $m \in M'$, $z \in \mathrm{Z}(M) \backslash M'$. So $z \in G' \setminus M'$ and $z \in \mathrm{Z}(M) = \mathrm{Z}(G)$. Let $b \in M \backslash A$. Then $G = \langle b, A\rangle$. By Lemma 2.13, there exists an $a \in A$ such that $z = [a, b]$. Since $z \notin M'$, we have $a \notin M$. Then $G = \langle b, a, A \cap M\rangle$. Let $N = \langle b, a, \Phi(G)\rangle$. Then N is maximal in G. Since N is not abelian, $\Phi(G) = \Phi(N)$ and hence $N = \langle b, a\rangle$. Note that $[a, b] = z \in \mathrm{Z}(G)$ and $N' = \langle [a, b]^g \mid g \in N\rangle = \langle z\rangle$. We have $c(N) = 2$ and hence $N \in \mathcal{A}_1$, which implies $\mathrm{Z}(N) =$

$\Phi(N)$. Hence $G' \leqslant \Phi(G) = \Phi(N) = \mathrm{Z}(N) \leqslant \mathrm{Z}(G)$ and $c(G) = 2$, a contradiction. Thus we proved $G \in \mathcal{A}_2$.

Conversely, assume that $G \in \mathcal{A}_2$ and $d(G) = 3$. Then every non-abelian subgroup of G is maximal and hence is minimal non-abelian, which can be generated by two elements. Thus $G \in \mathcal{D}_p(3)$. $\square$

5. The classification of $\mathcal{M}_p$-groups

Since all groups of order p^4 have an abelian maximal subgroup, $\mathcal{M}_p$-groups have order at least p^5. Let $M'_p = \{G \in \mathcal{M}_p \mid G$ is neither metacyclic nor 3-group of maximal class$\}$. Then we only need to classify $\mathcal{M}'_p$-groups.

Lemma 5.1. *Suppose that G is of maximal class. Then $G \notin \mathcal{M}'_p$.*

Proof. Let G be a p-group of maximal class and $G \in \mathcal{M}'_p$. By the definition of $\mathcal{M}'_p$-group, we have that $p \geqslant 5$ and G has no abelian maximal subgroup. Hence $d(M) = 2$ for all $M \lessdot G$. Since groups of order p^4 have an abelian maximal subgroup, $|G| \geqslant p^5$. Let $\overline{G} = G/G_4$. Then $|\overline{G}| = p^4$ and hence $\overline{G}$ has an abelian maximal subgroup M/G_4. Since $\overline{G} \cong \frac{G/G_5}{\mathrm{Z}(G/G_5)}$ and $|G/G_5| = p^5$, Theorem 2.6 gives that $\exp(\overline{G}) = p$. Then M/G_4 is an elementary abelian p-group. It follows that $d(M) \geqslant 3$ and $M \lessdot G$, a contradiction. $\square$

Theorem 5.2. *Suppose that $G \in \mathcal{M}'_2$. Then* (1) $d(G) = 3$; (2) *G has an abelian subgroup A such that $|G : A| = 4$ and $d(A) \geqslant 3$*; (3) $\Phi(G) = \mathrm{Z}(G)$; (4) *$G \in \mathcal{A}_2$ and is the following group*: $G = \langle a, b, c \mid a^4 = b^4 = c^4 = 1,\ [a, b] = c^2,\ [a, c] = b^2c^2,\ [b, c] = a^2b^2,\ [a^2, b] = [a^2, c] = [b^2, a] = [b^2, c] = [c^2, a] = [c^2, b] = 1\rangle$ (Suzuki 2-group of order 2^6).

Proof. (1) Otherwise, $d(G) = 2$. Since all subgroups of G of order 2^{n-1} and 2^n have two generators, Theorem 2.10 gives that G is metacyclic, which is a contradiction.

(2) If $|G| = 2^5$, since G has no abelian maximal subgroup, Lemma 2.1 gives that G has a non-metacyclic maximal subgroup, say M. By the classification of groups of order 2^4, M is minimal non-abelian and we may let

$$M = \langle a, b \mid a^{2^2} = b^2 = c^2 = 1,\ [a, b] = c,\ [c, a] = [c, b] = 1 \rangle.$$

Its subgroup $A = \langle a^2, b, c\rangle$ is abelian and of index 4 in G, and $d(A) = 3$.

