Computers and Mathematics with Applications 59 (2010) 161–175

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Characterizations of regular semigroups by (α, β)-fuzzy ideals

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ARTICLE INFO

Article history:
Received 21 September 2008
Accepted 28 July 2009

Keywords:
(α, β)-fuzzy subsemigroups
(α, β)-fuzzy ideals
(α, β)-fuzzy bi-ideals
(α, β)-fuzzy quasi-ideals

ABSTRACT

Using the ideas of belonging and quasi-coincidence of a fuzzy point with a fuzzy set, the concepts of (α, β)-fuzzy ideals and (α, β)-fuzzy generalized bi-ideals, which are generalization of fuzzy ideals and fuzzy generalized bi-ideals, in a semigroup are introduced, and related properties are investigated. We also define the lower and upper parts of fuzzy subsets of a semigroup. Characterizations of regular semigroups by the properties of the lower part of $(\in, \in \vee q)$-fuzzy left ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals and $(\in, \in \vee q)$-fuzzy generalized bi-ideals are given.

1. Introduction

The fundamental concept of a fuzzy set, introduced by L. A. Zadeh in his definitive paper [1] of 1965, provides a natural framework for generalizing several basic notions of algebra. Kuroki initiated the theory of fuzzy Semigroups in his papers [2,3]. A systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [4], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy Languages. Fuzziness has a natural place in the field of formal languages. The monograph by Mordeson and Malik [5] deals with the application of fuzzy approach to the concepts of automata and formal languages. Murali [6] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [7], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [8,9] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In [10] $(\in, \in \vee q)$-fuzzy subrings and ideals are defined. In [11] Davvaz define $(\in, \in \vee q)$-fuzzy subnearring and ideals of a near ring. In [12] Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup. In [13] Kazanci and Yamak study $(\in, \in \vee q)$-fuzzy bi-ideals of a semigroup. In [14] Bhakat define $(\in \vee q)$-level subset of a fuzzy set. In this paper we introduce the concept of (α, β)-fuzzy ideal, (α, β)-fuzzy generalized bi-ideal, and characterize regular semigroups by the properties of these ideals.

2. Preliminaries

A semigroup is an algebraic system $(S, .)$ consisting of a nonempty set S together with an associative binary operation ".". By a subsemigroup of S we mean a nonempty subset A of S such that $A^2 \subseteq A$. A nonempty subset A of S is called a left (right) ideal of S if $SA \subseteq A$ $(AS \subseteq A)$. A nonempty subset A of S is called a two-sided ideal or simply an ideal of S if it is both a left and a right ideal of S. A nonempty subset Q of S is called a quasi-ideal of S if $QS \cap SQ \subseteq Q$. A subsemigroup B of a

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2009.07.062

semigroup S is called a bi-ideal of S if $BSB \subseteq B$. A nonempty subset B of S is called a generalized bi-ideal of S if $BSB \subseteq B$. A subsemigroup A of a semigroup S is called an interior ideal of S if $SAS \subseteq A$. Obviously every one-sided ideal of a semigroup S is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal. An element a of a semigroup S is called a regular element if there exists an element x in S such that $a = axa$. A semigroup S is called regular if every element of S is regular. It is well known that for a regular semigroup the concepts of quasi-ideal, bi-ideal and generalized bi-ideal coincide.

A fuzzy subset f of a universe X is a function from X into the unit closed interval [0, 1], i.e. $f : X \to [0, 1]$. A fuzzy subset f in a universe X of the form

$$f(y) = \begin{cases} t \in (0, 1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set f in a set X, Pu and Liu [7] gave meaning to the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy set f written $x_t \in f$ (resp. $x_t q f$) if $f(x) \geq t$ (resp. $f(x) + t > 1$), and in this case, $x_t \in \vee q f$ (resp. $x_t \in \wedge q f$) means that $x_t \in f$ or $x_t q f$ ($resp. x_t \in f$ and $x_t q f$). To say that $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold. For any two fuzzy subsets f and g of S, $f \leq g$ means that, for all $x \in S$, $f(x) \leq g(x)$ (cf. [16]). The symbols $f \wedge g$, and $f \vee g$ will mean the following fuzzy subsets of S

$$(f \wedge g)(x) = f(x) \wedge g(x)$$
$$(f \vee g)(x) = f(x) \vee g(x).$$

for all $x \in S$.

Let f and g be two fuzzy subsets of a semigroup S. The product $f \circ g$ is defined by

$$(f \circ g)(x) = \begin{cases} \vee_{x=yz} \{f(y) \wedge g(z)\}, & \text{if } \exists y, z \in S, \text{ such that } x = yz \\ 0 & \text{otherwise}. \end{cases}$$

Definition 1 (*[4]*). A fuzzy subset f of S is called a fuzzy subsemigroup of S if for all $x, y \in S$

$$f(xy) \geq \min\{f(x), f(y)\}.$$

Definition 2 (*[4]*). A fuzzy subset f of S is called a fuzzy left (right) ideal of S if for all $x, y \in S$

$$f(xy) \geq f(y) \qquad (f(xy) \geq f(x)).$$

A fuzzy subset f of S is called a fuzzy ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.

Definition 3 (*[4]*). A fuzzy subset f of S is called a fuzzy quasi-ideal of S if

$$(f \circ \mathcal{S}) \wedge (\mathcal{S} \circ f) \leq f$$

where $\mathcal{S}$ is the fuzzy subset of S mapping every element of S on 1.

Definition 4 (*[4]*). A fuzzy subsemigroup f of S is called a fuzzy bi-ideal of S if for all $x, y, z \in S$,

$$f(xyz) \geq \min\{f(x), f(z)\}.$$

Definition 5 (*[4]*). A fuzzy subset f of S is called a fuzzy generalized bi-ideal of S if for all $x, y, z \in S$,

$$f(xyz) \geq \min\{f(x), f(z)\}.$$

Definition 6 (*[4,17]*). A fuzzy subsemigroup f of S is called a fuzzy interior ideal of S if for all $x, a, y \in S$,

$$f(xay) \geq f(a).$$

Definition 7 (*[12]*). A fuzzy subset f of S is called an (α, β)-fuzzy interior ideal of S, where $\alpha \neq \in \wedge q$, if it satisfies,

(i) For all $x, y \in S$ and for all $t_1, t_2 \in (0, 1]$, $x_{t_1} \alpha f$ and $y_{t_2} \alpha f \Rightarrow (xy)_{\min\{t_1, t_2\}} \beta f$,
(ii) For all $x, a, y \in S$ and for all $t \in (0, 1]$, $a_t \alpha f \Rightarrow (xay)_t \beta f$.

Theorem 1 (*[12]*). *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if it satisfies the following conditions.*

(1) $f(xy) \geq \min\{f(x), f(y), 0.5\}$
(2) $f(xay) \geq \min\{f(a), 0.5\}$.

Theorem 2 (*[15,18]*). *For a semigroup S the following conditions are equivalent.*

(1) *S is regular.*
(2) *$R \cap L = RL$ for every right ideal R and every left ideal L of S.*
(3) *$ASA = A$ for every quasi-ideal A of S.*

3. (α, β)-fuzzy ideals

Let S be a semigroup and α and β denote any one of $\in, q, \in\vee q$ or $\in\wedge q$ unless otherwise specified.

Definition 8. A fuzzy subset f of a semigroup S is called an (α, β)-fuzzy subsemigroup of S, where $\alpha \neq \in\wedge q$ if $x_{t_1}\alpha f$, and $y_{t_2}\alpha f \Rightarrow (xy)_{\min\{t_1,t_2\}}\beta f$.

Let f be a fuzzy subset of S such that $f(x) \leq 0.5$ for all $x \in S$. Let $x \in S$ and $t \in (0, 1]$ be such that$x_t \in \wedge qf$. Then $f(x) \geq t$ and $f(x) + t > 1$. It follows that $1 < f(x) + t \leq f(x) + f(x) = 2f(x)$, so that $f(x) \geq 0.5$. This means that $\{x_t : x_t \in \wedge q\} = \emptyset$. Therefore the case $\alpha = \in\wedge q$ in the above definition is omitted.

Definition 9. A fuzzy subset f of a semigroup S is called an (α, β)-fuzzy left (right) ideal of S, where $\alpha \neq \in\wedge q$ if it satisfies, $y_t\alpha f$ and $x \in S \Rightarrow (xy)_t\,\beta f$ $((yx)_t\,\beta f)$ for all $x, y \in S$.

A fuzzy subset f of a semigroup S is called an (α, β)-fuzzy ideal of S if it is both an (α, β)-fuzzy left ideal and (α, β)-fuzzy right ideal of S.

Definition 10. A fuzzy subset f of a semigroup S is called an (α, β)-fuzzy bi-ideal of S, where $\alpha \neq \in\wedge q$, if it satisfies the following two conditions.

