The index of coincidence Nielsen classes of maps between surfaces

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Abstract
For a given pair of closed orientable surfaces $S_h, S_g$ and given integers $d_1, d_2$, one would like to find bounds for the index of the Nielsen coincidence classes among all possible pairs of maps $(f_1, f_2) : S_h \to S_g$ where $|\text{deg}(f_1)| = d_1$ and $|\text{deg}(f_2)| = d_2$. We show that these bounds are infinite when $h > g = 1$, or when $h \geq g > 1$ and both $d_i < (h - 1)/(g - 1)$. We calculate these bounds when $h = g$ and $d_2 = 1$. We also consider the similar question for the root case, which is simpler, and we solve it completely. Few results are given when $d_i = (h - 1)/(g - 1)$ for either $i = 1$ or $i = 2$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Nielsen Fixed Point Theory has been successfully extended to coincidence theory for pairs of maps $(f_1, f_2) : M \to N$ where $M, N$ are closed orientable manifolds of equal dimension. See [14]. In a similar fashion as is done in fixed point theory, in the set $\text{Coin}(f_1, f_2) = \{x \in M \mid f_1(x) = f_2(x)\}$ we define the following equivalence relation: $x_0, x_1 \in \text{Coin}(f_1, f_2)$ are Nielsen related if there exists a path $\lambda : [0, 1] \to M$ such that $\lambda(0) = x_0$, $\lambda(1) = x_1$ and $f_1 \circ \lambda$ is homotopic to $f_2 \circ \lambda$ rel endpoints. This equivalence relation splits the set $\text{Coin}(f_1, f_2)$ into a finite number of isolated subsets called the Nielsen

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coincidence classes. For each such class, which we denote by \( C \), we can associate an integer, \( \text{ind}(C; f_1, f_2) \in \mathbb{Z} \), see [15, Chapter 6], called the local index of the class. The coincidence Nielsen number, \( N(f_1, f_2) \), is defined as the number of essential coincidence Nielsen classes, i.e., classes which have index different from zero. (When \( M = N \) and \( f_2 = \text{id} \) is the identity map, we get back to the fixed point theory.) The importance of these numbers lies in their invariance: the indices of the Nielsen classes (hence also the Nielsen number) do not change when the maps \( f_1 \) and/or \( f_2 \) are changed by homotopy. But it is a very difficult question to calculate the Nielsen number and also the local indices of the Nielsen classes.

Of particular interest is this question for maps between surfaces. The behavior of these indices has been observed in the fixed point situation. For example, for maps on the torus \( S^1 \times S^1 \), it was proved in [10] that the local indices of the essential Nielsen classes are all bounded and the bounds depend only on the genus of the surface. More precisely we have that

\[
3 - 4g \leq \text{ind}(C) \leq 1.
\]

This result has been extended in [9] to arbitrary map \( f : S_g \to S_g \) not necessarily homotopic to a diffeomorphism. In the present paper, we will consider the similar question for coincidence of maps between surfaces.

Given integers \( g, h, d_1, d_2 \), consider any pair of surface maps \( (f_1, f_2) : S_h \to S_g \) with \( |\deg(f_1)| = d_1 \) and \( |\deg(f_2)| = d_2 \), and look at the index of any Nielsen coincidence class \( C \). Are these indices bounded? In case the answer is yes one would like to find such bounds.

More precisely, we define

\[
B(g, h, d_1, d_2) := \text{sup} \{ \text{ind}(C) : |\deg(f_1)| = d_1, |\deg(f_2)| = d_2 \},
\]

where the supreme is taken for all coincidence classes \( C \) of all pairs of maps \( (f_1, f_2) : S_h \to S_g \) with the given degrees. It is an integer or \( \infty \).

Clearly: \( B(g, h, d_2, d_1) = B(g, h, d_1, d_2) \). So we may assume \( d_1 \leq d_2 \).

By Kneser’s formula, (see [11] or [17]): \( h - 1 \geq d_i(g - 1) \) when \( d_i \neq 0 \).

By [10]: If \( g = h > 1 \) (hence \( d_i \leq 1 \)), then \( B(g, g, 1, 1) \leq 4g - 3 \).

By [9]: If \( g = h > 1 \) (hence \( d_i \leq 1 \)), then \( B(g, g, 0, 1) \leq 4g - 3 \).

By [2]: \( B(1, 1, d_1, d_2) = 1 \).

We will show:

\[
B(g, g, 1, 1) = 4g - 3 \quad \text{(Examples 1 and 2)};
\]

\[
B(g, g, 0, 1) = 2g - 1 \quad \text{(Proposition 5 and Example 3)}.
\]

When \( g > 1 \) and \( d_1 \leq d_2 < (h - 1)/(g - 1) \), hence both \( f_1, f_2 \) are not covering maps, then

\[
B(g, h, d_1, d_2) = \infty \quad \text{(Theorem 10)}.
\]

When \( g = 1 \) and \( h > 1 \) then

\[
B(1, h, d_1, d_2) = \infty \quad \text{(Proposition 9)}.
\]

For the case when the target is the sphere \( S^2 \), we get \( B(0, h, d_1, d_2) = d_2 - d_1 \) because \( S^2 \) is simply connected and there is only one homotopy class of maps of every given degree.
There is a similar question for the root case. It is related to the coincidence question, at least if the target is the torus, but one does not follow from the other immediately. So we look at the root case separately. For this we define in a similar fashion

$$BR(g, h, d) := \sup \{|\text{ind}(R)| \mid |\deg(f)| = d \},$$

where the supreme is taken for all root classes $R$ of all maps $f : S_h \to S_g$ with the given degree.

