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An anti-periodic boundary value problem for the fractional differential equation with a p-Laplacian operator☆

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ABSTRACT

This paper studies the existence of solutions for an anti-periodic boundary value problem for the fractional p-Laplacian equation. Under certain nonlinear growth conditions of the nonlinearity, a new existence result is obtained by using Schaefer's fixed point theorem. As an application, an example to illustrate our result is given.

1. Introduction

Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, etc. [1–4]. Recently, fractional differential equations have been of great interest due to the intensive development of the theory of fractional calculus itself and its applications. For some recent works on fractional differential equations, see [5–10] and the references therein.

Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes [11,12] and have recently received considerable attention. For examples and details of anti-periodic boundary value problems, see [13–17] and the references therein.

Turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problem, Leibenson [18] introduced the p-Laplacian equation as follows:

$$(\phi_p(x'(t)))' = f(t, x(t), x'(t)), \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$. In the past few decades, many important results relative to Eq. (1.1) with certain boundary value conditions have been obtained. We refer the reader to [19–23] and the references cited therein. However, to the best of our knowledge, there are relatively few results on anti-periodic boundary value problems for fractional p-Laplacian equations.

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Motivated by the works mentioned previously, in this paper we investigate the existence of solutions for the anti-periodic boundary value problem (ABVP for short) of a fractional p-Laplacian equation with the following form:

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = f(t, x(t)), & t \in [0, 1], \\ x(0) = -x(1), \quad D_{0+}^{\alpha}x(0) = -D_{0+}^{\alpha}x(1), \end{cases} \tag{1.2}$$

where $0 < \alpha,\ \beta \leq 1,\ 1 < \alpha + \beta \leq 2$, D_{0+}^{α} is a Caputo fractional derivative, and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Note that, when $p = 2$, the nonlinear operator $D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha})$ reduces to the linear operator $D_{0+}^{\beta}D_{0+}^{\alpha}$.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, based on Schaefer's fixed point theorem, we establish a theorem on the existence of solutions for ABVP (1.2) under nonlinear growth restriction of f. Finally, in Section 4, an explicit example is given to illustrate the main result.

2. Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory, which can be found, for instance, in [2,4]. By $C[0, 1]$ we denote the Banach space of all continuous functions from $[0, 1]$ into $\mathbb{R}$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$.

Definition 2.1. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha}u(t) = I_{0+}^{n-\alpha}\frac{d^n u(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.1 ([1]). *Let $\alpha > 0$. Assume that $u, D_{0+}^{\alpha}u \in L(0, 1)$. Then the following equality holds:*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0, 1, \ldots, n-1$; here n is the smallest integer greater than or equal to α.

3. Existence result

As a consequence of Lemma 2.1, we have the following result that is useful in what follows.

Lemma 3.1. *Given $h \in C[0, 1]$, the unique solution of*

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = h(t), & t \in [0, 1], \\ x(0) = -x(1), \quad D_{0+}^{\alpha}x(0) = -D_{0+}^{\alpha}x(1) \end{cases} \tag{3.1}$$

is

$$\begin{aligned} x(t) &= I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}h(t) + Ah(t)) + Bh(t) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)d\tau + Ah(s)\right)ds + Bh(t), \end{aligned}$$

where

$$Ah(t) = -\frac{1}{2}I_{0+}^{\beta}h(t)\Big|_{t=1} = -\frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}h(s)ds, \quad \forall t \in [0, 1],$$

$$\begin{aligned} Bh(t) &= -\frac{1}{2}I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}h(t) + Ah(t))\Big|_{t=1} \\ &= -\frac{1}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)d\tau + Ah(s)\right)ds, \quad \forall t \in [0, 1]. \end{aligned}$$

Proof. Assume that $x(t)$ satisfies the equation of ABVP (3.1), then Lemma 2.1 implies that

$$\phi_p(D_{0+}^{\alpha}x(t)) = I_{0+}^{\beta}h(t) + c_0, \quad c_0 \in \mathbb{R}.$$