If $|G| \geqslant 2^6$ and G has no abelian subgroup A such that $|G : A| = 4$ and $d(A) \geqslant 3$, since all subgroups of order 2^{n-1} and 2^{n-2} have two generators, Theorem 2.10 gives that G is metacyclic, which is a contradiction.

(3) Let $A \lessdot M \lessdot G$. We claim that M is minimal non-abelian. Otherwise, $M \in D_2(2)$. By Corollary 3.8(2), M is metacyclic, contradicting $d(A) \geqslant 3$.

Since $d(G) = 3$ and $d(M) = 2$, $\Phi(M) = \Phi(G)$. Hence $\Phi(G) < A$ and G/A is elementary abelian. Let M_1/A and M_2/A be two maximal subgroups of G/A. Then M_1 and M_2 are maximal subgroups of G containing A, and hence they are minimal non-abelian by the above argument. It follows that $\Phi(G) = \Phi(M_1) = \Phi(M_2) = \mathrm{Z}(M_1) = \mathrm{Z}(M_2)$, and hence $\Phi(G) \leqslant \mathrm{Z}(G)$. If $\Phi(G) < \mathrm{Z}(G)$, then $|G : \mathrm{Z}(G)| = 4$ and G would have an abelian maximal subgroup. Hence $\Phi(G) = \mathrm{Z}(G)$.

(4) Let M be any maximal subgroup of G. Then M is not abelian and $d(M) = 2$. Since $Z(G) = \Phi(G) = \Phi(M)$, $\Phi(M) = Z(M)$. It follows from Lemma 2.2 that $M \in \mathcal{A}_1$ and hence $G \in \mathcal{A}_2$. The last conclusion is obtained by [12]. $\square$

Lemma 5.3. *Let $G \in \mathcal{M}'_p$ and $|G| = p^5$. Then $G \in \mathcal{A}_2$, $d(G) = 2$, $c(G) = 3$, and G is one of the following groups*:

(1) $\langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1,\ [a,b] = c,\ [c,a] = b^{\nu p},\ [c,b] = a^p\rangle$, *where $p \geqslant 5$, ν is a fixed quadratic non-residue* (mod p);
(2) $\langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1,\ [a,b] = c,\ [c,a] = a^{-p}b^{-kp},\ [c,b] = a^{-p}\rangle$, *where $p \geqslant 5$, $4k = g^{2r+1} - 1$ for $r = 1, 2, \ldots, \frac{1}{2}(p-1)$, g denotes the smallest positive integer which is a primitive root* (mod p);
(3) $\langle a, b \mid a^9 = b^9 = c^3 = 1,\ [a,b] = c,\ [c,a] = b^{-3},\ [c,b] = a^3\rangle$;
(4) $\langle a, b \mid a^9 = b^9 = c^3 = 1,\ [a,b] = c,\ [c,a] = a^3b^3,\ [c,b] = a^{-3}\rangle$.

Proof. It follows from Theorem 5.2 that p is odd. Then Theorem 2.8 gives that $d(G) = 2$. According to Lemma 5.1, G is not of maximal class. Let M be any maximal subgroup of G. Then M is also not of maximal class by Theorem 2.7. Since $|M| = p^4$, the classification of groups of p^4 gives that $M \in \mathcal{A}_1$. Hence $G \in \mathcal{A}_2$. The other conclusions are obtained by [12, Theorem 3.8]. $\square$

Theorem 5.4. *For p odd suppose that $G \in \mathcal{M}'_p$ and $|G| = p^n \geqslant p^6$. Then*:

