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One more pathology of C^*-algebraic tensor products

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Abstract

We define a collection of tensor product norms for C^*-algebras and show that a symmetric tensor product functor on the category of separable C^*-algebras need not be associative.

Keywords: C^*-algebra; Tensor product; Property T group

1. Introduction

There are several known pathologies for tensor products of C^*-algebras, e.g. [12,1,5]. In this paper we reveal one more pathology: C^*-tensor products need not be associative.

Following E. Kirchberg [4], we call a bifunctor $(A, B) \to A \otimes_\alpha B$ a *C^*-algebraic tensor product functor* if it is obtained by completing of the algebraic tensor product $A \odot B$ of C^*-algebras in a functional way with respect to a suitable C^*-norm $\|\cdot\|_\alpha$. We call such a functor *symmetric* if the standard isomorphism $A \odot B \cong B \odot A$ extends to an isomorphism $A \otimes_\alpha B \cong B \otimes_\alpha A$. Similarly, we call it *associative* if the standard isomorphism $A \odot (B \odot C) \cong (A \odot B) \odot C$ extends to an isomorphism $A \otimes_\alpha (B \otimes_\alpha C) \cong (A \otimes_\alpha B) \otimes_\alpha C$ for any C^*-algebras A, B, C. It is well known that both the minimal tensor product functor $\otimes_{\min}$ and the maximal tensor product functor $\otimes_{\max}$ are symmetric and associative.

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In this paper we construct a collection of symmetric C^*-algebraic tensor product functors related to asymptotic homomorphisms of C^*-algebras. For technical reasons we restrict ourselves to the category of *separable* C^*-algebras. Using C^*-algebras related to property T groups [11,13] we show that some of these tensor product functors are not associative.

Recall that asymptotic homomorphisms of C^*-algebras were first defined and studied in [3] in relation to topological properties of C^*-algebras. The most important and the best known case is the case of asymptotic homomorphisms from a suspended C^*-algebra SA to the C^*-algebra $\mathbb{K}$ of compact operators, since the homotopy classes of those are the K-homology of A, the E-theory. Asymptotic homomorphisms to other C^*-algebras are less known. For example, it is known that any asymptotic homomorphism to the Calkin algebra is homotopic to a genuine homomorphism [6,8]. Even less is known about asymptotic homomorphisms to $\mathbb{B}(H)$, where there is no topological obstruction (recall that the K-groups of $\mathbb{B}(H)$ are trivial). Such asymptotic homomorphisms are called *asymptotic representations* and were first studied in relation to the asymptotic tensor product of C^*-algebras [9] and to semi-invertibility of C^*-algebra extensions [10].

2. Definition of asymptotic C^*-tensor products

Recall [3] that an asymptotic homomorphism φ from a C^*-algebra A to a C^*-algebra D is a family of maps $\varphi = (\varphi_t)_{t\in[0,\infty)} : A \to D$ satisfying the following properties:

1. the map $t \mapsto \varphi_t(a)$ is continuous for any $a \in A$;
2. $\lim_{t\to\infty} \varphi_t(a + \lambda b) - \varphi_t(a) - \lambda\varphi_t(b) = \lim_{t\to\infty} \varphi_t(a^*) - \varphi_t(a)^* = \lim_{t\to\infty} \varphi_t(ab) - \varphi_t(a)\varphi_t(b) = 0$ for any $a, b \in A$ and any $\lambda \in \mathbb{C}$.

Let $\mathbb{L}(H)$ be the algebra of bounded operators on a separable Hilbert space H. Our point is that we would like to consider D as a C^*-subalgebra of $\mathbb{L}(H)$: $D \subset \mathbb{L}(H)$. We also view asymptotic homomorphisms to D as asymptotic representations on H taking values in D. We are mostly interested in the special case

$$D = \mathbb{K}^\infty = \prod_{n=1}^{\infty} \mathbb{K} = \prod_{n=1}^{\infty} \mathbb{K}(H_n),$$

where $H_n = H$ for all $n \in \mathbb{N}$ and $\mathbb{K} = \mathbb{K}(H)$ is the C^*-algebra of compact operators on H.