(i) For all $x, y \in S$ and for all $t_1, t_2 \in (0, 1]$, $x_{t_1}\alpha f, y_{t_2}\alpha f \Rightarrow (xy)_{\min\{t_1,t_2\}}\,\beta f$.
(ii) For all $x, y, z \in S$ and for all $t_3, t_4 \in (0, 1]$, $x_{t_3}\alpha f, z_{t_4}\alpha f \Rightarrow (xyz)_{\min\{t_3,t_4\}}\,\beta f$.

Definition 11. A fuzzy subset f of a semigroup S is called an (α, β)-fuzzy generalized bi-ideal of S, where $\alpha \neq \in\wedge q$, if it satisfies,
For all $x, y, z \in S$ and for all $t_3, t_4 \in (0, 1]$, $x_{t_3}\alpha f, z_{t_4}\alpha f \Rightarrow (xyz)_{\min\{t_3,t_4\}}\,\beta f$.

Lemma 1. *A fuzzy subset f of a semigroup S is a fuzzy subsemigroup of S if and only if it satisfies,*
For all $x, y \in S$ and for all $t_1, t_2 \in (0, 1]$, $x_{t_1} \in f, y_{t_2} \in f \Rightarrow (xy)_{\min\{t_1,t_2\}} \in f$.

Proof. Suppose f is a fuzzy subsemigroup of a semigroup S. Let $x, y \in S$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in f$ and $y_{t_2} \in f$. Then $f(x) \geq t_1$ and $f(y) \geq t_2$. Since f is a fuzzy subsemigroup of S. So $f(xy) \geq \min\{f(x), f(y)\} \geq \min\{t_1, t_2\}$. Hence $(xy)_{\min\{t_1,t_2\}} \in f$.
Conversely, assume that f satisfies the given condition. We show that $f(xy) \geq f(x) \wedge f(y)$. On the contrary assume that there exist $x, y \in S$ such that $f(xy) < f(x) \wedge f(y)$. Let $t \in (0, 1]$ be such that $f(xy) < t < f(x) \wedge f(y)$. Then $x_t \in f$ and $y_t \in f$ but $(xy)_t \overline{\in} f$. This contradicts our hypothesis. Thus $f(xy) \geq f(x) \wedge f(y)$. ■

Lemma 2. *A fuzzy subset f of a semigroup S is a fuzzy left (resp. right) ideal of S if and only if it satisfies,*

For all $x, y \in S$ and for all $t \in (0, 1]$, $y_t \in f \Rightarrow (xy)_t \in f$ $((yx)_t \in f)$.

Proof. Suppose f is a fuzzy left ideal of a semigroup S. Let $y_t \in f$, then $f(y) \geq t$. Since f is a fuzzy left ideal of S, so $f(xy) \geq f(y) \geq t$. Hence $(xy)_t \in f$.
Conversely, suppose that f satisfies the given condition. We show that $f(xy) \geq f(y)$. On the contrary assume that there exist $x, y \in S$ such that $f(xy) < f(y)$. Let $t \in (0, 1]$ be such that $f(xy) < t < f(y)$. Then $y_t \in f$ but $(xy)_t \overline{\in} f$. Which contradicts our hypothesis. Hence $f(xy) \geq f(y)$. ■

Remark 1. The above Lemma shows that every fuzzy left (right) ideal of S is an $(\in, \in)$-fuzzy left (right) ideal of S.

The proofs of the following Lemmas are similar to the proof of above Lemma.

Lemma 3. *A fuzzy subset f of a semigroup S is a fuzzy bi-ideal of S if and only if it satisfies,*

(1) *For all $x, y \in S$ and for all $t_1, t_2 \in (0, 1]$, $x_{t_1} \in f, y_{t_2} \in f \Rightarrow (xy)_{\min\{t_1,t_2\}} \in f$.*
(2) *For all $x, y, z \in S$ and for all $t_3, t_4 \in (0, 1]$, $x_{t_3} \in f, z_{t_4} \in f \Rightarrow (xyz)_{\min\{t_3,t_4\}} \in f$.*

Lemma 4. *A fuzzy subset f of a semigroup S is a fuzzy generalized bi-ideal of S if and only if it satisfies,*
For all $x, y, z \in S$ and for all $t_3, t_4 \in (0, 1]$, $x_{t_3} \in f, z_{t_4} \in f \Rightarrow (xyz)_{\min\{t_3,t_4\}} \in f$.

Theorem 3. *Let f be a nonzero (α, β)-fuzzy subsemigroup of S. Then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is a subsemigroup of S.*

Proof. Let $x, y \in f_0$. Then $f(x) > 0$ and $f(y) > 0$. Let $f(xy) = 0$. If $\alpha \in \{\in, \in\vee q\}$, then $x_{f(x)}\alpha f$ and $y_{f(y)}\alpha f$ but $f(xy) = 0 < \min\{f(x), f(y)\}$ and $f(xy) + \min\{f(x), f(z)\} \leq 0 + 1 = 1$. So $(xy)_{\min\{f(x),f(y)\}}\,\overline{\beta} f$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Hence $f(xy) > 0$, that is $xy \in f_0$. Also $x_1 qf$ and $y_1 qf$ but $(xy)_1\,\overline{\beta} f$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$. Hence $f(xy) > 0$, that is, $xy \in f_0$. Thus f_0 is a subsemigroup of S. ■

Theorem 4. *Let f be a nonzero (α, β)-fuzzy generalized bi-ideal of S. Then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is a generalized bi-ideal of S.*

Proof. Let $x, z \in f_0$ and $y \in S$. Then $f(x) > 0$ and $f(z) > 0$. Let $f(xyz) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{f(x)}\alpha f$ and $z_{f(z)}\alpha f$ but $f(xyz) = 0 < \min\{f(x), f(z)\}$ and $f(xyz) + \min\{f(x), f(z)\} \leq 0 + 1 = 1$. So $(xyz)_{\min\{f(x), f(z)\}} \overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $f(xyz) > 0$, that is, $xyz \in f_0$. Also $x_1 q f$ and $z_1 q f$ but $(xyz)_1 \overline{\beta} f$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence $f(xyz) > 0$, that is, $xyz \in f_0$. Thus f_0 is a generalized bi-ideal of S. ■

Theorem 5. *Let f be a nonzero (α, β)-fuzzy bi-ideal of S. Then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is a bi-ideal of S.*

Proof. Follows from Theorems 3 and 4. ■

Theorem 6. *Let f be a nonzero (α, β)-fuzzy left (resp. right) ideal of S. Then the set $f_0 = \{x \in S \mid f(x) > 0\}$ is a left (resp. right) ideal of S.*

Proof. Proof is similar to the proofs of Theorems 3 and 4. ■

Theorem 7. *Let L be a left (resp. right) ideal of S and let f be a fuzzy subset in S such that,*

$$f(x) = \begin{cases} 0 & \text{if } x \in S - L \\ \geq 0.5 & \text{if } x \in L. \end{cases}$$

Then

(1) *f is a $(q, \in \vee q)$-fuzzy left (resp. right) ideal of S.*
(2) *f is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S.*

Proof. (1) Let $x, y \in S$ and $t \in (0, 1]$ such that $y_t q f$. Then $f(y) + t > 1$. So $y \in L$. Therefore $xy \in L$. Thus if $t \leq 0.5$, then $f(xy) \geq 0.5 \geq t$ and so $(xy)_t \in f$. If $t > 0.5$, then $f(xy) + t > 0.5 + 0.5 = 1$ and so $(xy)_t\, q f$. Therefore $(xy)_t \in \vee q f$. Thus f is a $(q, \in \vee q)$-fuzzy left ideal of S.

(2) Let $x, y \in S$ and $t \in (0, 1]$ such that $y_t \in f$. Then $f(y) \geq t$. Thus $y \in L$, and so $xy \in L$. Thus if $t \leq 0.5$, then $f(xy) \geq 0.5 \geq t$ and so $(xy)_t \in f$. If $t > 0.5$, then $f(xy) + t > 0.5 + 0.5 = 1$ and so $(xy)_t\, q f$. Therefore $(xy)_t \in \vee q f$. Thus f is an $(\in, \in \vee q)$-fuzzy left ideal of S. ■

Similarly we can prove the following Theorems.

