The radius of \( k \)-connected planar graphs with bounded faces
Patrick Ali\(^1\), Peter Dankelmann\(^*\), Simon Mukwembi
School of Mathematical Sciences, University of KwaZulu-Natal, Durban, 4041, South Africa

**Abstract**

We prove that if \( G \) is a 3-connected plane graph of order \( p \), maximum face length \( l \) and radius \( \text{rad}(G) \), then the bound
\[
\text{rad}(G) \leq \frac{p}{6} + \frac{5l}{6} + \frac{2}{3}
\]
holds. For constant \( l \), our bound is shown to be asymptotically sharp and improves on a bound by Harant \((1990)\) \([6]\). Furthermore we extend these results to 4- and 5-connected planar graphs.

© 2012 Published by Elsevier B.V.

1. Introduction

Let \( G \) be a connected graph of order \( p \). The distance between two vertices \( u, v \) of \( G \), \( d_G(u, v) \) is the length of the shortest \( u-v \) path in \( G \). The diameter of \( G \), \( \text{diam}(G) \) is the greatest distance among all pairs of vertices. The eccentricity, \( \text{ex}(v) \), of a vertex \( v \in V(G) \) is the maximum distance between \( v \) and any other vertex in \( G \). The minimum eccentricity of \( G \) is the radius of \( G \), denoted by \( \text{rad}(G) \). The radius of a graph is an important measure of centrality. For example, in facility location problems the graph may model a community where the edges represent roads between locations (vertices). If one wishes to locate an emergency facility such as a police station, a hospital, or a fire station then the primary concern may be to choose a location such that the response/travel time from the emergency facility to a location farthest away is as small as possible. The radius is thus a good measure that indicates the response/travel time from an emergency facility to a location farthest away, if the best location for the emergency facility is chosen.

The degree, \( \text{deg}(v) \), of a vertex \( v \) of \( G \) is the number of edges incident with it. The minimum degree, \( \delta(G) \), of \( G \) is the smallest of the degrees of vertices in \( G \). The vertex-connectivity, \( \kappa(G) \), of \( G \) is defined as the minimum number of vertices whose deletion from \( G \) results in a disconnected or trivial graph. A graph \( G \) is \( k \)-connected if \( \kappa(G) \geq k \). A graph \( G \) is planar if it can be embedded into the plane with no crossing edges. A plane graph is a planar graph together with an embedding into the plane. A plane graph divides the plane into faces. The union of the vertices and edges of \( G \) incident with a face \( f \) of \( G \) is called the boundary of \( f \). Two vertices \( u \) and \( v \) share a face if they are on the boundary of a common face. The length of a face in a plane graph \( G \) is the length of the shortest walk in \( G \) that bounds it.

Several upper bounds on the radius in terms of other graph parameters are known. Erdős et al. \([5]\) proved that if \( G \) is a connected graph of order \( p \) and minimum degree \( \delta(G) \geq 2 \), then
\[
\text{rad}(G) \leq \frac{3(p - 3)}{2(\delta(G) + 1)} + 5.
\]
and also constructed graphs that show that the bound is best possible, apart from the value of the additive constant. In addition, they gave improved bounds for triangle-free and $C_4$-free graphs. However, using different methods, Dankelmann et al. [1] and Dlamini [3], obtained the slightly stronger bound

$$\text{rad}(G) \leq \frac{3p}{2(\delta(G) + 1)} + 1.$$  

In [8], Harant and Walther gave bounds on the radius in terms of order and vertex-connectivity. For even $\kappa(G)$, the well-known bound $\text{diam}(G) \leq (p + \kappa(G) - 2)/\kappa(G)$ on the diameter is also sharp for the radius. For odd $\kappa(G)$, Harant and Walther [8] proved that

$$\text{rad}(G) \leq \frac{p}{\kappa(G) + 1} + O(\log p),$$

and conjectured that $\text{rad}(G) \leq \frac{p}{\kappa(G) + 1} + C$ for some constant $C$. Harant [7] showed that for $\kappa(G) = 3$, the $O(\log p)$ term can be replaced by 8. Using different methods, Mukwembi [10] proved that for odd $\kappa(G) \geq 3$, the $O(\log p)$ term can be replaced by $1 + \frac{16}{\kappa(G) + 1}$. It has, however, been shown by Egawa and Inoue [4] that for odd $\kappa(G) \geq 3$, the $O(\log p)$ term can be replaced by $1 + \frac{9}{2\kappa(G)}$. On the other hand, Iida and Kobayashi [9] obtained a slightly better bound by showing that if $\kappa(G) \geq 3$, $\kappa(G)$ odd, then the $O(\log p)$ term can be replaced by $1 + \frac{1}{\kappa(G)}$. Vizing [11] determined the maximum size of a graph of given order and radius, which yields a bound on the radius in terms of order and size. A similar result for bipartite graphs is due to Dankelmann et al. [2].