(1) *$d(G) = 2$, G and all maximal subgroups of G are not of maximal class.*
(2) *$G' = \Phi(G)$ is abelian, $G_3 = \mho_1(G)$ and G/G_3 is of order p^3.*
(3) *If K is maximal in G, then $G_3 = \Phi(K)$ and $K \in \mathcal{A}_1$ or $K \in \mathcal{D}'_p(2)$.*
(4) *G has at most one minimal non-abelian maximal subgroup.*
(5) *$c(G) = n - 2$. G has lower central complexion $(2, 1, 2, \underbrace{1, 1, \ldots, 1}_{n-5})$. If $K \in \mathcal{D}'_p(2)$ is maximal in G, then $K_3 = G_4, K_4 = G_5, \ldots, K_{n-3} = G_{n-2}$.*
(6) *If G has no minimal non-abelian maximal subgroup, then $|G| = p^6$.*

Proof. (1) Theorem 2.8 gives that $d(G) = 2$. According to Lemma 5.1, G is not of maximal class. Let M be any maximal subgroup of G. Then M is also not of maximal class by Theorem 2.7.

(2) If G' is not abelian, then $d(G') = 2$, contradicting Theorem 2.11. Other conclusions follow from Theorem 2.9.

(3) Since $G/G_3 = G/\mho_1(G)$ has exponent p and is of order p^3, K/G_3 is elementary abelian and of order p^2. It follows that $G_3 = \Phi(K)$ since $d(K) = 2$. Since K is not of maximal class by (1) and G' is an abelian maximal subgroup of K, $K \in \mathcal{A}_1$ or $K \in \mathcal{D}'_p(2)$.

(4) Otherwise, let K_1, K_2 be two distinct minimal non-abelian maximal subgroups of G. Then $G_3 = \Phi(K_1) = \Phi(K_2) = Z(K_1) = Z(K_2)$ by (3). It follows that $G_3 = \mho_1(G) \leqslant Z(G)$. Since $z^p \in \mho_1(G) \leqslant Z(G)$ for any $z \in G$, $1 = [x, y, z^p] = [x, y, z]^p$ for any $x, y, z \in G$, and hence $\exp(G_3) = p$. Let $G = \langle a, b\rangle$. Then $G_3 = \langle [a, b, a], [a, b, b]\rangle$ gives that $|G_3| \leqslant p^2$ and hence $|G| \leqslant p^5$, contradicting $|G| \geqslant p^6$.

(5) Suppose that $K \in \mathcal{D}'_p(2)$ is maximal in G. As in Lemma 3.2, we let $c(K) = c$ and $|K| = p^{m+c}$ where $m \geqslant 2, c \geqslant 3$. Then $|G| = p^{m+c+1}$, $|G'| = p^{m+c-1}$, $|G_3| = p^{m+c-2}$. By Lemma 3.2, $|K'| = p^{c-1}$ and K/K' has type invariants (p^m, p). Let $\overline{G} = G/K'$. Then $|\overline{G}| = p^{m+2}$ and $\overline{G}$ has an abelian maximal subgroup K/K'. Since $|\overline{G}/\overline{G}'| = p^2$, Theorem 2.5(1) gives that $\overline{G}$ is of maximal class. Since $\overline{G}_3 = G_3/K' = \Phi(K)/K' = \Phi(K/K')$ is cyclic and of order p^{m-1}, and groups of maximal class have no normal cyclic subgroup of order p^2 (Theorem 2.4(6)), we have $m = 2$. Thus $c(K) = n - m - 1 = n - 3$, $|K'| = p^{n-4}$, $|\overline{G}| = p^4$ and $G_4 \leqslant K'$. Since $K' \leqslant \Phi(K) = G_3$, $1 \neq K_{n-3} \leqslant G_{n-2}$. By (1), G is not of maximal class. Hence $c(G) = n - 2$.

Let $M \in \mathcal{D}'_p(2)$ be a maximal subgroup of G and $M \neq K$. Then $c(M) = n - 3$, $|M'| = p^{n-4}$ and $G_4 \leqslant M'$ by the above argument. Since M/K' is not abelian, $|M'K'/K'| = \frac{|M'|}{|M' \cap K'|} = p$, and hence $|M' \cap K'| = p^{n-5}$. Since $G_4 \leqslant M' \cap K'$, $|G_4| \leqslant p^{n-5}$. In the other hand, $|G_4| = |G_4/G_5||G_5/G_6| \dots |G_{n-2}| \geqslant p^{n-5}$. Hence $|G_4| = p^{n-5}$, $|G_3/G_4| = p^2$ and G has lower central complexion $(2, 1, 2, \underbrace{1, 1, \dots, 1}_{n-5})$.