Let A, B be separable C^*-algebras and let $\varphi = (\varphi_t)_{t\in[0,\infty)}$, $\psi = (\psi_t)_{t\in[0,\infty)}$ be asymptotic representations of A and B respectively, taking values in D.

Let $A \odot B$ be the algebraic tensor product of A and B. For each $a \in A$ and $b \in B$, we can define elements $a^{\varphi\otimes\psi}, b^{\varphi\otimes\psi} \in C_b([0,\infty), \mathbb{L}(H \otimes H))$ by

$$a^{\varphi\otimes\psi}(t) = \varphi_t(a) \otimes 1_H \quad \text{and} \quad b^{\varphi\otimes\psi}(t) = 1_H \otimes \psi_t(b).$$

Note that $a^{\varphi\otimes\psi}(t) \cdot b^{\varphi\otimes\psi}(t) \in C_b([0,\infty), D \otimes_{\min} D))$, where $\otimes_{\min}$ denotes the minimal tensor product of C^*-algebras.

We can then define a $*$-homomorphism

$$\varphi \otimes \psi : A \odot B \to C_b\big([0,\infty), D \otimes_{\min} D\big)/C_0\big([0,\infty), D \otimes_{\min} D\big)$$

such that

$$\varphi \otimes \psi\left(\sum_i a_i \otimes b_i\right) = q\left(\sum_i a_i^{\varphi\otimes\psi} \cdot b_i^{\varphi\otimes\psi}\right),$$

where

$$q : C_b\big([0,\infty), D \otimes_{\min} D\big) \to C_b\big([0,\infty), D \otimes_{\min} D\big)/C_0\big([0,\infty), D \otimes_{\min} D\big)$$

is the quotient map. Note that

$$\big\|\varphi \otimes \psi(c)\big\| = \lim\sup_{t\to\infty}\left\|\sum_i \varphi(a_i) \otimes \psi(b_i)\right\|$$

for any $c = \sum_i a_i \otimes b_i \in A \odot B$. We can now define a seminorm $\|\cdot\|_{D,0}$ on $A \odot B$ by

$$\|c\|_{D,0} = \sup_{\varphi,\psi}\big\|\varphi \otimes \psi(c)\big\|,$$

where we take the supremum over all pairs (φ, ψ), where φ and ψ are asymptotic representations of A and B, respectively, taking values in D.

Note that a genuine $*$-homomorphism from A to D can be considered as an asymptotic representation in the obvious way. So, if $D = \mathbb{L}(H)$ then $\|\cdot\|_{D,0} \geqslant \|\cdot\|_{\min}$, and $\|\cdot\|_{D,0}$ is a norm. This norm coincides with the symmetric asymptotic tensor norm defined in [10]. More generally, the seminorm $\|\cdot\|_{D,0}$ is a norm if there exist *faithful* asymptotic representations of A and B taking values in D. Remark that there are other C^*-algebras D, besides $\mathbb{L}(H)$, that admit faithful asymptotic representations of any separable C^*-algebra. For example, it follows from [8] that one can take the *coarse Roe algebra* of $\mathbb{Z}$ as D.

In general, the seminorm $\|\cdot\|_{D,0}$ may be degenerate (e.g. it may happen that any asymptotic representation of a C^*-algebra A taking values in some D may be asymptotically equivalent to zero, see Lemma 2 below), so let us define the norm $\|\cdot\|_D$ on $A \odot B$ by

$$\|c\|_D = \max\big\{\|c\|_{\min}, \|c\|_{D,0}\big\},$$

where $c \in A \odot B$. Clearly, $\|\cdot\|_D$ is a C^*-norm, hence a cross-norm, and

$$\|\cdot\|_{\min} \leqslant \|\cdot\|_D \leqslant \|\cdot\|_{\max}.$$

We denote by $A \otimes_D B$ the C^*-algebra obtained by completing $A \odot B$ with respect to the norm $\|\cdot\|_D$. Obviously the correspondence $(A, B) \mapsto A \otimes_D B$ is a C^*-algebraic tensor product functor on the category of separable C^*-algebras.