For 3-connected planar graphs, Harant [6] proved an upper bound on the radius in terms of order and maximum face length. It was shown that

$$\text{rad}(G) \leq \frac{p}{6} + l + \frac{3}{2},$$

where $l$ is the maximum face length. No graphs which attain the bound were constructed. In this paper we strengthen this bound to

$$\text{rad}(G) \leq \frac{p}{6} + \frac{5l}{6} + \frac{5}{6}.$$  

We also prove that for 4-connected planar graphs of order $p$, maximum face length $l$ and radius $\text{rad}(G)$ the bound

$$\text{rad}(G) \leq \frac{p}{8} + \frac{5l}{4} + 1$$

holds and for 5-connected planar graphs of order $p$, maximum face length $l$ and radius $\text{rad}(G)$ the bound

$$\text{rad}(G) \leq \frac{p}{10} + \frac{8l}{5} + 1$$

holds. We furthermore show that for large $p$ and constant $l$ our bounds are sharp, apart from an additive constant.

2. Results

Let $G$ be a connected plane graph of order $p$. From now on let $z$ be a fixed, not necessarily central, vertex of $G$ and let $\text{ex}(z) = r$. For each $i = 0, 1, \ldots, r$ let

$$N_i := \{x \in V(G) | d_G(x, z) = i\}.$$  

A vertex $x \in N_i$ is active if $i \leq r - 1$ and $x$ has a neighbour in $N_{i+1}$. We denote by $A_i$ the set of active vertices in $N_i$. For $i \in \mathbb{N}$ and $1 \leq i \leq r - 1$ we define $\hat{H}_i$ to be the graph with vertex set $A_i$, where two vertices are adjacent in $\hat{H}_i$ if and only if they share a face in $G$.

**Lemma 2.1.** Let $G$, $z$, $A_i$ and $\hat{H}_i$ be as above and let $1 \leq i \leq r - 1$.

(a) If $G$ is 3-connected and $u$ a vertex of $\hat{H}_i$, then $u$ has two distinct neighbours $v, w \in A_i - \{u\}$ in $\hat{H}_i$.

(b) If $G$ is 4-connected and $u, v, w$ are three distinct vertices of $\hat{H}_i$, then at least one of $u, v, w$ has a neighbour in $A_i - \{u, v, w\}$ in $\hat{H}_i$.

**Proof.** (a) Since $u$ is a vertex of $A_i$, it has neighbours in $N_{i+1}$ and in $N_{i+1}$. Number the neighbours of $u$ as $x_0, x_1, \ldots, x_t$ such that the edges $ux_i$ appear in clockwise order, $x_0$ is in $N_{i+1}$ and, say, $x_t$ is in $N_{i+1}$. Denote the face containing $u, x_i$ and $x_{i+1}$ by $f_j$ for $j = 0, 1, \ldots, t$ where subscripts are taken modulo $t + 1$. Let $P_j$ be the $x_i - x_{i+1}$ path of the vertices on the boundary of $f_j$ except $u$ in clockwise order.

We show that there exists an $j$ such that $f_j$ contains a vertex $v \in A_i - \{u\}$. Consider the walk, $W := x_0 P_0 x_1 P_1 x_2 \cdots x_{k-1} P_{k-1} x_t$ i.e., the $x_0 - x_t$ walk that traverses the vertices of $P_0$ then $P_1, P_2, \ldots, P_{k-1}$.
Let $b$ be the first vertex of $W$ in $N_{i+1}$ and let $v$ be the predecessor of $b$ in $W$. Then $v$ is in $N_i$. Since $v$ has a neighbour in $N_{i+1}$, we have $v \in A_i$. Furthermore, $v$ is on the boundary of $f_{j_1}$ for some $j_1 \in \{0, 1, \ldots, k - 1\}$. Similarly we can show that there exists a $j_2 \in \{k, k+1, \ldots, t\}$ such that the boundary of $f_{j_2}$ contains a vertex $w \in A_i \setminus \{u\}$. It remains to show that $v \neq w$. Suppose $v = w$. Join $u$ and $v$ by an edge that goes through face $f_{j_1}$, and another edge through face $f_{j_2}$, thus creating a plane multigraph. The new edges form a 2-cycle, $C_2$. Since the last $x_i$ that precedes $v$ on $W$ and the first $x_i$ that succeeds $v$ on $W$ are on different sides of $C_2$, the inside and the outside of $C_2$ both contain vertices. Any path between vertices inside $C_2$ and those vertices outside has to pass through $u$ or $v$, and hence $u$ and $v$ form a cutset, a contradiction to the 3-connectedness of $G$.