Since $K' \leqslant G_3$, $K_i \leqslant G_{i+1}$ for $3 \leqslant i \leqslant n-3$. It follows that $K_i = G_{i+1}$ for $3 \leqslant i \leqslant n-3$ by comparing their order.

(6) Otherwise, $n \geqslant 7$ and hence $n - 3 \geqslant 4$. By (5), $|G_{n-3}| = p^2$. Let $M = C_G(G_{n-3})$. Then $G/M \lesssim \mathrm{Aut}(G_{n-3})$ and hence M is maximal in G. By assumption, $M \in \mathcal{D}'_p(2)$. Then (5) gives that $M_{n-4} = G_{n-3}$ and $c(M) = n - 3$. Since $G_{n-3} \leqslant \mathrm{Z}(M)$, $M_{n-4} \leqslant \mathrm{Z}(M)$ and hence $c(M) \leqslant n - 4$, a contradiction. $\square$

Theorem 5.5. *For $p \geqslant 5$, suppose that $G \in \mathcal{M}'_p$ and G has no minimal non-abelian maximal subgroup. Then G is one of following non-isomorphic groups*:

$$G = \langle a, b \mid a^{p^2} = b^{p^2} = c^{p^2} = 1,\ [a, b] = c,\ [c, b] = a^p c^{mp},\ [c, a] = b^{\nu p} c^{np},$$
$$[a, b^p] = [a^p, b] = c^p,\ [c, a^p] = [c, b^p] = [c^p, a] = [c^p, b] = 1\rangle,$$

where ν is fixed quadratic non-residue module p. The parameters m, n are the smallest positive integers satisfying $(m-1)^2 - \nu^{-1}(n+\nu)^2 \equiv r \pmod p$, for $r = 0, 1, \dots, p-1$.

Proof. According to Theorem 5.4, G' is abelian, $G_3 = \mho_1(G)$, $c(G) = 4$ and $|G| = p^6$. Let $\overline{G} = G/G_4$ and $K \lessdot G$. Then $\overline{G}$ is not metacyclic and $K/G_4 = K/K_3 \in \mathcal{A}_1$. It follows that $\overline{G}$ is a group of Lemma 5.3. Let $G = \langle a, b\rangle$. Then we may let

(1) $\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^2} = \bar{b}^{p^2} = \bar{c}^p = 1,\ [\bar{a}, \bar{b}] = \bar{c},\ [\bar{c}, \bar{b}] = \bar{a}^p,\ [\bar{c}, \bar{a}] = \bar{b}^{\nu p}\rangle$, where ν is a fixed quadratic non-residue (mod p); or
(2) $\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^2} = \bar{b}^{p^2} = \bar{c}^p = 1,\ [\bar{a}, \bar{b}] = \bar{c},\ [\bar{c}, \bar{b}] = \bar{a}^{-p},\ [\bar{c}, \bar{a}] = \bar{a}^{-p}\bar{b}^{-kp}\rangle$, where $p \geqslant 5$, $4k = g^{2r+1} - 1$ for $r = 1, 2, \dots, \frac{1}{2}(p-1)$, g denotes the smallest positive integer which is a primitive root (mod p).

We claim that $\exp(G') \neq p$. Otherwise, $\mho_1(G) \leqslant \mathrm{Z}(G)$ and hence $G_3 \leqslant \mathrm{Z}(G)$, contradicting $c(G) = 4$. We also have $\exp(G_3) = p$ since $1 = [x, y, z^p] = [x, y, z]^p$ for all $x, y, z \in G$. Hence we may assume that $G = \langle a, b\rangle$ such that $a^{p^2} = b^{p^2} = c^{p^2} = 1$, $[a, b] = c$, $G_4 = \langle c^p\rangle$ and

(1) $[c,b]=a^pc^{mp}$, $[c,a]=b^{\nu p}c^{np}$; or
(2) $[c,b]=a^{-p}c^{mp}$, $[c,a]=a^{-p}b^{-kp}c^{np}$, where $4k=g^{2r+1}-1$ for $r=1,2,\dots,\frac{1}{2}(p-1)$, g denotes the smallest positive integer which is a primitive root (mod p).