Lemma 1. *The functor $\otimes_D$ is symmetric.*

Proof. Obvious. □

3. Asymptotic representations taking values in $\mathbb{K}^\infty$

Let G be a residually finite infinite property T group, let π_n be the sequence of all non-equivalent irreducible unitary representations on finite-dimensional Hilbert spaces H_n and let $\overline{A}$ be the C^*-algebra generated by operators $\bigoplus_{n=1}^{\infty} \pi_n(g)$, $g \in G$. We denote by E the C^*-subalgebra in $\mathbb{L}(\bigoplus_{n=1}^{\infty} H_n)$ generated by $\overline{A}$ and by compact operators: $E = \overline{A} + \mathbb{K}$. Put $A = E/\mathbb{K}$. This C^*-algebra was first considered by S. Wassermann and we refer to his paper [13] for more details.

Lemma 2. *Let $\varphi = (\varphi_t)_{t\in[0,\infty)} : A \to \mathbb{K}^\infty$ be an asymptotic homomorphism. Then φ is asymptotically equivalent to zero, i.e.* $\lim_{t\to\infty} \varphi_t(a) = 0$ *for any $a \in A$.*

Proof. Without loss of generality we may assume that φ is self-adjoint (e.g. by changing $\varphi_t(a)$ by $\frac{1}{2}(\varphi_t(a) + \varphi_t(a^*)^*)$).

Let $q_n : \prod_{n=1}^{\infty} \mathbb{K} \to \mathbb{K}$ be the projection onto the nth copy. Then $\varphi_t^{(n)} = q_n \circ \varphi_t$ is an asymptotic homomorphism to $\mathbb{K}$.

There exists t_0 such that

$$\left\| \left(\varphi_t^{(n)}(1)^2\right) - \varphi_t^{(n)}(1) \right\| < \frac{2}{9} \tag{1}$$

for all $t > t_0$, hence the spectrum of $\varphi_t^{(n)}(1)$ lies in $[-\frac{1}{3}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{4}{3}]$. Let f be the continuous function, which equals 0 on $(-\infty, \frac{1}{3}]$, 1 on $[\frac{2}{3}, \infty)$ and which is linear on $[\frac{1}{3}, \frac{2}{3}]$. Then $p_n(t) = f(\varphi_t^{(n)}(1))$ is a continuous family of finite rank projections, and $\lim_{t\to\infty} \|\varphi_t^{(n)}(1) - p_n(t)\| = 0$.

Continuity of the family $p_n(t)$, $t \in (t_0, \infty)$, implies that there is a continuous family $(u_t)_{t\in(t_0,\infty)}$ of unitaries such that $\mathrm{Ad}_{u_t} p_n(t) = p_n(t_0) = p_n$ is a constant finite rank projection. Then $\mathrm{Ad}_{u_t^*} \varphi_t^{(n)}$ is an asymptotic homomorphism such that $\lim_{t\to\infty} \|\mathrm{Ad}_{u_t^*} \varphi_t^{(n)}(1) - p_n\| = 0$.

Then the formula $\psi_t(a) = p_n(\mathrm{Ad}_{u_t^*} \varphi_t^{(n)}(a))p_n$ defines an asymptotic homomorphism from A to the matrix algebra of the fixed dimension $N_n = \dim p_n$.

The group G with the stated properties is known to be finitely generated, so without loss of generality we may assume that $\psi_t(g_i)$ are unitaries, where $g_i \in G$, $i = 1, \dots, k$, are generators for G.

Since the direct product of k copies of the unitary group U_{N_n} is compact, so the set $\{(\psi_t(g_1), \dots, \psi_t(g_k))\colon t \in [0, \infty)\}$ has an accumulation point $(u_1, \dots, u_k) \in U_{N_n}^k$. If we put $\sigma(g_i) = u_i$ then this map extends to a genuine representation of G of dimension N_n. Indeed, G is a quotient of the free group $\mathbb{F}_k$ generated by $g_1, \dots, g_k$ modulo some relations and each ψ_t and σ obviously define representations of $\mathbb{F}_k$, which we denote by the same characters. If $r \in \mathbb{F}_k$ is a relation then $\lim_{t\to\infty} \|\psi_t(r) - p_n\| = 0$. Therefore, $\sigma(r) = p_n$, hence σ factorizes through a representation of G.