(b) Suppose that none of $u$, $v$, $w$ shares a face with a vertex in $A_i \setminus \{u, v, w\}$. By the proof and notation of Lemma 2.1(a), we have $v$ on the boundary of $f_{j_1}$ for some $j_1 \in \{0, 1, \ldots, k - 1\}$ and $w$ on the boundary of $f_{j_2}$ for some $j_2 \in \{k, k+1, \ldots, t\}$. Also $v \neq w$. By Lemma 2.1(a) and the assumption that none of $u$, $v$, $w$ shares a face with a vertex in $A_i \setminus \{u, v, w\}$, we conclude that $u$ and $v$ share a face, $u$ and $w$ share a face, and $v$ and $w$ share a face. So we can add new edges between $u$ and $v$ through face $f_{j_1}$, between $u$ and $w$ through face $f_{j_2}$, and between $v$ and $w$ through $f_{j_1}$, thus creating a plane multigraph. The three new edges form a 3-cycle, $C_3$. Since the last $x_i$ that precedes $v$ on $W$ and the first $x_i$ that succeeds $v$ on $W$ are on different sides of $C_3$, the inside and the outside of $C_3$ both contain vertices. Thus any path between vertices inside $C_3$ and those vertices outside has to pass through $u$ or $v$ or $w$, and hence $u$, $v$, $w$ form a cutset, a contradiction to the 4-connectedness of $G$.

Lemma 2.2. Let $G, z, A_i$ and $\hat{H}_i$ be as above and let $1 \leq i \leq r - 1$. Let $u$ be a vertex of $\hat{H}_i$. If $G$ is 5-connected, then $u$ has two neighbours $v$ and $w$ in $\hat{H}_i$ that have no common neighbour in $\hat{H}_i$ other than $u$.

Proof. By the proof and notation of Lemma 2.1, we have $v$ on the boundary of $f_{j_1}$ for some $j_1 \in \{0, 1, \ldots, k - 1\}$ and $w$ on the boundary of $f_{j_2}$ for some $j_2 \in \{k, k+1, \ldots, t\}$. Also $v \neq w$. Suppose that $v$ and $w$ share a neighbour $a \neq u$ in $\hat{H}_i$, so $v$ and $w$ share a face $f'$ and $a$ share a face $f''$. As above we can add edges to $G$: between $u$ and $v$ through face $f_{j_1}$, between $u$ and $w$ through face $f_{j_2}$, between $v$ and $w$ through face $f'$, and between $u$ and $a$ through face $f''$, thus creating a plane multigraph. Now the four edges $uv, uw, va$ and $wa$ form a 4-cycle, $C_4$. Since the last $x_i$ that precedes $v$ on $W$ and the first $x_i$ that succeeds $v$ on $W$ are on different sides of $C_4$, the inside and the outside of $C_4$ both contain vertices. Thus any path between vertices inside $C_4$ and those vertices outside has to pass through $u$, $v$, $w$ or $a$, and hence $u$, $v$, $w$ and $a$ form a cutset, a contradiction to the 5-connectedness of $G$.

Corollary 2.3. Let $\hat{H}_i$ be as above. If $G$ is 3-connected, then $\delta(\hat{H}_i) \geq 2$. Moreover,

(a) each component of $\hat{H}_i$ has at least three vertices
(b) if $G$ is 4-connected, then each component of $\hat{H}_i$ has at least four vertices
(c) if $G$ is 5-connected, then each component of $\hat{H}_i$ has at least five vertices.

Lemma 2.4. Let $G$ be 3-connected and $z$ as above. Let $i \in \{1, 2, \ldots, r - 1\}$.

(a) If $|A_i| = 3$ then there exists a vertex $z_i \in A_i$ with $d_G(z_i, v) \leq \lfloor \frac{k}{2} \rfloor$ for all $v \in A_i$.
(b) If $4 \leq |A_i| \leq 5$ then there exists a vertex $z_i \in A_i$ with $d_G(z_i, v) \leq l$ for all $v \in A_i$.