Since G is metabelian, $1=[c,b,a]=[c,a,b]$, implying G is one group of type (1).

Conversely, if G is one group of type (1), then $[c,a^ib^j]\equiv a^{jp}b^{i\nu p}\pmod{G_4}$ and $[c,a^ib^j,a^ib^j]=c^{(j^2-i^2\nu)p}$. Let K be any maximal subgroup of G. Then we may let $K=\langle a^ib^j,c,a^p,b^p\rangle$ where $p\nmid ij$. It is easy to see that $K'=\langle a^{jp}b^{i\nu p},c^p\rangle$ and $G'=\langle c,a^p,b^p\rangle$ is an abelian maximal subgroup of K. Since K/K' has type invariants (p^2,p) and G'/K' has type invariants (p,p), Theorem 3.3 gives that $K\in\mathcal{D}'_p(2)$ and hence $G\in\mathcal{M}'_p$.

Let $\tilde{G}$ be such a group with parameters $(\tilde{m},\tilde{n})$. Using the similar method as in the proof of Theorem 3.9, we can get $\tilde{G}\cong G$ if and only if $(\tilde{m}-1)^2-\nu^{-1}(\tilde{n}+\nu)^2\equiv(m-1)^2-\nu^{-1}(n+\nu)^2\pmod p$. So there are p non-isomorphic groups depending only on the value of $(m-1)^2-\nu^{-1}(n+\nu)^2$. □

Theorem 5.6. *Suppose that $G\in\mathcal{M}'_3$ and G has no minimal non-abelian maximal subgroup. Then G is one of following non-isomorphic groups* (*where the trivial commutators are omitted*):

(1) $\langle a,b\mid a^9=b^9=c^3=d^3=1,\ [a,b]=c,\ [c,b]=a^3,\ [c,a]=b^{-3},\ [a^3,b]=[a,b^3]=d\rangle$;
(2) $\langle a,b\mid a^9=b^9=c^3=d^3=1,\ [a,b]=c,\ [c,b]=a^3d,\ [c,a]=b^{-3}d,\ [a^3,b]=[a,b^3]=d\rangle$.

Proof. According to Theorem 5.4, G' is abelian, $G_3=\mho_1(G)$, $c(G)=4$ and $|G|=3^6$. Let $\overline{G}=G/G_4$ and $K\lessdot G$. Then $\overline{G}$ is not metacyclic and $K/G_4=K/K_3\in\mathcal{A}_1$. It follows that $\overline{G}$ is a group of Lemma 5.3. Let $G_4=\langle d\rangle$, $G=\langle a,b\rangle$ and $\bar{a}$, $\bar{b}$ have the relations described in Lemma 5.3. Then G has one of following relations

(a) $a^9=d^{n_1}, b^9=d^{n_2}, c^3=d^{n_3}, d^3=1, [a,b]=cd^l, [c,b]=a^3d^m, [c,a]=b^{-3}d^n$; or
(b) $a^9=d^{n_1}, b^9=d^{n_2}, c^3=d^{n_3}, d^3=1, [a,b]=cd^l, [c,b]=a^{-3}d^m, [c,a]=a^3b^3d^n$

and $G'=\langle c,a^3,b^3,d\rangle$, $G_3=\langle a^3,b^3,d\rangle$. Since $\mho_1(G)=G_3$, $1=[x,y,z^3]=[x,y,z]^3$ for any $x,y,z\in G$, implying $\exp(G_3)=3$. Then $a^9=b^9=1$. We claim that G is not a group of type (b). Otherwise, since G is metabelian, $1=[c,b,a]=[c,a,b]=[a^3,b]$, implying $a^3\in Z(G)$. Then $K=\langle b,c,a^3,d\rangle\in\mathcal{A}_1$ and K is maximal in G, a contradiction.