Suppose that $p_n \neq 0$ for some n. This implies that the representation σ is non-zero, hence it contains at least one of π_j. Then $\|\sigma(a)\| \geqslant \|\pi_j(a)\|$ for any $a \in \mathbb{C}[G]$. Let $\|a\|_A$ denote the norm (in A) of $a \in \mathbb{C}[G]$ as an element of A. Since

$$\|a\|_A \geqslant \limsup_{t\to\infty} \left\|\psi_t(a)\right\| \geqslant \left\|\sigma(a)\right\| = \left\|\pi_j(a)\right\|,$$

we have $\|a\|_A = \limsup_{n\to\infty} \|\pi_n(a)\| \geqslant \|\pi_j(a)\|$ for any $a \in A$. Then the identity map of G extends to a $*$-homomorphism $i : A \to C^*_{\pi_j}(G)$, where $C^*_\pi(G)$ denotes the C^*-algebra generated by the representation π. Tensoring it by $\mathrm{id}_{C^*_{\bar\pi_j}(G)}$, where $\bar\pi$ denotes the contragredient representation for a representation π, we get a $*$-homomorphism

$$i \otimes \mathrm{id}_{C^*_{\bar\pi_j}(G)} : A \otimes C^*_{\bar\pi_j}(G) \to C^*_{\pi_j}(G) \otimes C^*_{\bar\pi_j}(G). \tag{2}$$

We do not specify the tensor product norm here because $C^*_{\bar\pi_j}(G)$ is finite-dimensional, hence nuclear. It was shown in [13] that the norm on the left-hand side of (2) is strictly smaller than the norm on the right-hand side, so this $*$-homomorphism cannot exist. This contradiction shows that $p_n = 0$ for all n, hence (1) implies that $\lim_{t\to\infty} \|q_n \circ \varphi_t(1)\| = 0$ uniformly in n, therefore, $\lim_{t\to\infty} \|q_n \circ \varphi_t(a)\| = 0$ uniformly in n for any $a \in A$. □

Corollary 3. *For A defined above, one has $A \otimes_{\mathbb{K}^\infty} B = A \otimes_{\min} B$ for any C^*-algebra B.*

Proof. Since

$$\|\varphi \otimes \psi(a \otimes b)\| = \lim\sup_{t\to\infty} \|\varphi_t(a) \otimes \psi_t(b)\| = \lim\sup_{t\to\infty} \|\varphi_t(a)\| \cdot \|\psi_t(b)\| = 0$$

for any $a \in A$, $b \in B$ and for any asymptotic representations φ and ψ, one has

$$\|a \otimes b\|_{\mathbb{K}^\infty,0} = \sup_{\varphi,\psi} \|\varphi \otimes \psi(a \otimes b)\| = 0,$$

hence $\|c\|_{\mathbb{K}^\infty,0} = 0$ for any $c \in A \odot B$, therefore,

$$\|c\|_{\mathbb{K}^\infty} = \max\{\|c\|_{\min}, 0\} = \|c\|_{\min}. \qquad \square$$

4. An example of an asymptotic representation taking values in $\mathbb{K}^\infty$

Let $C = C_0(0,1]$. We are going to construct an asymptotic representation ϕ of $C \otimes A$ taking values in $\mathbb{K}^\infty$. (We do not specify here the tensor norm since C is nuclear.) This construction is based on results from [7].

Let $\chi : A \to E$ be a continuous homogeneous self-adjoint selection map, cf. [2]. We denote by P_n the projection in $\bigoplus_{n=1}^\infty H_n$ onto H_n. For $a \in A$ put $\alpha(a) = \iota \circ \chi(a)$, where $\iota : E \to \mathbb{L}(\bigoplus_{n=1}^\infty H_n)$ is the standard inclusion.

Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a *dense* sequence of points in $(0,1)$. For $t = k \in \mathbb{N}$ and for $f \in C$, put

$$\beta_k(f) = \sum_{n=k+1}^\infty f(\tau_n) P_n,$$

where the sum is $*$-strongly convergent. If $k < t < k+1$ then put

$$\beta_t(f) = f\big((1 - t + k)\tau_{k+1}\big) P_{k+1} + \beta_{k+1}(f).$$

Let $F \in C \otimes A$. One can consider F as a continuous function on $[0, 1]$ taking values in A such that $F(0) = 0$. Put

$$\phi_k(F) = \sum_{n=k+1}^{\infty} P_n \alpha\big(F(\tau_n)\big) P_n$$

(this sum also is convergent with respect to the $*$-strong topology), and

$$\phi_t(F) = P_{k+1} \alpha\big(F\big((1-t+k)\tau_{k+1}\big)\big) P_{k+1} + \phi_{k+1}(F)$$

for $k < t < k+1$.