From the boundary value condition $D^{\alpha}_{0+}x(0) = -D^{\alpha}_{0+}x(1)$, one has

$$c_0 = -\frac{1}{2} I^{\beta}_{0+} h(t)|_{t=1} = Ah(t).$$

Thus, we have

$$x(t) = I^{\alpha}_{0+}\phi_q(I^{\beta}_{0+}h(t) + Ah(t)) + c_1, \quad c_1 \in \mathbb{R}.$$

By condition $x(0) = -x(1)$, we get

$$c_1 = -\frac{1}{2} I^{\alpha}_{0+}\phi_q(I^{\beta}_{0+}h(t) + Ah(t))|_{t=1} = Bh(t).$$

The proof is complete. □

Define the operator $F : C[0, 1] \to C[0, 1]$ by

$$\begin{aligned}
Fx(t) &= I^{\alpha}_{0+}\phi_q(I^{\beta}_{0+}Nx(t) + ANx(t)) + BNx(t)\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau))d\tau - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1} f(\tau, x(\tau))d\tau\right)ds\\
&\quad - \frac{1}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau))d\tau\right.\\
&\quad \left. - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1} f(\tau, x(\tau))d\tau\right)ds, \quad \forall t \in [0, 1],
\end{aligned}$$

where $N : C[0, 1] \to C[0, 1]$ is the Nemytskii operator defined by

$$Nx(t) = f(t, x(t)), \quad \forall t \in [0, 1].$$

Clearly, the fixed points of the operator F are solutions of ABVP (1.2).

Our main result, based on Schaefer's fixed point theorem, is stated as follows.

Theorem 3.1. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that*

(H) *there exist nonnegative functions $a, b \in C[0, 1]$ such that*

$$|f(t, u)| \le a(t) + b(t)|u|^{p-1}, \quad \forall t \in [0, 1],\ u \in \mathbb{R}.$$

Then ABVP (1.2) has at least one solution, provided that

$$\frac{3^q \|b\|_{\infty}^{q-1}}{2^q \Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} < 1. \tag{3.2}$$

Proof. The proof will be given in the following two steps.

Step 1: $F : C[0, 1] \to C[0, 1]$ is completely continuous.

Let $\Omega \subset C[0, 1]$ be an open bounded subset. By the continuity of f, we can get that F is continuous and $F(\overline{\Omega})$ is bounded. Moreover, there exists a constant $T > 0$ such that $|I^{\beta}_{0+}Nx + ANx| \le T,\ \forall x \in \overline{\Omega}, t \in [0, 1]$. Thus, in view of the Arzelá–Ascoli theorem, we need only prove that $F(\overline{\Omega}) \subset C[0, 1]$ is equicontinuous.

For $0 \le t_1 < t_2 \le 1$, $x \in \overline{\Omega}$, we have

$$\begin{aligned}
&|Fx(t_2) - Fx(t_1)|\\
&= |I^{\alpha}_{0+}\phi_q(I^{\beta}_{0+}Nx(t) + ANx(t))|_{t=t_2} - I^{\alpha}_{0+}\phi_q(I^{\beta}_{0+}Nx(t) + ANx(t))|_{t=t_1}|\\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2 - s)^{\alpha-1}\phi_q(I^{\beta}_{0+}Nx(s) + ANx(s))ds - \int_0^{t_1}(t_1 - s)^{\alpha-1}\phi_q(I^{\beta}_{0+}Nx(s) + ANx(s))ds\right|\\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]\phi_q(I^{\beta}_{0+}Nx(s) + ANx(s))ds\right.\\
&\quad \left. + \int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}\phi_q(I^{\beta}_{0+}Nx(s) + ANx(s))ds\right|\\
&\le \frac{T^{q-1}}{\Gamma(\alpha)}\left\{\int_0^{t_1}[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}]ds + \int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}ds\right\}\\
&= \frac{T^{q-1}}{\Gamma(\alpha+1)}[t_1^{\alpha} - t_2^{\alpha} + 2(t_2 - t_1)^{\alpha}].
\end{aligned}$$

Since t^{α} is uniformly continuous on $[0, 1]$, we can obtain that $F(\overline{\Omega}) \subset C[0, 1]$ is equicontinuous.