Without loss of generality we may assume that $[a,b]=c$, that is, $l=0$. Then $[a^3,b]=c^3[c,a,a]=c^3[b^{-3},a]=c^3[a,b^3]=c^6[c,b,b]=c^6[a^3,b]$. It follows that $c^3=1$ and $[c,a,a]=[c,b,b]$. Since $[c,a,b]=[c,b,a]=1$, $[c,a,a]\neq 1$. Let $[c,a,a]=[c,b,b]=d$. Then $G=\langle a,b\rangle$ has the defining relations

$$a^9=b^9=c^3=d^3=1,\quad [a,b]=c,\quad [c,b]=a^3d^m,$$

$$[c,a]=b^{-3}d^n,\quad [c,a,a]=[c,b,b]=d,$$

where the trivial commutators are omitted.

Conversely, suppose G has the above defining relations. Let $K_1 = \langle \Phi(G), a\rangle$, $K_2 = \langle \Phi(G), b\rangle$, $K_3 = \langle \Phi(G), ba\rangle$ and $K_4 = \langle \Phi(G), ba^{-1}\rangle$. Then maximal subgroups of G are K_s ($s = 1, 2, 3, 4$). It is easy to see that $|K_s'| = 9$. According to Lemma 5.3, $d(\bar{K}_s) = 2$ and hence $d(K_s) = 2$. Since $c(K_s) = 3$, $\Phi(K_s)/K_s'$ is cyclic and K_s have abelian maximal subgroup G' such that G'/K_s' is not cyclic, Corollary 3.4 gives that $K_s \in \mathcal{D}_3'(2)$, and hence $G \in \mathcal{M}_3'$.

Next we determine non-isomorphic types. By substitution, we can get that G is the group of type (2) for $m = n = 1$ and G is the group of type (1) for others. (Details are omitted.)

Finally, using the similar method as Theorem 3.9, we can show that the two groups of types (1) and (2) are not isomorphic to each other. (Details are omitted.) □

Theorem 5.7. *Suppose that G has a minimal non-abelian maximal subgroup. Then $G \in \mathcal{M}_p'$ if and only if $G = \langle a_1, b\rangle$ has following defining relations*:

(a) $a_i = [a_{i-1}, b]$, $i = 2, 3, \dots, n$;
(b) $[a_1, a_2] = c$;
(c) $c^3 = 1$;
(d) $[c, a_1] = [c, b] = 1$;
(e) $[a_{n-2}, b] = 1$;
(f) $[a_1, a_i] = 1$, $i = 3, 4, \dots, n-2$;
(g) $[a_i, a_j] = 1$, $i, j = 2, 3, \dots, n-2$;
(h) $b^3 = ca_{n-2}^{\delta}$;
(i) $a_1^3 a_2^3 a_3 = a_{n-2}^{\gamma}$;
(j) $a_i^3 a_{i+1}^3 a_{i+2} = 1$, $i = 2, 3, \dots, n-2$,

where $n \geqslant 6$, $0 \leqslant \delta, \gamma \leqslant 2$.

Proof. Let $G \in \mathcal{M}_p'$, $|G| = p^n$ and G_1 be the minimal non-abelian maximal subgroup of G. By Theorem 5.4, $n \geqslant 6$, $d(G) = 2$, $G' = \Phi(G)$ is abelian, $G_3 = \mho_1(G) = \Phi(G_1)$ and $c(G) = n-2$.

First we claim that $p = 3$. Otherwise, $p \geqslant 5$. Let $\overline{G} = G/G_1'$. Since $\overline{G}$ has an abelian maximal subgroup $\overline{G}_1 = G_1/G_1'$ and $|\overline{G}/\overline{G}'| = p^2$, $\overline{G}$ is of maximal class by Theorem 2.5. Since $c(\overline{G}) = n-2 \geqslant 4$, Corollary 3.8(3) gives that $d(\overline{G}_1) \geqslant 3$, a contradiction.