Proof. (a) Since $\hat{H}_i$ has minimum degree two and exactly three vertices, $\hat{H}_i$ is connected. Fix a vertex $z_i$ of $\hat{H}_i$ and let $v \in A_i$ be arbitrary. Since any two vertices that are adjacent in $\hat{H}_i$ are joined by a path of length at most $\lfloor \frac{k}{2} \rfloor$ in $G$, the $z_i-v$ path in $\hat{H}_i$ yields a $z_i-v$ path in $G$ of length at most $\lfloor \frac{k}{2} \rfloor$. Hence $d_G(z_i, v) \leq \lfloor \frac{k}{2} \rfloor$, as desired.
(b) Since $\hat{H}_i$ has minimum degree at least two and at most five vertices, $\hat{H}_i$ is connected and has a vertex $z_i$ of eccentricity at most two. As in (a), this implies that $d_G(z_i, v) \leq l$ for all $v \in A_i$.

Lemma 2.5. Let $G$ be 4-connected and $z$ as above. Let $i \in \{1, 2, \ldots, r - 1\}$. If $6 \leq |A_i| \leq 7$ then there exists a vertex $z_i \in A_i$ with $d_G(z_i, v) \leq \lfloor \frac{3l}{2} \rfloor$ for all $v \in A_i$.

Proof. Since $\hat{H}_i$ has at most seven vertices, it follows by Corollary 2.3 that $\hat{H}_i$ is connected. By Lemma 2.1(a), $\hat{H}_i$ has minimum degree at least two. Hence $\hat{H}_i$ has a vertex $z_i$ of eccentricity at most three. As in Lemma 2.4, this implies that $d_G(z_i, v) \leq \lfloor \frac{3l}{2} \rfloor$ for all $v \in A_i$.

Lemma 2.6. Let $G$ be 5-connected and $z$ as above. Let $i \in \{1, 2, \ldots, r - 1\}$. If $8 \leq |A_i| \leq 9$ then there exists a vertex $z_i \in A_i$ with $d_G(z_i, v) \leq 2l$ for all $v \in A_i$.

Proof. Since $\hat{H}_i$ has at most nine vertices, it follows by Corollary 2.3(c) that $\hat{H}_i$ is connected. Also $\hat{H}_i$ has minimum degree at least two by Lemma 2.1(a). So $\hat{H}_i$ has a vertex $z_i$ of eccentricity at most four. As in Lemma 2.4, this implies that $d_G(z_i, v) \leq 2l$ for all $v \in A_i$.
From now on let \( z \) be a central vertex of \( G \), i.e., a vertex of eccentricity \( r = \text{rad}(G) \). We employ the notation

\[
N_{2i} = \bigcup_{0 \leq j \leq i} N_j \quad \text{and} \quad N_{2i+1} = \bigcup_{i \leq j} N_j.
\]

Form a spanning tree \( T \) of \( G \) that is distance preserving from \( z \). For a vertex \( y \in V(G) \), denote by \( T(z, y) \), the set of vertices on the path connecting \( z \) and \( y \) in \( T \).

**Theorem 2.7.** Let \( G \) be a 3-connected plane graph of order \( p \), maximum face length \( l \) and radius \( r \). Then

\[
r \leq \frac{p}{6} + \frac{5}{6}l + \frac{2}{3}.
\]  

(1)

**Proof.** We first bound the cardinalities of the \( N_i \) from below. The following claim immediately follows from the 3-connectedness of \( G \):

**Claim 1.** Let \( i \in \{1, 2, \ldots, r - 1\} \). Then \( |N_i| \geq 3 \).

This bound can be improved if \( i \) is not too close to 0 or \( r \).

**Claim 2.** Let \( i \in \{\left\lfloor \frac{r}{2} \right\rfloor + 1, \left\lfloor \frac{r}{2} \right\rfloor + 2, \ldots, r - \left\lfloor \frac{r}{2} \right\rfloor - 1\} \). Then \( |N_i| \geq 4 \).

**Proof of Claim 2.** By way of contradiction suppose \( |N_i| = 3 \) for some \( i \in \{\left\lfloor \frac{r}{2} \right\rfloor + 1, \left\lfloor \frac{r}{2} \right\rfloor + 2, \ldots, r - \left\lfloor \frac{r}{2} \right\rfloor - 1\} \). Let \( z_i \in A_i \) be as in Lemma 2.4. Let \( x \) denote the unique vertex of \( T(z, z_i) \) which belongs to \( N_{\left\lfloor \frac{r}{2} \right\rfloor + 1} \). We show that \( \text{ex}(x) \leq r - 1 \). First let \( y \in N_{2i-1} \).