Theorem 8. *Let A be a subsemigroup of S and let f be a fuzzy subset in S such that*

$$f(x) = \begin{cases} 0 & \text{if } x \in S - A \\ \geq 0.5 & \text{if } x \in A. \end{cases}$$

Then

(1) *f is a $(q, \in \vee q)$-fuzzy subsemigroup of S.*
(2) *f is an $(\in, \in \vee q)$-fuzzy subsemigroup of S.*

Theorem 9. *Let B be a generalized bi-ideal of S and let f be a fuzzy subset in S such that*

$$f(x) = \begin{cases} 0 & \text{if } x \in S - B \\ \geq 0.5 & \text{if } x \in B. \end{cases}$$

Then

(1) *f is a $(q, \in \vee q)$-fuzzy generalized bi-ideal of S.*
(2) *f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S.*

Theorem 10. *Let B be a bi-ideal of S and let f be a fuzzy subset in S such that*

$$f(x) = \begin{cases} 0 & \text{if } x \in S - B \\ \geq 0.5 & \text{if } x \in B. \end{cases}$$

Then

(1) *f is a $(q, \in \vee q)$-fuzzy bi-ideal of S.*
(2) *f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

4. $(\in, \in \vee q)$-fuzzy ideals

Lemma 5. *Let f be a fuzzy subset of a semigroup S. Then f is an $(\in, \in \vee q)$-fuzzy left (right) ideal of S if and only if $f(xy) \geq \min\{f(y), 0.5\}$ $(f(xy) \geq \min\{f(x), 0.5\})$.*

Proof. Let f be an $(\in, \in \vee q)$-fuzzy left ideal of S. On the contrary assume that $f(xy) < \min\{f(y), 0.5\}$. Choose $t \in (0, 1]$ such that $f(xy) < t < \min\{f(y), 0.5\}$. Then $y_t \in f$ but $(xy)_t \overline{\in \vee q} f$, which is a contradiction. Hence $f(xy) \geq \min\{f(y), 0.5\}$.

Conversely, assume that $f(xy) \geq \min\{f(y), 0.5\}$. Let $y_t \in f$ then $f(y) \geq t$. Now $f(xy) \geq \min\{f(y), 0.5\} \geq \min\{t, 0.5\}$. If $t \leq 0.5$, then $f(xy) \geq t$. So $(xy)_t \in f$. If $t > 0.5$ then $f(xy) \geq 0.5$. So $f(xy) + t > 0.5 + 0.5 = 1$. Thus $(xy)_t\, q f$. Hence $(xy)_t \in \vee q f$. Thus f is an $(\in, \in \vee q)$-fuzzy left ideal of S. ■

Corollary 1. *Let f be a fuzzy subset of a semigroup S. Then f is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S if and only if $f(xy) \geq \min\{f(y), 0.5\}$ and $f(xy) \geq \min\{f(x), 0.5\}$.*

Theorem 11. *If f is an $(\in, \in \vee q)$-fuzzy left ideal and g is an $(\in, \in \vee q)$-fuzzy right ideal of a semigroup S then $f \circ g$ is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S.*

Proof. Let $x, y \in S$. Then

$$\begin{aligned}(f \circ g)(y) \wedge 0.5 &= (\vee_{y=pq}\{f(p) \wedge g(q)\}) \wedge 0.5 \\ &= \vee_{y=pq}\{f(p) \wedge g(q) \wedge 0.5\} \\ &= \vee_{y=pq}\{f(p) \wedge 0.5 \wedge g(q)\}.\end{aligned}$$

(If $y = pq$, then $xy = x(pq) = (xp)q$. Since f is an $(\in, \in \vee q)$-fuzzy left ideal so by Lemma 5 $f(xp) \geq \min\{f(p), 0.5\}$.)
Thus

$$\begin{aligned}(f \circ g)(y) \wedge 0.5 &= \vee_{y=pq}\{f(p) \wedge 0.5 \wedge g(q)\} \\ &\leq \vee_{y=pq}\{f(xp) \wedge g(q)\} \\ &\leq \vee_{xy=ab}\{f(a) \wedge g(b)\} \\ &= (f \circ g)(xy).\end{aligned}$$

So

$$\min\{(f \circ g)(y), 0.5\} \leq (f \circ g)(xy).$$

Similarly we can show that $(f \circ g)(xy) \geq \min\{(f \circ g)(x), 0.5\}$.
Thus $f \circ g$ is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S. ■

Lemma 6. *Intersection of $(\in, \in \vee q)$-fuzzy left ideals of a semigroup S is an $(\in, \in \vee q)$-fuzzy left ideal of S.*

Proof. Let $\{f_i\}_{i\in I}$ be a family of $(\in, \in \vee q)$-fuzzy left ideals of S. Let $x, y \in S$.
Then $(\wedge_{i\in I} f_i)(xy) = \wedge_{i\in I}(f_i(xy))$
(Since each f_i is an $(\in, \in \vee q)$-fuzzy left ideal of S, so $f_i(xy) \geq f_i(y) \wedge 0.5$ for all $i \in I$)
Thus

$$\begin{aligned}(\wedge_{i\in I} f_i)(xy) &= \wedge_{i\in I}(f_i(xy)) \\ &\geq \wedge_{i\in I}(f_i(y) \wedge 0.5) \\ &= (\wedge_{i\in I} f_i(y)) \wedge 0.5 \\ &= (\wedge_{i\in I} f_i)(y) \wedge 0.5.\end{aligned}$$

Hence $\wedge_{i\in I} f_i$ is an $(\in, \in \vee q)$-fuzzy left ideal of S. ■

Similarly we can prove that intersection of $(\in, \in \vee q)$-fuzzy right ideals of a semigroup S is an $(\in, \in \vee q)$-fuzzy right ideal of S. Thus intersection of $(\in, \in \vee q)$-fuzzy two-sided ideals of a semigroup S is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S.

Now we show that if f and g are $(\in, \in \vee q)$-fuzzy ideals of a semigroup S, then $f \circ g \nleq f \cap g$.

Example 1. Consider the semigroup $S = \{a, b, c, d\}$.

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Fact 1. *A fuzzy subset f of S is an $(\in, \in \vee q)$-fuzzy left ideal of S if and only if*

(i) $f(a) \geq f(x) \wedge 0.5$ *for every* $x \in S$,
(ii) $f(b) \geq f(y) \wedge 0.5$ *for* $y = c$ *and* $y = d$.

Proof. Suppose f is an $(\in, \in \vee q)$-fuzzy left ideal of S. Now $f(a) = f(ax) \geq f(x) \wedge 0.5$ for every $x \in S$. As $b = cc$ or $b = dc$ or $b = dd$. Thus $f(b) = f(cc) \geq f(c) \wedge 0.5$ and $f(b) = f(dd) \geq f(d) \wedge 0.5$. ■

Conversely, assume that (i) and (ii) hold. Since $b = cc$, $b = dc$ and $b = dd$, thus by given conditions $f(xy) \geq f(x) \wedge 0.5$ for every $x, y \in S$. Hence f is an $(\in, \in \vee q)$-fuzzy left ideal of S.

Similarly we can prove that

Fact 2. *A fuzzy subset f of S is an $(\in, \in \vee q)$-fuzzy right ideal of S if and only if*

(iii) $f(a) \geq f(x) \wedge 0.5$ *for every* $x \in S$,
(iv) $f(b) \geq f(y) \wedge 0.5$ *for* $y = c$ *and* $y = d$.

Fact 3. *A fuzzy subset f of S is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S if and only if*

(v) $f(a) \geq f(x) \wedge 0.5$ *for every* $x \in S$,
(vi) $f(b) \geq f(y) \wedge 0.5$ *for* $y = c$ *and* $y = d$.

Let f and g be fuzzy subsets of S such that

$$f(a) = 0.5, \quad f(b) = 0.6, \quad f(c) = 0.7, \quad f(d) = 0$$
$$g(a) = 0.7, \quad g(b) = 0.5, \quad g(c) = 0.6, \quad g(d) = 0.2.$$

Then f and g are $(\in, \in \vee q)$-fuzzy ideals of S.

Now

$$\begin{aligned} f \circ g(b) &= \vee_{b=xy} \{f(x) \wedge g(y)\} \\ &= \vee \{0.6, 0, 0\} \\ &= 0.6 \nleq (f \wedge g)(b) = 0.5 \end{aligned}$$

Hence $f \circ g \nleq f \wedge g$ in general.

Theorem 12. *A fuzzy subset f of a semigroup S is an $(\in, \in \vee q)$-fuzzy subsemigroup of S if and only if $f(xy) \geq \min\{f(x), f(y), 0.5\}$ for all $x, y \in S$.*

Lemma 7. *A fuzzy subset f of a semigroup S is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S if and only if $f(xyz) \geq \min\{f(x), f(z), 0.5\}$ for all $x, y, z \in S$.*

Proof. Suppose f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. On the contrary suppose that there exist $x, y, z \in S$ such that $f(xyz) < \min\{f(x), f(z), 0.5\}$. Choose $t \in (0, 1]$ such that $f(xyz) < t < \min\{f(x), f(z), 0.5\}$. Then $x_t \in f$ and $z_t \in f$ but $(xyz)_{\min\{t,t\}} = (xyz)_t \overline{\in \vee q} f$, which is a contradiction. Thus $f(xyz) \geq \min\{f(x), f(z), 0.5\}$ for all $x, y, z \in S$.