Let $b \in G \backslash G_1$ and $a_1 \in G_1 \backslash G'$. Then $G = \langle a_1, b\rangle$. Let $a_i = [a_{i-1}, b]$, where $2 \leqslant i \leqslant n$ and $[a_1, a_2] = c$. Then $G' = \langle a_2, G_3\rangle$ and hence $G_1 = \langle a_1, a_2\rangle$. Since $|G_1'| = 3$, (c) holds. Since G_1' char G_1 char G, $G_1' \leqslant Z(G)$. It follows that $G_3 = \langle c, a_3, G_4\rangle$, $G_4 = \langle a_4, \dots, a_{n-2}\rangle, \dots$, $G_{n-2} = \langle a_{n-2}\rangle$ and $Z(G) = \langle c, a_{n-2}\rangle$. Then (d), (e) holds.

Since $G_3 = Z(G_1)$, $a_i \in Z(G_1)$ for $i = 3, 4, \dots, n-2$. Then (f) holds. Since G' is abelian, (g) holds. Since $b^3 \in G_3 = Z(G_1)$, $[a_i, b^3] = 1$ for $i = 1, 2, \dots, n-2$. Then Proposition 2.12(1) gives that

(j) $a_i^3 a_{i+1}^3 a_{i+2} = 1$, $i = 2, 3, \dots, n-2$.

By Proposition 2.12(1), we have $[a_1^3, b] = [a_1, b]^3 [a_1, b, a_1]^3 [a_1, b, a_1, a_1] = a_2^3$ and hence $[a_1^3 a_2^3 a_3, b] = a_2^3 a_3^3 a_4 = 1$. Obviously, $[a_1^3 a_2^3 a_3, a_1] = 1$. Hence $a_1^3 a_2^3 a_3 \in Z(G)$ and we may assume $a_1^3 a_2^3 a_3 = c^{\alpha} a_{n-2}^{\gamma}$ for $0 \leqslant \alpha, \gamma \leqslant 2$. Replacing b by $b' = ba_1^{\alpha}$, we have $a_1^3 (a_2')^3 a_3' = a_1^3 a_2^3 a_3 c^{-\alpha} = a_{n-2}^{\gamma}$. So we get (i) (changing the generator b when necessary).

Finally, since $b^3 \in Z(G)$, we may assume that $b^3 = c^{\beta} a_{n-2}^{\delta}$, $0 \leqslant \beta, \delta \leqslant 2$. By Proposition 2.12(2), we have $(ba_1^{-1})^3 = b^3[b, a_1]^3[b, a_1, b][b, a_1, a_1]a_1^{-3} = b^3 c(a_1^3 a_2^3 a_3)^{-1} = c^{\beta+1} a_{n-2}^{\delta-\gamma}$. Let $M = \langle b, G' \rangle$. Then $M \neq G_1$ and hence M is a $\mathcal{D}'_3(2)$-group by Theorem 5.4. According to Theorem 3.2(4), $b^3 \notin G_{n-2} = M_{n-3}$. Similarly, $(ba_1^{-1})^3 \notin G_{n-2}$. Then $\beta \neq 0, 2$. Hence we have $b^3 = ca_{n-2}^{\delta}$, (h) holds.

Conversely, if $G = \langle b, a_1 \rangle$ satisfies the above relations (a)–(j), we show that $G \in \mathcal{M}'_p$. (Note that (j) can be derived from (a)–(g) and (i), so the relations (a)–(i) are already the defining relations of G.)

Let $M = \langle b, G' \rangle = \langle b, a_2 \rangle$ and $A = \langle c, a_2, \ldots, a_{n-2} \rangle$. According to Theorem 3.6, $M \in \mathcal{D}'_3(2)$, $M' = \langle a_3, \ldots, a_{n-2} \rangle$, $|M| = 3^{n-1}$ and A is the unique abelian maximal subgroup of M. By Lemma 3.11, $M' = \langle a_3, a_4 \rangle$. By (a), (b), (d), (f), (i), a_1 induce an automorphism of M and $a_1^3 a_2^3 a_3$ induce an identical automorphism of M. Hence $G = \langle b, a_1 \rangle$ and $|G| = 3^n$ by cyclic extension theory. Moreover, $G' = A$, $G_3 = \langle c, a_3, G_4 \rangle$, $G_4 = \langle a_4, G_5 \rangle$ and so on. Let $G_1 = \langle a_1, G' \rangle$. Since $M' = \langle a_3, a_4 \rangle$, by (i), (j), (b) and (c), $G_1 = \langle a_1, a_2 \rangle$ is a minimal non-abelian maximal subgroup of G.