Then

\[
d(x, y) \leq d(x, z) + d(z, y)
\]

\[
\leq \left\lfloor \frac{l}{2} \right\rfloor + 1 + i - 1
\]

\[
\leq \left\lfloor \frac{l}{2} \right\rfloor + 1 + r - \left\lfloor \frac{l}{2} \right\rfloor - 1 - 1
\]

\[
= r - 1.
\]

Now let \( y \in N_{2i} \). Let \( y_i \in T(z, y) \cap N_i \) so that \( d(x, z_i) = i - \left\lfloor \frac{r}{2} \right\rfloor - 1 \). By Lemma 2.4 we have \( d(z_i, y_i) \leq \left\lfloor \frac{l}{2} \right\rfloor \). Also \( d(y_i, y) \leq r - i \).

It follows that

\[
d(x, y) \leq d(x, z_i) + d(z_i, y_i) + d(y_i, y)
\]

\[
\leq i - \left\lfloor \frac{l}{2} \right\rfloor - 1 + \left\lfloor \frac{l}{2} \right\rfloor + r - i
\]

\[
= r - 1.
\]

Therefore, \( \text{ex}(x) \leq r - 1 \), contradicting the fact that \( r \) is the radius of \( G \). \( \Box \)

**Claim 3.** Let \( i \in \{l + 1, l + 2, \ldots, r - l - 1\} \). Then \( |N_i| \geq 6 \).

**Proof of Claim 3.** Suppose to the contrary that \( |N_i| \leq 5 \) for some \( i \in \{l + 1, l + 2, \ldots, r - l - 1\} \). Let \( z_i \in A_i \) be as in Lemma 2.4. Let \( x \) denote the unique vertex of \( T(z, z_i) \) which belongs to \( N_{l+1} \). We show that \( \text{ex}(x) \leq r - 1 \). First let \( y \in N_{2i-1} \).

Then

\[
d(x, y) \leq d(x, z) + d(z, y)
\]

\[
\leq l + 1 + i - 1
\]

\[
\leq l + 1 + r - l - 1 - 1
\]

\[
= r - 1.
\]

Now let \( y \in N_{2i} \). Let \( y_i \in T(z, y) \cap N_i \) so that \( d(x, z_i) = i - l - 1 \). By Lemma 2.4 we have \( d(z_i, y_i) \leq l \). Also \( d(y_i, y) \leq r - i \).

It follows that

\[
d(x, y) \leq d(x, z_i) + d(z_i, y_i) + d(y_i, y)
\]

\[
\leq i - l - 1 + l + r - i
\]

\[
= r - 1.
\]

Therefore, \( \text{ex}(x) \leq r - 1 \), contradicting the fact that \( r \) is the radius of \( G \). \( \Box \)
We now complete the proof of the theorem. If \( r \geq 2l + 2 \), then we have by Claims 1, 2 and 3,
\[
P = |N_0| + \left( |N_1| + \cdots + |N_{\lfloor \frac{l}{2} \rfloor}| \right) + \left( |N_{\lfloor \frac{l}{2} \rfloor + 1}| + \cdots + |N_l| \right) + \left( |N_{l+1}| + \cdots + |N_{r-1}| \right) + |N_r|
\]
\[
\geq 1 + 3 \left( \frac{l}{2} \right) + 4 \left( l - \frac{l}{2} \right) + 6(r - 2l - 1) + 4 \left( - \frac{l}{2} + l \right) + 3 \left( \frac{l}{2} \right) + 1
\]
\[
= -4 - 4l - 2 \left( \frac{l}{2} \right) + 6r,
\]
and (1) follows. If \( 2 \lfloor \frac{l}{2} \rfloor + 2 \leq r \leq 2l + 1 \), then Claims 1 and 2 yield a lower bound on \( p \), and if \( r \leq 2 \lfloor \frac{l}{2} \rfloor + 1 \) then Claim 1 yields again a slightly stronger bound on \( p \). It is easy to verify that both bounds are slightly stronger than the above bound on \( p \), and that each of these bounds implies (1).

\[\square\]