Conversely, assume that $f(xyz) \geq \min\{f(x), f(z), 0.5\}$ for all $x, y, z \in S$. Let $x_{t_1} \in f$ and $z_{t_2} \in f$ for $t_1, t_2 \in (0, 1]$. Then $f(x) \geq t_1$ and $f(z) \geq t_2$. So $f(xyz) \geq \min\{f(x), f(z), 0.5\} \geq \min\{t_1, t_2, 0.5\}$. Now if $\min\{t_1, t_2\} \leq 0.5$, then $f(xyz) \geq \min\{t_1, t_2\}$. So $(xyz)_{\min\{t_1,t_2\}} \in f$. If $\min\{t_1, t_2\} > 0.5$. Then $f(xyz) \geq 0.5$. So $f(xyz) + \min\{t_1, t_2\} > 0.5 + 0.5 = 1$, which implies that $(xyz)_{\min\{t_1,t_2\}}\, q f$. Hence $(xyz)_{\min\{t_1,t_2\}} \in \vee q f$. Thus f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. ■

Theorem 13 ([10]). *A fuzzy subset f of a semigroup S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if it satisfies the following conditions,*

(1) $f(xy) \geq \min\{f(x), f(y), 0.5\}$ *for all* $x, y \in S$,
(2) $f(xyz) \geq \min\{f(x), f(z), 0.5\}$ *for all* $x, y, z \in S$.

Proof. Follows from Theorem 12 and Lemma 7. ■

It is clear that every $(\in, \in \vee q)$-fuzzy bi-ideal of a semigroup S is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. The next example shows that the fuzzy generalized bi-ideal of S is not necessarily a fuzzy bi-ideal of S.

Example 2. Consider the semigroup $S = \{a, b, c, d\}$.

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let f be a fuzzy subset of S such that

$$f(a) = 0.5, \quad f(b) = 0, \quad f(c) = 0.2, \quad f(d) = 0.$$

Then f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. Because $f(xyz) = f(a) = 0.5 \geq f(x) \wedge f(z) \wedge 0.5$. But f is not $(\in, \in \vee q)$-fuzzy bi-ideal of S. Because $f(cc) = f(b) = 0 \ngeq 0.2 = f(c) \wedge f(c) \wedge 0.5$.

Lemma 8. *Every $(\in, \in \vee q)$-fuzzy generalized bi -ideal of a regular semigroup S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

Proof. Let f be any $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S and let a, b be any elements of S. Then there exists an element $x \in S$ such that $b = bxb$. Thus we have $f(ab) = f(a(bxb)) = f(a(bx)b) \geq \min\{f(a), f(b), 0.5\}$. This shows that f is an $(\in, \in \vee q)$-fuzzy subsemigroup of S and so f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. ■

Definition 12. A fuzzy subset f of a semigroup S is called an $(\in, \in\vee q)$-fuzzy quasi-ideal of S, if it satisfies,

$$f(x) \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x), 0.5\}$$

where $\mathcal{S}$ is the fuzzy subset of S mapping every element of S on 1.

Theorem 14. *Let f be an $(\in, \in\vee q)$-fuzzy quasi-ideal of a semigroup S, then the set $f_0 = \{x \in S : f(x) > 0\}$ is a quasi-ideal of S.*

Proof. In order to show that f_0 is a quasi-ideal of S, we have to show that $Sf_0 \cap f_0S \subseteq f_0$. Let $a \in Sf_0 \cap f_0S$. This implies that $a \in Sf_0$ and $a \in f_0S$. So $a = sx$ and $a = yt$ for some $s, t \in S$ and $x, y \in f_0$. Thus $f(x) > 0$ and $f(y) > 0$. Now $f(a) \geq \min\{(f \circ \mathcal{S})(a), (\mathcal{S} \circ f)(a), 0.5\}$.

Since

$$\begin{aligned}(\mathcal{S} \circ f)(a) &= \vee_{a=pq}\{\mathcal{S}(p) \wedge f(q)\} \\ &\geq \{\mathcal{S}(s) \wedge f(x)\} \quad \text{because } a = sx \\ &= f(x).\end{aligned}$$

Similarly $(f \circ \mathcal{S})(a) \geq f(y)$.

Thus

$$\begin{aligned}f(a) &\geq \min\{(f \circ \mathcal{S})(a), (\mathcal{S} \circ f)(a), 0.5\} \\ &\geq \min\{f(x), f(y), 0.5\} \\ &> 0 \quad \text{because } f(x) > 0 \text{ and } f(y) > 0.\end{aligned}$$

Thus $a \in f_0$. Hence f_0 is a quasi-ideal of S. ■

Remark 2. Every fuzzy quasi-ideal of S is an $(\in, \in\vee q)$-fuzzy quasi-ideal of S.

Lemma 9. *A nonempty subset Q of a semigroup S is a quasi-ideal of S if and only if the characteristic function C_Q is an $(\in, \in\vee q)$-fuzzy quasi-ideal of S.*

Proof. Suppose Q is a quasi-ideal of S. Let C_Q be the characteristic function of Q. Let $x \in S$. If $x \notin Q$ then $x \notin SQ$ or $x \notin QS$. If $x \notin SQ$ then $\left(\mathcal{S} \circ C_Q\right)(x) = 0$ and so $\min\left\{\left(C_Q \circ \mathcal{S}\right)(x), \left(\mathcal{S} \circ C_Q\right)(x), 0.5\right\} = 0 = C_Q(x)$. If $x \in Q$ then $C_Q(x) = 1 \geq \min\left\{\left(C_Q \circ \mathcal{S}\right)(x), \left(\mathcal{S} \circ C_Q\right)(x), 0.5\right\}$. Hence C_Q is an $(\in, \in\vee q)$-fuzzy quasi-ideal of S.

Conversely, assume that C_Q is an $(\in, \in\vee q)$-fuzzy quasi-ideal of S. Let $a \in QS \cap SQ$. Then there exist $b, c \in S$ and $x, y \in Q$ such that $a = xb$ and $a = cy$. Then

$$\begin{aligned}\left(C_Q \circ \mathcal{S}\right)(a) &= \vee_{a=pq}\left\{C_Q(p) \wedge \mathcal{S}(q)\right\} \\ &\geq C_Q(x) \wedge \mathcal{S}(b) \\ &= 1 \wedge 1 \\ &= 1.\end{aligned}$$

So $\left(C_Q \circ \mathcal{S}\right)(a) = 1$. Similarly $\left(\mathcal{S} \circ C_Q\right)(a) = 1$.

Hence $C_Q(a) \geq \min\left\{\left(C_Q \circ \mathcal{S}\right)(a), \left(\mathcal{S} \circ C_Q\right)(a), 0.5\right\} = 0.5$. Thus $C_Q(a) = 1$, which implies that $a \in Q$. Hence $SQ \cap QS \subseteq Q$, that is Q is a quasi-ideal of S. ■

The proof of the following Lemma is straight forward.

Lemma 10. *The characteristic function C_L is an $(\in, \in\vee q)$-fuzzy left ideal of S if and only if L is a left ideal of S.*

Similarly the characteristic function C_R is an $(\in, \in\vee q)$-fuzzy right ideal of S if and only if R is a right ideal of S. Hence it follows that characteristic function C_I is an $(\in, \in\vee q)$-fuzzy two-sided ideal of S if and only if I is a two-sided ideal of S.

Theorem 15. *Every $(\in, \in\vee q)$-fuzzy left ideal of S is an $(\in, \in\vee q)$-fuzzy quasi-ideal of S.*

Proof. Let $x \in S$, then

$$(\mathcal{S} \circ f)(x) = \vee_{x=yz}\{\mathcal{S}(y) \wedge f(z)\} = \vee_{x=yz} f(z)$$

This implies that

$$\begin{aligned}(\mathcal{S} \circ f)(x) \wedge 0.5 &= (\vee_{x=yz} f(z)) \wedge 0.5 \\ &= \vee_{x=yz}(f(z) \wedge 0.5) \\ &\leq f(yz) \\ &= f(x) \quad (\text{because } f \text{ is an } (\in, \in\vee q)\text{-fuzzy left ideal of } S.).\end{aligned}$$

Thus $(\mathcal{S} \circ f)(x) \wedge 0.5 \leq f(x)$. Hence $f(x) \geq (\mathcal{S} \circ f)(x) \wedge 0.5 \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x), 0.5\}$. Thus f is an $(\in, \in\vee q)$-fuzzy quasi-ideal of S. ■

Similarly we can show that every $(\in, \in\vee q)$-fuzzy right ideal of S is an $(\in, \in\vee q)$-fuzzy quasi-ideal of S.