Lemma 4. *The family of maps $(\phi_t)_{t \in [0,\infty)}$ is an asymptotic representation of $C \otimes A$ taking values in $\prod_{n=1}^{\infty} \mathbb{L}(H_n) \subset \mathbb{K}^{\infty}$.*

Proof. By the definition, the maps ϕ_t, $t \in [0, \infty)$, take values in $\prod_{n=1}^{\infty} \mathbb{L}(H_n)$, so we only need to check that algebraic properties hold asymptotically. Let us check that for multiplication, as other properties can be checked in the same way. Let $F_1, F_2 \in C \otimes A$. Set

$$\kappa_{F_1,F_2} : \tau \mapsto \alpha\big(F_1(\tau)F_2(\tau)\big) - \alpha\big(F_1(\tau)\big)\alpha\big(F_2(\tau)\big).$$

Then κ_{F_1,F_2} is a continuous map from $[0, 1]$ to $\mathbb{L}(\bigoplus_{n=1}^{\infty} H_n)$. Since χ is a lifting for the quotient map $E \to E/\mathbb{K}$, $\kappa_{F_1,F_2}(\tau) \in \mathbb{K} \cap \iota(E)$ for any $\tau \in [0, 1]$.

Set $\mathcal{K} = \kappa_{F_1,F_2}([0, 1]) \subset \mathbb{K} \cap \iota(E)$. Then $\mathcal{K}$ is compact, hence, for any increasing sequence $\{Q_n\}_{n \in \mathbb{N}}$ of projections, $\lim_{n\to\infty} \sup_{K \in \mathcal{K}} \|(1 - Q_n)K(1 - Q_n)\| = 0$.

Then

$$\lim_{k\to\infty} \phi_k(F_1F_2) - \phi_k(F_1)\phi_k(F_2) = \lim_{k\to\infty} \sum_{n=k+1}^{\infty} P_n \kappa_{F_1,F_2}(\tau_n) P_n = 0.$$

Finally, we easily pass to the continuous parameter: as

$$\lim_{k\to\infty} P_{k+1} \kappa_{F_1,F_2}\big((1-t+k)\tau_k\big) P_{k+1} = 0$$

for $k < t < k+1$, so we conclude that

$$\lim_{t\to\infty} \phi_t(F_1F_2) - \phi_t(F_1)\phi_t(F_2) = 0. \qquad \square$$

Note that if $F = f \otimes a \in C \otimes A$ then

$$\phi_t(f \otimes a) = \sum_{n=k}^{\infty} P_n \alpha(a) P_n \cdot \beta_t(f).$$

5. Comparing tensor norms

Let B be the C^*-algebra generated by operators $\bigoplus_{n=1}^{\infty} \bar{\pi}_n(g)$, $g \in G$, where $\bar{\pi}$ denotes the contragredient representation for π.

Theorem 5. *The tensor products $C \otimes_{\mathbb{K}^\infty} (A \otimes_{\mathbb{K}^\infty} B)$ and $(C \otimes_{\mathbb{K}^\infty} A) \otimes_{\mathbb{K}^\infty} B$ are not canonically isomorphic.*

Proof. Let $f \in C$ be the identity function, $f(\tau) = \tau$, and let $\{g_1, \dots, g_m\}$ be a symmetric set of generators of the group G as above. We identify the group elements with the corresponding unitaries in C^*-algebras generated by representations of G (like B) and in their quotients (like A). Let

$$d = \sum_{i=1}^{m} f \otimes g_i \otimes g_i \in C \odot A \odot B.$$

Denote by $\|\cdot\|_1$ and by $\|\cdot\|_2$ the norms on $C \odot A \odot B$ inherited from $C \otimes_{\mathbb{K}^\infty} (A \otimes_{\mathbb{K}^\infty} B)$ and $(C \otimes_{\mathbb{K}^\infty} A) \otimes_{\mathbb{K}^\infty} B$ respectively. Our aim is to show that $\|d\|_1 \neq \|d\|_2$.