**Corollary 2.8.** Let \( G \) be a 3-connected maximal planar graph. Then
\[
r \leq \frac{p}{6} + \frac{19}{6}.
\]

The following graphs show that for fixed \( l \) the bound in Theorem 2.7 is best possible, apart from the value of the additive constant. For an even integer \( k \geq 4 \), let \( G_1, G_2, \ldots, G_k \) be disjoint copies of the cycle \( C_k \), and let \( a_i, b_i, c_i \in V(G_i) \). Let \( G_k' \) be the graph obtained from the union of \( G_1, G_2, \ldots, G_k \) by adding the edges \( a_i+a_i, b_i+b_i, c_i+c_i, a_{i+1}b_i, c_{i+1}b_i, a_{i+1}c_i \) for \( i = 1, 2, \ldots, k-1 \). Furthermore let \( C_l \) be a cycle with vertices \( j_1, j_2, \ldots, j_l \). Now join the graphs \( C_l \) and \( G_k' \) by adding the edges \( j_1a_1, j_2a_2, j_3b_1, j_4a_1 \) and \( j_1c_1 \) for \( i = 2, 3, \ldots, l \) thus obtaining a planar graph \( H_k \). Clearly, \( p(H_k) = 3k + l \) so that \( k = \frac{p(H_k)-l}{3} \).

By a simple calculation, \( \text{rad}(H_k) = \text{ex}(G_k/2) = \frac{k}{2} + 1 \) and so \( \text{rad}(H_k) = \frac{p(H_k)}{6} - \frac{1}{6} + 1 \).

**Theorem 2.9.** Let \( G \) be a 4-connected plane graph of order \( p \), maximum face length \( l \) and radius \( r \). Then
\[
r \leq \frac{p}{8} + \frac{5}{4}l + \frac{3}{4}.
\]

**Proof.** Recall that \( z \) is a central vertex of \( G \). We first bound the cardinalities of the \( N_i \) from below. The following claim immediately follows from the 4-connectedness of \( G \):

Claim 1. Let \( i \in \{1, 2, \ldots, r-1\} \). Then \( |N_i| \geq 4 \).

In Claim 2 we improve this bound if \( i \) is not too close to 0 or \( r \). We omit the proof since it is identical to the proof of Claim 3 of Theorem 2.7.

Claim 2. Let \( i \in \{l + 1, l + 2, \ldots, r - l - 1\} \). Then \( |N_i| \geq 6 \).

Claim 3. Let \( i \in \{\frac{l}{2} + 1, \frac{l}{2} + 2, \ldots, r - \lfloor \frac{l}{2} \rfloor - 1\} \). Then \( |N_i| \geq 8 \).

**Proof of Claim 3.** Suppose to the contrary that \( |N_i| \leq 7 \) for some \( i \in \{\lfloor \frac{l}{2} \rfloor + 1, \lfloor \frac{l}{2} \rfloor + 2, \ldots, r - \lfloor \frac{l}{2} \rfloor - 1\} \). Let \( z_i \in A_i \) be as in Lemma 2.5. Let \( x \) denote the unique vertex of \( T(z, z_i) \) which belongs to \( N_{\lfloor \frac{l}{2} \rfloor + 1} \). We show that \( \text{ex}(x) \leq r - 1 \). First let \( y \in N_{\lfloor \frac{l}{2} \rfloor - 1} \). Then
\[
d(x, y) \leq d(x, z) + d(z, y)
\]
\[
\leq \left\lfloor \frac{3l}{2} \right\rfloor + 1 + i - 1
\]
\[
\leq \left\lfloor \frac{3l}{2} \right\rfloor + 1 + r - \left\lfloor \frac{3l}{2} \right\rfloor - 1 - 1
\]
\[
= r - 1.
\]
Now let \( y \in N_{\geq i} \). Let \( y_i \in T(z, y) \cap N_i \) so that \( d(x, z_i) = i - \left\lfloor \frac{3l}{2} \right\rfloor - 1 \). By Lemma 2.5 we have \( d(z_i, y_i) \leq \left\lfloor \frac{3l}{2} \right\rfloor \). Also \( d(y_i, y) \leq r - i \). It follows that
\[
d(x, y) \leq d(x, z_i) + d(z_i, y_i) + d(y_i, y)
\]
\[
\leq i - \left\lfloor \frac{3l}{2} \right\rfloor - 1 + \left\lfloor \frac{3l}{2} \right\rfloor + r - i
\]
\[
= r - 1.
\]
Therefore, \( \text{ex}(x) \leq r - 1 \), contradicting the fact that \( r \) is the radius of \( G \). \[\square\]
We now complete the proof of the theorem. If \( r \geq 2\left\lfloor \frac{3l}{2} \right\rfloor + 2 \), then by Claims 1, 2 and 3 we have