Lemma 11. *Every $(\in, \in \vee q)$-fuzzy quasi-ideal of S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

Proof. Suppose f is an $(\in, \in \vee q)$-fuzzy quasi-ideal of a semigroup S. Now

$$\begin{aligned} f(xy) &\geq (f \circ \mathcal{S})(xy) \wedge (\mathcal{S} \circ f)(xy) \wedge 0.5 \\ &= [\vee_{xy=ab}\{f(a) \wedge \mathcal{S}(b)\}] \wedge [\vee_{xy=pq}\{\mathcal{S}(p) \wedge f(q)\}] \wedge 0.5 \\ &\geq [f(x) \wedge \mathcal{S}(y)] \wedge [\mathcal{S}(x) \wedge f(y)] \wedge 0.5 \\ &\geq [f(x) \wedge 1] \wedge [1 \wedge f(y)] \wedge 0.5 \\ &= f(x) \wedge f(y) \wedge 0.5. \end{aligned}$$

So $f(xy) \geq \min\{f(x), f(y), 0.5\}$.
Also

$$\begin{aligned} f(xyz) &\geq (f \circ \mathcal{S})(xyz) \wedge (\mathcal{S} \circ f)(xyz) \wedge 0.5 \\ &= [\vee_{xyz=ab}\{f(a) \wedge \mathcal{S}(b)\}] \wedge [\vee_{xyz=pq}\{\mathcal{S}(p) \wedge f(q)\}] \wedge 0.5 \\ &\geq [f(x) \wedge \mathcal{S}(yz)] \wedge [\mathcal{S}(xy) \wedge f(z)] \wedge 0.5 \\ &\geq [f(x) \wedge 1] \wedge [1 \wedge f(z)] \wedge 0.5 \\ &= f(x) \wedge f(z) \wedge 0.5. \end{aligned}$$

So $f(xyz) \geq \min\{f(x), f(z), 0.5\}$. Thus f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. ■

Lemma 12. *Every $(\in, \in \vee q)$-fuzzy two-sided ideal of S is an $(\in, \in \vee q)$-fuzzy interior ideal of S.*

Proof. Let f be an $(\in, \in \vee q)$-fuzzy two-sided ideal of S. Now $f(xy) \geq f(y) \wedge 0.5 \geq f(x) \wedge f(y) \wedge 0.5$. So $f(xy) \geq f(x) \wedge f(y) \wedge 0.5$. Also for all $x, a, y \in S$. $f(xay) \geq f(x(ay)) \geq f(ay) \wedge 0.5 \geq f(a) \wedge 0.5$. So $f(xay) \geq f(a) \wedge 0.5$. Hence f is an $(\in, \in \vee q)$-fuzzy interior ideal of S. ■

The following example shows that the converse of Lemma 12 does not hold in general.

Example 3. Consider the semigroup $S = \{0, a, b, c\}$.

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Let f be a fuzzy subset of S such that

$$f(0) = 0.7, \quad f(a) = 0.4, \quad f(b) = 0.6, \quad f(c) = 0.$$

Then f is an $(\in, \in \vee q)$-fuzzy interior ideal of S which is not an $(\in, \in \vee q)$-fuzzy two-sided ideal of S. In fact $f(xyz) = f(0) = 0.7 > 0.5 \geq f(y) \wedge 0.5$. Also if $xy = 0$, then $f(xy) = f(0) = 0.7 > 0.5 \geq f(x) \wedge f(y) \wedge 0.5$. If $xy = a$, then $f(xy) = f(a) = 0.4 > 0 = f(x) \wedge f(y) \wedge 0.5$. And if $xy = b$, then $f(xy) = f(b) = 0.6 > 0 = f(x) \wedge f(y) \wedge 0.5$ for every $x, y, z \in S$. Thus f is an $(\in, \in \vee q)$-fuzzy interior ideal of S. But since $f(bc) = f(a) = 0.4 < 0.5 = f(b) \wedge 0.5$. So f is not an $(\in, \in \vee q)$-fuzzy right ideal of S, that is, it is not an $(\in, \in \vee q)$-fuzzy two-sided ideal of S.

5. Lower and upper parts of $(\in, \in \vee q)$-fuzzy ideals

Definition 13. Let f be a fuzzy subset of a semigroup S. We define the upper part f^+ and the lower part f^- of f as follows, $f^+(x) = f(x) \vee 0.5$ and $f^-(x) = f(x) \wedge 0.5$.

Lemma 13. *Let f and g be fuzzy subsets of a semigroup S. Then the following holds.*

(1) $(f \wedge g)^- = (f^- \wedge g^-)$
(2) $(f \vee g)^- = (f^- \vee g^-)$
(3) $(f \circ g)^- = (f^- \circ g^-)$

Proof. For all $a \in S$.
(1)

$$\begin{aligned} (f \wedge g)^-(a) &= (f \wedge g)(a) \wedge 0.5 \\ &= f(a) \wedge g(a) \wedge 0.5 \\ &= (f(a) \wedge 0.5) \wedge (g(a) \wedge 0.5) \\ &= f^-(a) \wedge g^-(a) \\ &= (f^- \wedge g^-)(a). \end{aligned}$$

(2)

$$\begin{aligned}(f \vee g)^{-}(a) &= (f \vee g)(a) \wedge 0.5\\ &= (f(a) \vee g(a)) \wedge 0.5\\ &= (f(a) \wedge 0.5) \vee (g(a) \wedge 0.5)\\ &= f^{-}(a) \vee g^{-}(a)\\ &= \left(f^{-} \vee g^{-}\right)(a).\end{aligned}$$

(3) If a is not expressible as $a = bc$ for some $b, c \in S$, then $(f \circ g)(a) = 0$. Thus $(f \circ g)^{-}(a) = (f \circ g)(a) \wedge 0.5 = 0$. Since a is not expressible as $a = bc$, so $\left(f^{-} \circ g^{-}\right)(a) = 0$. Thus in this case $(f \circ g)^{-} = \left(f^{-} \circ g^{-}\right)$. And if a is expressible $a = xy$ for some $x, y \in S$. Then

$$\begin{aligned}(f \circ g)^{-}(a) &= (f \circ g)(a) \wedge 0.5\\ &= \vee_{a=xy}\{f(x) \wedge g(y)\} \wedge 0.5\\ &= \vee_{a=xy}\{f(x) \wedge 0.5 \wedge g(y) \wedge 0.5\}\\ &= \vee_{a=xy}\{f^{-}(x) \wedge g^{-}(y)\}\\ &= \left(f^{-} \circ g^{-}\right)(a). \quad \blacksquare\end{aligned}$$

Lemma 14. *Let f and g be fuzzy subsets of a semigroup S. Then the following holds.*

(1) $(f \wedge g)^{+} = \left(f^{+} \wedge g^{+}\right)$
(2) $(f \vee g)^{+} = \left(f^{+} \vee g^{+}\right)$
(3) $(f \circ g)^{+} \geq \left(f^{+} \circ g^{+}\right)$.

If every element x of S is expressible as $x = bc$, then $(f \circ g)^{+} = \left(f^{+} \circ g^{+}\right)$.

Proof. For all $a \in S$.

$$\begin{aligned}(1)\ (f \wedge g)^{+}(a) &= (f \wedge g)(a) \vee 0.5\\ &= (f(a) \wedge g(a)) \vee 0.5\\ &= (f(a) \vee 0.5) \wedge (g(a) \vee 0.5)\\ &= f^{+}(a) \wedge g^{+}(a)\\ &= \left(f^{+} \wedge g^{+}\right)(a).\end{aligned}$$

(2)

$$\begin{aligned}(f \vee g)^{+}(a) &= (f \vee g)(a) \vee 0.5\\ &= f(a) \vee g(a) \vee 0.5\\ &= (f(a) \vee 0.5) \vee (g(a) \vee 0.5)\\ &= f^{+}(a) \vee g^{+}(a)\\ &= \left(f^{+} \vee g^{+}\right)(a).\end{aligned}$$

(3) If a is not expressible as $a = bc$ for some $b, c \in S$, then $(f \circ g)(a) = 0$. Thus $(f \circ g)^{+}(a) = (f \circ g)(a) \vee 0.5 = 0.5$. But $\left(f^{+} \circ g^{+}\right)(a) = 0$. So $f^{+} \circ g^{+} \leq (f \circ g)^{+}$. But if a is expressible as $a = bc$ for some $b, c \in S$, then

$$\begin{aligned}(f \circ g)^{+}(a) &= (f \circ g)(a) \vee 0.5\\ &= (\vee_{a=xy}\{f(x) \wedge g(y)\}) \vee 0.5\\ &= \vee_{a=xy}\{(f(x) \wedge g(y)) \vee 0.5\}\\ &= \vee_{a=xy}\{(f(x) \vee 0.5) \wedge (g(y) \wedge 0.5)\}\\ &= \vee_{a=xy}\{f^{+}(x) \wedge g^{+}(y)\}\\ &= \left(f^{+} \circ g^{+}\right)(a). \quad \blacksquare\end{aligned}$$

Definition 14. Let A be a nonempty subset of a semigroup S. Then the lower and upper parts of the characteristic function is,

$$C_A^{-}(a) = \begin{cases} 0.5 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}$$

and

$$C_A^{+}(a) = \begin{cases} 1 & \text{if } a \in A \\ 0.5 & \text{if } a \notin A. \end{cases}$$

Lemma 15. *Let A and B be nonempty subsets of a semigroup S. Then the following properties hold.*

(1) $(C_A \wedge C_B)^- = C^-_{A\cap B}$
(2) $(C_A \vee C_B)^- = C^-_{A\cup B}$
(3) $(C_A \circ C_B)^- = C^-_{AB}$.