Let $M_j = \langle ba_1^j, a_2, G_3 \rangle$, $j = 1, 2$. Then M_j are other maximal subgroups of G, $M'_j = \langle a_3, a_4, \ldots, a_{n-2} \rangle$ and $G' = A$ is an abelian maximal subgroup of M_j. By calculation, $(ba_1^j)^3 = b^3 ca_{n-2}^{j\gamma} = c^2 a_{n-2}^{\delta+j\gamma}$ (details omitted), which implies $\Phi(M_j) = \langle c, M'_j \rangle = G_3$. Hence M_j/M'_j has type invariants $(3^2, 3)$, and A/M'_j has type invariants $(3, 3)$. By Theorem 3.3, $M_j \in \mathcal{D}'_3(2)$. Therefore $G \in \mathcal{M}'_3$. $\square$

By using similar calculations and arguments as in the proof of Theorem 3.9, we get the following theorem. The details are omitted.

Theorem 5.8. *Let G and $\tilde{G}$ be two groups having defining relations in Theorem* 5.7 *with parameters (δ, γ) and $(\tilde{\delta}, \tilde{\gamma})$, respectively. Then G and $\tilde{G}$ are isomorphic if and only if there exist integers s, t_1 with $p \nmid st_1$ such that*

(1) $s^{n-4} t_1 \tilde{\delta} \equiv \delta \pmod 3$;
(2) $\tilde{\gamma} s^{n-3} \equiv \gamma \pmod 3$.

As an immediate consequence of the above theorem, the number of non-isomorphic groups in Theorem 5.7 is 6 if n is odd, or 4 if n is even.

Theorem 5.9. *Suppose that $G \in \mathcal{M}'_p$ and G possess a minimal non-abelian maximal subgroup. Then $p = 3$, $|G| \geqslant 3^6$ and*:

(1) *If $|G| = 3^{2q+4}$, where $q \geqslant 1$, then $G = \langle a_1, b \rangle$ and*

$$a_1^{3^{q+1}} = a_2^{3^{q+1}} = b^{3^2} = c^3 = 1, \quad [a_1, b] = a_2, \quad [a_2, b] = a_3,$$
$$[a_1, a_2] = c, \quad [c, a_1] = [c, a_2] = 1, \quad a_1^3 a_2^3 a_3 = a_2^{k(-3)^q}, \quad b^3 = a_2^{s(-3)^q} c,$$

where $s = 0, 1$, $k = 0, 1$.

(2) *If* $|G| = 3^{2q+5}$, *where* $q \geqslant 1$, *then* $G = \langle a_1, b\rangle$ *and*

$$a_1^{3^{q+2}} = a_2^{3^{q+1}} = b^{3^2} = c^3 = 1, \quad [a_1, b] = a_2, \quad [a_2, b] = a_3,$$
$$[a_1, a_2] = c, \quad [c, a_1] = [c, a_2] = 1, \quad a_1^3 a_2^3 a_3 = a_1^{k(-3)^{q+1}}, \quad b^3 = a_1^{s(-3)^{q+1}} c,$$

where $s = 0, 1$, $k = 0, 1, 2$.

Summarizing, we have the following

Main Theorem. *Suppose that all non-abelian proper subgroups of a* p*-group* G *are generated by two elements. Then one of the following holds*:

(1) *All subgroups of* G *of index* p^2 *are abelian*;
(2) G *is metacyclic*;
(3) G *is of maximal class and has an abelian maximal subgroup*;
(4) G *is* 3*-group of maximal class*;
(5) G *is a* $\mathcal{D}_p'(2)$*-group, namely one group of Theorem* 3.13;
(6) G *is an* $\mathcal{M}_3'$*-group and* G *has a unique minimal non-abelian maximal subgroup, namely one group of Theorem* 5.9;
(7) G *is one group of Theorems* 5.5, 5.6.

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