\[
p = |N_0| + (|N_1| + \cdots + |N_i|) + \left( |N_{i+1}| + \cdots + |N_{\left\lfloor \frac{3l}{2} \right\rfloor}| \right) + \left( \left| N_{\left\lfloor \frac{3l}{2} \right\rfloor+1} \right| + \cdots + \left| N_{r-\left\lfloor \frac{3l}{2} \right\rfloor-1} \right| \right)
\]
\[
\geq 1 + 4l + 6\left( \left\lfloor \frac{3l}{2} \right\rfloor - 1 \right) + 8\left( r - 2 \left\lfloor \frac{3l}{2} \right\rfloor - 1 \right) + 6\left( -l + \left\lfloor \frac{3l}{2} \right\rfloor \right) + 4l + 1
\]
\[
\geq -6 - 10l + 8r.
\]

and (2) follows. If \( 2l + 2 \leq r \leq 2\left\lfloor \frac{3l}{2} \right\rfloor + 1 \), then Claims 1 and 2 yield a lower bound on \( p \), and if \( r \leq 2l + 1 \), then again Claim 1 yields a lower bound on \( p \). It is easy to verify that both bounds are slightly stronger than the above bound on \( p \), and that each of them implies (2). \( \square \)

The following graphs show that for fixed \( l \) the bound in Theorem 2.9 is the best possible, apart from the value of the additive constant. For an even integer \( i \in 1, 2, \ldots, r \) let \( N_i \) be the graph obtained from the union of \( G_1, G_2, \ldots, G_k \) by adding the edges \( a_{i-1}a_i, b_{i-1}b_i, c_{i-1}c_i, d_{i-1}d_i, a_{i+1}a_i, b_{i+1}b_i, c_{i+1}c_i, d_{i+1}d_i \) for \( i = 1, 2, \ldots, k-1 \) and \( a_0c_0 \). Furthermore let \( C_i \) be a cycle with vertices \( j_1, j_2, \ldots, j_l \). Now join the graphs \( C_i \) and \( G_k \) by adding the edges \( j_1a_1, j_1b_1, j_2b_1, j_3b_1, j_4c_1 \) and \( j_l \) for \( i = 3, 4, \ldots, l \) thus obtaining a planar graph \( H'_k \). Clearly, \( p(H'_k) = 4k + l \) so that \( k = \frac{p(H'_k) - l}{4} \). By a simple calculation, \( \text{rad}(H'_k) = \text{ex}(d_{k/2}) = \frac{k}{2} \) and so \( \text{rad}(H'_k) = \frac{p(H'_k) - l}{8} - \frac{l}{8} \).

**Theorem 2.10.** Let \( G \) be a 5-connected plane graph of order \( p \), maximum face length \( l \) and radius \( r \). Then

\[
r \leq \frac{p}{10} + \frac{8l + 4}{5}.
\]

**Proof.** Recall that \( z \) is a central vertex of \( G \). We first bound the cardinalities of the \( N_i \) from below. The following claim immediately follows from the 5-connectedness of \( G \):

**Claim 1.** Let \( i \in \{1, 2, \ldots, r-1\} \). Then \( |N_i| \geq 5 \).

In Claims 2 and 3 we improve this bound if \( i \) is not too close to 0 or \( r \). We omit the proofs since they are identical to the proofs of Claim 3 of Theorem 2.7 and Claim 3 of Theorem 2.9, respectively.

**Claim 2.** Let \( i \in \{1 + l, 2l + 2, \ldots, r-l - 1\} \). Then \( |N_i| \geq 6 \).

**Claim 3.** Let \( i \in \{\left\lfloor \frac{l}{2} \right\rfloor + 1, \left\lfloor \frac{3l}{2} \right\rfloor + 2, \ldots, r - \left\lfloor \frac{3l}{2} \right\rfloor - 1\} \). Then \( |N_i| \geq 8 \).

**Claim 4.** Let \( i \in \{2l + 1, 2l + 2, \ldots, r - 2l - 1\} \). Then \( |N_i| \geq 10 \).

**Proof of Claim 4.** Suppose to the contrary that \( |N_i| \leq 9 \) for some \( i \in \{2l + 1, 2l + 2, \ldots, r - 2l - 1\} \). Let \( z_i \in A_i \) be as in Lemma 2.6. Let \( x \) denote the unique vertex of \( T(z, z_i) \) which belongs to \( N_{2l+1} \). We show that \( \text{ex}(x) \leq r - 1 \). First let \( y \in N_{2l+1} \).