Lemma 16. *The lower part of characteristic function C_L^- is an $(\in, \in \vee q)$-fuzzy left ideal of S if and only if L is a left ideal of S.*

Proof. Let L be a left ideal of S. Then by Theorem 7 C_L^- is an $(\in, \in \vee q)$-fuzzy left ideal of S.

Conversely, assume that C_L^- is an $(\in, \in \vee q)$-fuzzy left ideal of S. Let $y \in L$. Then $C_L^-(y) = 0.5$. So $y_{0.5} \in C_L^-$. Since C_L^- is an $(\in, \in \vee q)$-fuzzy left ideal of S. So $(xy)_{0.5} \in \vee q C_L^-$, which implies that $(xy)_{0.5} \in C_L^-$ *or* $(xy)_{0.5}\, q C_L^-$. Hence $C_L^-(xy) \geq 0.5$ *or* $C_L^-(xy) + 0.5 > 1$. If $C_L^-(xy) + 0.5 > 1$, this is not possible, because $C_L^-(xy) \leq 0.5$. Thus $C_L^-(xy) \geq 0.5$, which implies that $C_L^-(xy) = 0.5$. Hence $xy \in L$. Thus L is a left ideal of S. ■

Similarly we can prove *that the lower part of characteristic function C_R^-* is an $(\in, \in \vee q)$-fuzzy right ideal of S if and only if R is a right ideal of S. Thus *the lower part of characteristic function C_I^-* is an $(\in, \in \vee q)$-fuzzy two-sided ideal of S if and only if I is a two-sided ideal of S.

Lemma 17. *Let Q be a nonempty subset of a semigroup S. Then Q is a quasi-ideal of S if and only if the lower part of characteristic function C_Q^- is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S.*

We have shown in Lemma 2 that every fuzzy left (right) ideal of a semigroup S is an $(\in, \in)$-fuzzy left (right) ideal of S. Obviously every $(\in, \in)$-fuzzy left (right) ideal of S is an $(\in, \in \vee q)$-fuzzy left ideal of S. But $(\in, \in \vee q)$-fuzzy left (right) ideal of S need not be fuzzy left (right) ideal of S.

Example 4. Consider the semigroup given in Example 1. Then the fuzzy subset f of S defined by $f(a) = 0.5, f(b) = 0.6, f(c) = 0.7$ and $f(d) = 0$ is an $(\in, \in \vee q)$-fuzzy left ideal of S but f is not fuzzy left ideal of S. Because $f(a) = f(ab) = 0.5$ and $f(b) = 0.6$, so $f(ab) \ngeq f(b)$.

Next we show that if f is an $(\in, \in \vee q)$-fuzzy left (right) ideal of S then f^- is a fuzzy left (right) ideal of S.

Proposition 1. *Let f be an $(\in, \in \vee q)$-fuzzy left (right) ideal of S, then f^- is a fuzzy left (right) ideal of S.*

Proof. Let f be an $(\in, \in \vee q)$-fuzzy left ideal of S, then for all $a, b \in S$, we have
$f(ab) \geq f(b) \wedge 0.5$. This implies that $f(ab) \wedge 0.5 \geq f(b) \wedge 0.5$. So $f^-(ab) \geq f^-(b)$. Thus f^- is a fuzzy left ideal of S.
Similarly if f is an $(\in, \in \vee q)$-fuzzy right ideal of S, then for all $a, b \in S$, we have
$f(ab) \geq f(a) \wedge 0.5$ implies that $f(ab) \wedge 0.5 \geq f(a) \wedge 0.5$. So $f^-(ab) \geq f^-(a)$. Thus f^- is a fuzzy right ideal of S. ■

Next we show that every fuzzy left ideal of S is not of the form f^- for some $(\in, \in \vee q)$-fuzzy left ideal f of S.

Example 5. Consider the semigroup of Example 1. A fuzzy subset f of S is a fuzzy left ideal of S if and only if (i) $f(a) \geq f(x)$ for all $x \in S$ and (ii) $f(b) \geq f(y)$ for $y = c$ or d. Thus $f(a) = 0.9, f(b) = 0.8, f(c) = 0.7, f(d) = 0.7$ is a fuzzy left ideal of S but this is not of the form g^- for some $(\in, \in \vee q)$-fuzzy left ideal g of S.

In [4] regular semigroups are characterized by the properties of their fuzzy ideals, fuzzy bi-ideals and fuzzy generalized bi-ideals. Next we are characterizing the regular semigroups by the properties of lower parts of $(\in, \in \vee q)$-fuzzy ideals, bi-ideals and generalized bi-ideals.

Theorem 16. *For a semigroup S the following conditions are equivalent.*

(1) *S is regular.*
(2) $(f \wedge g)^- = (f \circ g)^-$ *for every $(\in, \in \vee q)$-fuzzy right ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*

Proof. First assume that (1) holds. Let f be an $(\in, \in \vee q)$-fuzzy right ideal and g be an $(\in, \in \vee q)$-fuzzy left ideal of S. Now $a \in S$, we have

$$\begin{aligned}(f \circ g)^-(a) &= (f \circ g)(a) \wedge 0.5 \\ &= (\vee_{a=yz}\{f(y) \wedge g(z)\}) \wedge 0.5 \\ &= \vee_{a=yz}\{f(y) \wedge g(z) \wedge 0.5\} \\ &= \vee_{a=yz}(\{f(y) \wedge 0.5\} \wedge \{g(z) \wedge 0.5\} \wedge 0.5) \\ &\leq \vee_{a=yz}(\{f(yz) \wedge (g(yz)\} \wedge 0.5) \\ &= f(a) \wedge g(a) \wedge 0.5) \\ &= (f \wedge g)(a) \wedge 0.5 \\ &= (f \wedge g)^-(a).\end{aligned}$$

So $(f \circ g)^- \leq (f \wedge g)^-$.
Since S is regular, so there exists an element $x \in S$ such that $a = axa$.

So

$$\begin{aligned}(f\circ g)^{-}(a) &= (f\circ g)(a)\wedge 0.5\\ &= (\vee_{a=yz}\{f(y)\wedge g(z)\})\wedge 0.5\\ &\geq \{f(ax)\wedge g(a)\}\wedge 0.5\\ &\geq \{f(a)\wedge 0.5\wedge g(a)\}\wedge 0.5\\ &= f(a)\wedge g(a)\wedge 0.5\\ &= (f\wedge g)(a)\wedge 0.5\\ &= (f\wedge g)^{-}(a).\end{aligned}$$

So $(f\circ g)^{-}\geq (f\wedge g)^{-}$. Thus $(f\wedge g)^{-}=(f\circ g)^{-}$ and so (1) implies (2).