Step 2: *A priori bounds.*
Set

$$\Omega = \{x \in C[0,1] | x = \lambda^{q-1} Fx,\ \lambda \in (0,1)\}.$$

Now it remains to show that the set Ω is bounded.
By (H), we have

$$\begin{aligned} |ANx(t)| &\le \frac{1}{2\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f(s,x(s))| ds \\ &\le \frac{1}{2\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (a(s) + b(s)|x(s)|^{p-1}) ds \\ &\le \frac{1}{2\Gamma(\beta)} (\|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1}) \cdot \frac{1}{\beta} \\ &= \frac{1}{2\Gamma(\beta+1)} (\|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1}), \quad \forall t \in [0,1], \end{aligned}$$

which, together with the monotonicity of s^{q-1}, yields that

$$\begin{aligned} |BNx(t)| &\le \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |I_{0+}^{\beta} Nx(s) + ANx(s)|^{q-1} ds \\ &\le \frac{1}{2\Gamma(\alpha)} \cdot \frac{3^{q-1}}{2^{q-1}(\Gamma(\beta+1))^{q-1}} (\|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1})^{q-1} \cdot \frac{1}{\alpha} \\ &= \frac{3^{q-1}}{2^q \Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} (\|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1})^{q-1}, \quad \forall t \in [0,1]. \end{aligned} \tag{3.3}$$

For $x \in \Omega$, we get $x(t) = \lambda^{q-1} Fx(t)$. Thus, from (3.3), we obtain that

$$\begin{aligned} |x(t)| &\le |I_{0+}^{\alpha} \phi_q (I_{0+}^{\beta} Nx(t) + ANx(t))| + |BNx(t)| \\ &\le \frac{3^q}{2^q \Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} (\|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1})^{q-1}, \quad \forall t \in [0,1]. \end{aligned} \tag{3.4}$$

In view of (3.2), from (3.4), we can see that there exists a constant $M > 0$ such that

$$\|x\|_\infty \le M.$$

As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is the solution of ABVP (1.2). The proof is complete. □

4. An example

In this section, we will give an example to illustrate our main result.

Example 4.1. Consider the following ABVP for the fractional p-Laplacian equation:

$$\begin{cases} D_{0+}^{\frac{1}{2}} \phi_3 \left(D_{0+}^{\frac{3}{4}} x(t) \right) = \frac{1}{10} x^2(t) + \cos t, \quad t \in [0,1], \\ x(0) = -x(1), \quad D_{0+}^{\frac{3}{4}} x(0) = -D_{0+}^{\frac{3}{4}} x(1). \end{cases} \tag{4.1}$$

Corresponding to ABVP (1.2), we get that $p = 3$, $q = \frac{3}{2}$, $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$ and

$$f(t,u) = \frac{1}{10} u^2 + \cos t.$$

Choose $a(t) = 1$, $b(t) = \frac{1}{10}$. By a simple calculation, we can obtain that $\|b\|_\infty = \frac{1}{10}$ and

$$\frac{3^{\frac{3}{2}} \left(\frac{1}{10}\right)^{\frac{1}{2}}}{2^{\frac{3}{2}} \Gamma\left(\frac{3}{4}+1\right) \left(\Gamma\left(\frac{1}{2}+1\right)\right)^{\frac{1}{2}}} < 1.$$

Obviously, ABVP (4.1) satisfies all assumptions of Theorem 3.1. Hence, it has at least one solution.

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