Then

\[
d(x, y) \leq d(x, z) + d(z, y)
\]
\[
\leq 2l + 1 + i - 1
\]
\[
\leq 2l + 1 + r - 2l - 1 - 1
\]
\[
= r - 1.
\]

Now let \( y \in N_{2l+1} \). Let \( y \in T(z, y) \cap N_i \) so that \( d(x, z_i) = i - 2l - 1 \). By Lemma 2.6 we have \( d(z_i, y_i) \leq 2l \). Also \( d(y_i, y) \leq r - i \). It follows that

\[
d(x, y) \leq d(x, z_i) + d(z_i, y_i) + d(y_i, y)
\]
\[
\leq i - 2l - 1 + 2l + r - i
\]
\[
= r - 1.
\]

Therefore, \( \text{ex}(x) \leq r - 1 \), contradicting the fact that \( r \) is the radius of \( G \). \( \square \)
We now complete the proof of the theorem. If \( r \geq 4l + 2 \), then we have by Claims 1, 2, 3 and 4, we have

\[
p = |N_0| + (|N_1| + \cdots + |N_l|) + \left(|N_{l+1}| + \cdots + \left|N_{\frac{3l}{2}}\right|\right)
+ \left(|N_{\frac{3l}{2}}| + \cdots + |N_{2l}|\right) + (|N_{2l+1}| + \cdots + |N_{r-2l-1}|)
+ \left(|N_{r-2l}| + \cdots + \left|N_{r-\frac{3l}{2}}\right|\right)
+ \left(|N_{r-\frac{3l}{2}}| + \cdots + \left|N_{r-l-1}\right|\right)
+ (|N_{r-l}| + \cdots + |N_{r-1}|) + |N_r|
\geq 1 + 5l + 6(\frac{3l}{2} - 1) + 8(2l - \frac{3l}{2}) + 10(r - 4l - 1) + 8(2l - \frac{3l}{2}) + 6(\frac{3l}{2} - l) + 5l + 1
\geq -8 - 16l + 10r,
\]

and (3) follows. If \( 2\frac{3l}{2} + 2 \leq r \leq 4l + 1 \), then Claims 1, 2, and 3 yield a lower bound on \( p \), if \( 2l + 2 \leq r \leq 2\frac{3l}{2} + 1 \) then Claims 1 and 2 yield a lower bound on \( p \), and if \( r \leq 2l + 1 \) then Claim 1 yields a lower bound on \( p \). It is easy to verify that these bounds are stronger than the lower bound on \( p \) above, and that each of them implies (3). \( \square \)

The following graphs show that for fixed \( l \) the bound in Theorem 2.10 is best possible, apart from the value of the additive constant. For an even integer \( k \geq 8 \) let \( G_1, G_2, \ldots, G_k \) be disjoint copies of the 5-cycle, \( C_5 \), and let \( a_i, b_i, c_i, d_i, u_i \in V(G_i) \). Let \( G''_k \) be the graph obtained from the union of \( G_1, G_2, \ldots, G_k \) by adding the edges \( a_i + a_i, b_i + b_i, c_i + c_i, d_i + d_i, u_i + u_i, a_i + b_i, a_i + c_i, a_i + d_i, b_i + c_i, b_i + d_i, c_i + d_i, u_i + d_i \) for \( i = 1, 2, \ldots, k - 1 \) and a new vertex \( v_k \) adjacent to \( a_k, b_k, c_k, d_k \) and \( v_k \). Furthermore let \( G_l \) be a cycle with vertices \( j_1, j_2, \ldots, j_l \). Now join the graphs \( G_l \) and \( G''_k \) by adding the edges \( j_1w_1, j_1a_1, j_1b_1, j_1c_1, j_1d_1, j_2b_1, j_2b_1, j_2c_1, j_2d_1 \), and \( j_id_i \) for \( i = 3, 4, \ldots, l \) thus obtaining a planar graph \( H''_k \). Clearly, \( p(H''_k) = 5k + l \) so that \( k = \frac{p(H''_k) - l}{5} \). By a simple calculation, \( \text{rad}(H''_k) = \text{ex}(d_{k/2}) \frac{k}{2} \) and so \( \text{rad}(H''_k) = \frac{p(H''_k)}{10} - \frac{l}{10} \).

Acknowledgments

The first and third authors were financial supported by the National Research Foundation and the University of KwaZulu-Natal is acknowledged.

References