Conversely, assume that (2) holds. Let R and L be right ideal and left ideal of S, respectively. In order to see that $R\cap L=RL$ holds. Let a be any element of $R\cap L$. Then by Lemma 16, the lower part of characteristic functions C_R^{-} and C_L^{-} of R and L are $(\in,\in\vee q)$-fuzzy right ideal and $(\in,\in\vee q)$-fuzzy left ideal of S, respectively. Thus we have

$$\begin{aligned}C_{RL}^{-} &= (C_R\circ C_L)^{-} \quad \text{by Lemma 15}\\ &= (C_R\wedge C_L)^{-} \quad \text{by (1)}\\ &= C_{R\cap L}^{-} \quad \text{by Lemma 15}.\end{aligned}$$

Thus $R\cap L=RL$. Hence it follows from Theorem 2 that S is regular and so (2) implies (1). ■

Theorem 17. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*
(2) *$(h\wedge f\wedge g)^{-}\leq (h\circ f\circ g)^{-}$ for every $(\in,\in\vee q)$-fuzzy right ideal h, every $(\in,\in\vee q)$-fuzzy generalized bi-ideal f and every $(\in,\in\vee q)$-fuzzy left ideal g of S.*
(3) *$(h\wedge f\wedge g)^{-}\leq (h\circ f\circ g)^{-}$ for every $(\in,\in\vee q)$-fuzzy right ideal h, every $(\in,\in\vee q)$-fuzzy bi-ideal f, and every $(\in,\in\vee q)$-fuzzy left ideal g of S.*
(4) *$(h\wedge f\wedge g)^{-}\leq (h\circ f\circ g)^{-}$ for every $(\in,\in\vee q)$-fuzzy right ideal h, every $(\in,\in\vee q)$-fuzzy quasi-ideal f, and every $(\in,\in\vee q)$-fuzzy left ideal g of S.*

Proof. (1) $\Rightarrow$ (2) : Let h, f and g be any $(\in,\in\vee q)$-fuzzy right ideal, $(\in,\in\vee q)$-fuzzy generalized bi-ideal, and any $(\in,\in\vee q)$-fuzzy left ideal of S, respectively. Let a be any element of S. Since S is regular, so there exists an element $x\in S$ such that $a=axa$. Hence we have

$$\begin{aligned}(h\circ f\circ g)^{-}(a) &= (\vee_{a=yz}\{h(y)\wedge (f\circ g)(z)\})\wedge 0.5\\ &\geq h(ax)\wedge (f\circ g)(a)\wedge 0.5\\ &\geq (h(a)\wedge 0.5)\wedge \left(\vee_{a=pq}\{f(p)\wedge g(q)\}\right)\wedge 0.5\\ &\geq h(a)\wedge (f(a)\wedge g(xa))\wedge 0.5\\ &\geq h(a)\wedge (f(a)\wedge g(a)\wedge 0.5)\wedge 0.5\\ &= h(a)\wedge f(a)\wedge g(a)\wedge 0.5\\ &= (h\wedge f\wedge g)^{-}.\end{aligned}$$

So (1) implies (2). (2) $\Rightarrow$ (3) $\Rightarrow$ (4) straight forward.

(4) $\Rightarrow$ (1) : Let h and g be any $(\in,\in\vee q)$-fuzzy right ideal and any $(\in,\in\vee q)$-fuzzy left ideal of S, respectively. Since $\mathcal{S}$ is an $(\in,\in\vee q)$-fuzzy quasi-ideal of S, by the assumption, we have

$$\begin{aligned}(h\wedge g)^{-}(a) &= (h\wedge g)(a)\wedge 0.5\\ &= (h\wedge \mathcal{S}\wedge g)(a)\wedge 0.5\\ &= (h\wedge \mathcal{S}\wedge g)^{-}(a)\\ &\leq (h\circ \mathcal{S}\circ g)^{-}(a)\\ &\leq (h\circ \mathcal{S}\circ g)(a)\wedge 0.5\\ &= (\vee_{a=bc}\{(h\circ \mathcal{S})(b)\wedge g(c)\})\wedge 0.5\\ &= (\vee_{a=bc}\{(\vee_{b=pq}\{h(p)\wedge \mathcal{S}(q)\})\wedge g(c)\})\wedge 0.5\\ &= (\vee_{a=bc}\{(\vee_{b=pq}\{h(p)\wedge 1\})\wedge g(c)\})\wedge 0.5\\ &= (\vee_{a=bc}\{(\vee_{b=pq}h(p))\wedge g(c)\})\wedge 0.5\\ &= (\vee_{a=bc}\{(\vee_{b=pq}h(p))\wedge g(c)\}\wedge 0.5)\wedge 0.5\\ &= (\vee_{a=bc}\{(\vee_{b=pq}\{h(p)\wedge 0.5\})\wedge g(c)\})\wedge 0.5\end{aligned}$$

$$
\begin{aligned}
&\le \left(\vee_{a=bc}\{\vee_{b=pq}\{h(pq)\} \wedge g(c)\}\right) \wedge 0.5\\
&= \left(\vee_{a=bc}\{h(b) \wedge g(c)\}\right) \wedge 0.5\\
&= (h \circ g)(a) \wedge 0.5\\
&= (h \circ g)^{-}(a).
\end{aligned}
$$

Thus it follows that $(h \wedge g)^{-} \le (h \circ g)^{-}$ for every $(\in, \in \vee q)$-fuzzy right ideal h and every $(\in, \in \vee q)$-fuzzy left ideal g of S. But $(h \circ g)^{-} \le (h \wedge g)^{-}$ always. So $(h \circ g)^{-} = (h \wedge g)^{-}$. Hence it is follows from Theorem 16 that S is regular. ■

Theorem 18. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*
(2) *$f^{-} = (f \circ \mathcal{S} \circ f)^{-}$ for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal f of S.*
(3) *$f^{-} = (f \circ \mathcal{S} \circ f)^{-}$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f of S.*
(4) *$f^{-} = (f \circ \mathcal{S} \circ f)^{-}$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal f of S.*

Proof. $(1) \Rightarrow (2)$: Let f be an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S and let a be any element of S. Since S is regular, so there exists an element $x \in S$ such that $a = axa$. Hence we have

$$
\begin{aligned}
(f \circ \mathcal{S} \circ f)^{-}(a) &= (f \circ \mathcal{S} \circ f)(a) \wedge 0.5\\
&= \left(\vee_{a=yz}\{(f \circ \mathcal{S})(y) \wedge f(z)\}\right) \wedge 0.5\\
&\ge (f \circ \mathcal{S})(ax) \wedge f(a) \wedge 0.5\\
&= \left(\vee_{ax=pq}\{(f(p) \wedge \mathcal{S}(q)\}) \wedge f(a)\right) \wedge 0.5\\
&\ge (f(a) \wedge \mathcal{S}(x)) \wedge f(a) \wedge 0.5\\
&= (f(a) \wedge 1) \wedge f(a) \wedge 0.5\\
&= f(a) \wedge 0.5\\
&= f^{-}(a).
\end{aligned}
$$

Thus $(f \circ \mathcal{S} \circ f)^{-} \ge f^{-}$.

Since f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. So we have

$$
\begin{aligned}
(f \circ \mathcal{S} \circ f)^{-}(a) &= (f \circ \mathcal{S} \circ f)(a) \wedge 0.5\\
&= \left(\vee_{a=yz}\{(f \circ \mathcal{S})(y) \wedge f(z)\}\right) \wedge 0.5\\
&= \left(\vee_{a=yz}\{(\vee_{y=pq}\{f(p) \wedge \mathcal{S}(q)\}) \wedge f(z)\}\right) \wedge 0.5\\
&= \left(\vee_{a=yz}\{(\vee_{y=pq}\{f(p) \wedge 1\}) \wedge f(z)\}\right) \wedge 0.5\\
&= \left(\vee_{a=yz}\{\vee_{y=pq}\{f(p)\} \wedge f(z)\}\right) \wedge 0.5\\
&= \vee_{a=yz}\{\vee_{y=pq} f(p) \wedge f(z) \wedge 0.5\}\\
&\le \vee_{a=(pq)z}\{f(pqz) \wedge 0.5\} \quad \text{(because } f \text{ is an } (\in, \in \vee q)\text{-fuzzy generalized bi-ideal of } S.)\\
&= f(a) \wedge 0.5\\
&= f^{-}(a).
\end{aligned}
$$

So, $(f \circ \mathcal{S} \circ f)^{-} \le f^{-}$. Thus $f^{-} = (f \circ \mathcal{S} \circ f)^{-}$. Now $(2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (1)$: Let A be any quasi-ideal of S. Then we have $ASA \subseteq A(SS) \cap (SS)A \subseteq AS \cap SA \subseteq A$. Let a be any element of A. Since by Lemma 9 C_A is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. So we have

$$
\begin{aligned}
\left(\vee_{a=yz}\{(C_A \circ \mathcal{S})(y) \wedge C_A(z)\}\right) \wedge 0.5 &= ((C_A \circ \mathcal{S}) \circ C_A)(a) \wedge 0.5\\
&= ((C_A \circ \mathcal{S}) \circ C_A)^{-}(a)\\
&= C_A^{-}(a)\\
&= 0.5
\end{aligned}
$$

implies that

$$\vee_{a=yz}\{(C_A \circ \mathcal{S})(y) \wedge C_A(z)\} \ge 0.5$$

since

$$\vee_{a=yz}\{(C_A \circ \mathcal{S})(y) \wedge C_A(z)\} \ne 0.5$$

So

$$\vee_{a=yz}\{(C_A \circ \mathcal{S})(y) \wedge C_A(z)\} > 0.5.$$

Hence

$$\vee_{a=yz}\{(C_A \circ \mathcal{S})(y) \wedge C_A(z)\} = 1.$$

This implies that there exist elements b and c of S such that $(C_A \circ \mathcal{S})(b) = 1$ and $C_A(c) = 1$ with $a = bc$. Thus we have

$$\vee_{b=pq}\{C_A(p) \wedge \mathcal{S}(q)\} = (C_A \circ \mathcal{S})(b) = 1.$$