It follows from Lemma 3 and from amenability of C that

$$C \otimes_{\mathbb{K}^\infty} (A \otimes_{\mathbb{K}^\infty} B) = C \otimes_{\min} (A \otimes_{\min} B),$$

so

$$\|d\|_1 = \|f\| \cdot \left\| \sum_{i=1}^{m} g_i \otimes g_i \right\|_{\min} = \left\| \sum_{i=1}^{m} g_i \otimes g_i \right\|_{\min}.$$

It was shown in [13] that the latter norm is strictly smaller than m, so

$$\|d\|_1 < m. \tag{3}$$

When estimating the norm $\|\cdot\|_2$ from below, we may use two special asymptotic representations instead of taking the supremum over all of them. Let us take ϕ_t for $C \otimes A$ and the identity representation for B. Then

$$\begin{aligned}
\|d\|_2 &\geqslant \limsup_{t\to\infty} \left\| \sum_{i=1}^{m} \phi_t(f \otimes g_i) \otimes \sum_{n=1}^{\infty} \bar{\pi}_n(g_i) P_n \right\| \\
&= \limsup_{t\to\infty} \left\| \sum_{i=1}^{m} \sum_{n=1}^{\infty} P_n \beta_t(f) \alpha(g_i) P_n \otimes \sum_{n=1}^{\infty} \bar{\pi}_n(g_i) P_n \right\| \\
&\geqslant \limsup_{t\to\infty} \sup_n \left\| P_n \beta_t(f) \sum_{i=1}^{m} \alpha(g_i) P_n \otimes \bar{\pi}_n(g_i) P_n \right\| \\
&\geqslant \limsup_{n\to\infty} \left\| P_n \beta_n(f) \sum_{i=1}^{m} \alpha(g_i) P_n \otimes \bar{\pi}_n(g_i) P_n \right\|
\end{aligned}$$

$$= \limsup_{n\to\infty} f(\tau_n)\cdot\left\| \sum_{i=1}^{m} P_n\alpha(g_i)P_n\otimes\bar\pi_n(g_i)P_n\right\|$$
$$\geqslant \limsup_{j\to\infty} f(\tau_{n_j})\cdot\left\| \sum_{i=1}^{m} P_{n_j}\alpha(g_i)P_{n_j}\otimes\bar\pi_{n_j}(g_i)\right\|,$$

where $\{n_j\}$ is any increasing subsequence of integers. Since the sequence $\{\tau_n\}_{n=1}^{\infty}$ is dense in $[0,1]$, we can find a subsequence $\{n_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty}\tau_{n_j}=1$. Recall that, as α is a lifting for the quotient map $E\to E/\mathbb{K}$, so $\lim_{n\to\infty}\|P_n\alpha(g)P_n-\pi_n(g)\|$ for any $g\in G$. Then

$$\|d\|_2 \geqslant \limsup_{j\to\infty} f(\tau_{n_j})\cdot\left\| \sum_{i=1}^{m} P_{n_j}\alpha(g_i)P_{n_j}\otimes\bar\pi_{n_j}(g_i)\right\|$$
$$= \limsup_{j\to\infty}\left\| \sum_{i=1}^{m} P_{n_j}\alpha(g_i)P_{n_j}\otimes\bar\pi_{n_j}(g_i)\right\|$$
$$= \limsup_{j\to\infty}\left\| \sum_{i=1}^{m} \pi_{n_j}(g_i)\otimes\bar\pi_{n_j}(g_i)\right\|$$
$$= \left\| \sum_{i=1}^{m} \pi_{n_j}(g_i)\otimes\bar\pi_{n_j}(g_i)\right\| = \sum_{i=1}^{m} 1 = m.$$

On the other hand, $\|d\|_2\leqslant\sum_{i=1}^{m}\|f\otimes g_i\otimes g_i\|_2=m$, so we have

$$\|d\|_2 = m. \tag{4}$$

Comparing (3) and (4), we conclude that these two norms are different. □

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