This implies that there exist elements d and e of S such that

$$C_A(d) = 1 \quad \textit{and} \quad \mathcal{S}(e) = 1$$

with $b = de$. Thus $d, c \in A$ and $e \in S$ and so $a = bc = (de)c \in ASA$. Therefore, $A \subseteq ASA$, and so $A = ASA$. Hence it follows from Theorem 2 that S is regular. ∎

Theorem 19. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*
(2) *$(f \wedge g)^- = (f \circ g \circ f)^-$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal f and every $(\in, \in \vee q)$-fuzzy two-sided ideal g of S.*
(3) *$(f \wedge g)^- = (f \circ g \circ f)^-$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal f and every $(\in, \in \vee q)$-fuzzy interior ideal g of S.*
(4) *$(f \wedge g)^- = (f \circ g \circ f)^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f and every $(\in, \in \vee q)$-fuzzy two-sided ideal g of S.*
(5) *$(f \wedge g)^- = (f \circ g \circ f)^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f and every $(\in, \in \vee q)$-fuzzy interior ideal g of S.*
(6) *$(f \wedge g)^- = (f \circ g \circ f)^-$ for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal f and every $(\in, \in \vee q)$-fuzzy two-sided ideal g of S.*
(7) *$(f \wedge g)^- = (f \circ g \circ f)^-$ for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal f and every $(\in, \in \vee q)$-fuzzy interior ideal g of S.*

Proof. $(1) \Rightarrow (7)$: Let f and g be any $(\in, \in \vee q)$-fuzzy generalized bi-ideal and any $(\in, \in \vee q)$-fuzzy interior ideal of S, respectively. Then

$$\begin{aligned}
(f \circ g \circ f)^-(a) &= (f \circ g \circ f)(a) \wedge 0.5 \\
&\leq (f \circ \mathcal{S} \circ f)(a) \wedge 0.5 \\
&= \vee_{a=yz}\{(f \circ \mathcal{S})(y) \wedge f(z)\} \wedge 0.5 \\
&= \vee_{a=yz}\{\vee_{y=pq}\{f(p) \wedge \mathcal{S}(q)\} \wedge f(z)\} \wedge 0.5 \\
&= \vee_{a=yz}\{\vee_{y=pq}\{f(p) \wedge 1\} \wedge f(z)\} \wedge 0.5 \\
&= \vee_{a=yz}\{\vee_{y=pq} f(p) \wedge f(z)\} \wedge 0.5 \\
&= \vee_{a=yz}\{\vee_{y=pq} f(p) \wedge f(z) \wedge 0.5\} \\
&= \vee_{a=(pq)z}\{f(p) \wedge f(z) \wedge 0.5 \wedge 0.5\} \\
&\leq \vee_{a=(pq)z} f(pqz) \wedge 0.5 \\
&= f(a) \wedge 0.5 \\
&= f^-(a).
\end{aligned}$$

and

$$\begin{aligned}
(f \circ g \circ f)^-(a) &\leq (\mathcal{S} \circ g \circ \mathcal{S})^-(a) \\
&= (\mathcal{S} \circ g \circ \mathcal{S})(a) \wedge 0.5 \\
&= (\vee_{a=yz}\{(\mathcal{S} \circ g)(y) \wedge \mathcal{S}(z)\}) \wedge 0.5 \\
&= (\vee_{a=yz}\{(\vee_{y=pq}\{\mathcal{S}(p) \wedge g(q)\}) \wedge \mathcal{S}(z)\}) \wedge 0.5 \\
&= (\vee_{a=yz}\{(\vee_{y=pq}\{1 \wedge g(q)\}) \wedge 1\}) \wedge 0.5 \\
&= (\vee_{a=yz}\{\vee_{y=pq} g(q)\}) \wedge 0.5 \\
&= \vee_{a=yz}\{\vee_{y=pq} g(q) \wedge 0.5\} \\
&\leq \vee_{a=(pq)z}\{g(pqz) \wedge 0.5\} \\
&= g(a) \wedge 0.5 \\
&= g^-(a).
\end{aligned}$$

Thus $(f \circ g \circ f)^- \leq \left(f^- \wedge g^-\right) = (f \wedge g)^-$. Now let a be any element of S. Then, since S is regular, there exists an element $x \in S$ such that $a = axa \,(= axaxa)$. Since g is an $(\in, \in \vee q)$-fuzzy interior ideal of S, we have

$$\begin{aligned}
(f \circ g \circ f)^-(a) &= (f \circ g \circ f)(a) \wedge 0.5 \\
&= \vee_{a=yz}\{f(y) \wedge (g \circ f)(z)\} \wedge 0.5 \\
&\geq f(a) \wedge (g \circ f)(xaxa) \wedge 0.5 \\
&= f(a) \wedge \left(\vee_{xaxa=pq}\{g(p) \wedge f(q)\}\right) \wedge 0.5 \\
&\geq f(a) \wedge (g(xax) \wedge f(a)) \wedge 0.5 \\
&= f(a) \wedge (g(xax) \wedge f(a)) \wedge 0.5 \\
&\geq f(a) \wedge (g(a) \wedge 0.5 \wedge f(a)) \wedge 0.5
\end{aligned}$$

$$= f(a) \wedge g(a) \wedge 0.5$$
$$= (f \wedge g)(a) \wedge 0.5$$
$$= (f \wedge g)^{-}(a).$$

So $(f \circ g \circ f)^{-} \leq (f \wedge g)^{-}$. Hence $(f \circ g \circ f)^{-} = (f \wedge g)^{-}$. (7) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (1) : Let f be any $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Then, since $\mathcal{S}$ itself is an $(\in, \in \vee q)$-fuzzy two-sided of S, we have

$$\begin{aligned} f^{-}(a) &= f(a) \wedge 0.5 \\ &= (f \wedge \mathcal{S})(a) \wedge 0.5 \\ &= (f \wedge \mathcal{S})^{-}(a) \\ &= (f \circ \mathcal{S} \circ f)^{-}(a). \end{aligned}$$

Thus it follows from Theorem 18 that S is regular. ■

Theorem 20. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*
(2) *$(f \wedge g)^{-} \leq (f \circ g)^{-}$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*
(3) *$(f \wedge g)^{-} \leq (f \circ g)^{-}$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*
(4) *$(f \wedge g)^{-} \leq (f \circ g)^{-}$ for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*

Proof. (1) $\Rightarrow$ (4) : Let f and g be any $(\in, \in \vee q)$-fuzzy generalized bi-ideal and any $(\in, \in \vee q)$-fuzzy left ideal of S respectively. Let a be any element of S. Then there exists an element $x \in S$ such that $a = axa$. Thus we have

$$\begin{aligned} (f \circ g)^{-}(a) &= (f \circ g)(a) \wedge 0.5 \\ &= (\vee_{a=yz}\{f(y) \wedge g(z)\}) \wedge 0.5 \\ &\geq f(a) \wedge g(xa) \wedge 0.5 \\ &\geq f(a) \wedge g(a) \wedge 0.5 \wedge 0.5 \\ &= (f \wedge g)(a) \wedge 0.5 \\ &= (f \wedge g)^{-}(a). \end{aligned}$$

So $(f \circ g)^{-} \geq (f \wedge g)^{-}$. (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) : Let f be an $(\in, \in \vee q)$-fuzzy right ideal and g be an $(\in, \in \vee q)$-fuzzy left ideal of S. Since every $(\in, \in \vee q)$-fuzzy right ideal of S is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. So $(f \circ g)^{-} \geq (f \wedge g)^{-}$.

Now

$$\begin{aligned} (f \circ g)^{-}(a) &= (f \circ g)(a) \wedge 0.5 \\ &= (\vee_{a=yz}\{f(y) \wedge g(z)\}) \wedge 0.5 \\ &= \vee_{a=yz}(\{f(y) \wedge g(z)\} \wedge 0.5) \\ &= \vee_{a=yz}(\{f(y) \wedge 0.5\} \wedge \{g(z) \wedge 0.5\} \wedge 0.5) \\ &\leq \vee_{a=yz}(\{f(yz) \wedge g(yz)\} \wedge 0.5) \\ &= f(a) \wedge g(a) \wedge 0.5 \\ &= (f \wedge g)(a) \wedge 0.5 \\ &= (f \wedge g)^{-}(a). \end{aligned}$$

So $(f \circ g)^{-} \leq (f \wedge g)^{-}$. Hence $(f \circ g)^{-} = (f \wedge g)^{-}$ for every $(\in, \in \vee q)$-fuzzy right ideal f of S, and every $(\in, \in \vee q)$-fuzzy left ideal g of S. Thus by Theorem 16 that S is regular. ■

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