We will show:

$$BR(g, h, 0) = 0 \quad \text{(Introduction of Section 5)}.$$  

When $d > 0$, then

$$BR(1, 1, d) = 1 \quad \text{(Proposition 6)}.$$

When $h > 1$, then

$$BR(1, h, d) = d \quad \text{(Proposition 6)}.$$  

When $g > 1$ and $d < (h − 1)/(g − 1)$, then

$$BR(g, h, d) = d \quad \text{(Theorem 8)};$$

when $d = (h − 1)/(g − 1)$, then

$$BR(g, h, d) = 1 \quad \text{(Theorem 8)}.$$

The paper is divided into 7 sections. Section 2 contains some useful elementary facts about Nielsen coincidence classes in relation to composition of maps. We also recall a convenient formula for computing the Lefschetz coincidence number for maps between closed surfaces. In Section 3 we describe a cyclic covering of a surface in terms of the fundamental group. This will be used to study the Reidemeister classes. The index bounds $B(g, g, 1, 1)$ and $B(g, g, 0, 1)$ are determined in Section 4. The index bounds for the root classes are obtained in Section 5. The unboundedness results described above are given in Section 6. Section 7 gives a boundedness result.

Finally we point out that, except for few special situations, it remains open the question of deciding if $B(g, h, d_1, d_2)$ is bounded or not when $d_2 = (h − 1)/(g − 1)$.

### 2. Nielsen coincidence classes

Some useful elementary facts about Nielsen coincidence classes:

**Lemma 1.** Suppose $f, g : X \to Y$ and $h : Y \to Z$ are maps. Then:

(i) $\text{Coin}(f, g) \subset \text{Coin}(h \circ f, h \circ g)$.

(ii) If $h$ is an embedding, then $\text{Coin}(f, g) = \text{Coin}(h \circ f, h \circ g)$.

(iii) Every Nielsen coincidence class $C$ of $(f, g)$ is contained in a Nielsen coincidence class of $(h \circ f, h \circ g)$. 


Lemma 2. Suppose $h$ are both liftings of $h$ hence $C$ transversality, up to homotopy of $f$ $g(x)$ a homotopy $(f, g)$ $h$ covering map $x$ need to show that $\lambda$ path $X, Y, Z$ $h$ in fact, the homotopy lifting property of the covering map $h$ guarantees the existence of a homotopy $f \circ \lambda \simeq \gamma$, where $\gamma : I \to Y$ is a lifting of the path $h \circ g \circ \lambda$. Since $gamma$ and $g \circ \lambda$ are both liftings of $h \circ g \circ \lambda$ and they start at the same point $\gamma(0) = f \circ \lambda(0) = f(x) = g(x) = g \circ \lambda(0)$, hence $\gamma = g \circ \lambda$. Thus $f \circ \lambda \simeq g \circ \lambda$ as desired.

When $X, Y, Z$ are orientable closed smooth manifolds of equal dimension, then by transversality, up to homotopy of $f$ and $g$ we may assume that $\text{Coin}(f, g)$ is finite set, hence $C$ is also finite. The index equality is now clear. □

Lemma 2. Suppose $f, g : X \to Y$ and $h : W \to X$ are maps. Then:

(i) $\text{Coin}(f \circ h, g \circ h) = h^{-1}(\text{Coin}(f, g))$.

(ii) For every Nielsen coincidence class $C'$ of $(f \circ h, g \circ h)$, $h(C')$ is contained in a Nielsen coincidence class of $(f, g)$.

(iii) If $h$ is a covering map, then for every Nielsen coincidence class $C'$ of $(f \circ h, g \circ h)$, $h(C')$ is a Nielsen coincidence class of $(f, g)$, and $C'$ covers all points of $h(C')$ the same number of times. Moreover, when $W, X, Y$ are orientable closed smooth manifolds of equal dimension, then

$$\text{ind}(C'; f \circ h, g \circ h) = m \cdot \text{ind}(h(C'); f, g),$$

where $m$ is the number of times that $C'$ covers $h(C')$.

(iv) If $h$ is a primitive map (i.e., the induced homomorphism $h_* : \pi_1(W) \to \pi_1(X)$ is surjective) onto $X$, then for every Nielsen coincidence class $C$ of $(f, g)$, $h^{-1}(C)$ is a Nielsen coincidence class of $(f \circ h, g \circ h)$.

In other words, $h : \text{Coin}(f \circ h, g \circ h) \to \text{Coin}(f, g)$ induces a bijection of the Nielsen classes. Moreover, when $W, X, Y$ are orientable closed smooth manifolds of equal dimension, then

$$\text{ind}(h^{-1}(C); f \circ h, g \circ h) = \deg(h) \cdot \text{ind}(C; f, g).$$

Proof. The proofs of (i) and (ii) are straightforward. To prove (iv), we need to show the following: For any Nielsen coincidence class $C$ of $(f, g)$ and any two points $x_0, x_1 \in C$, $h^{-1}(x_0) \cap C'$ and $h^{-1}(x_1) \cap C'$ have the same number of points. In fact, since $C$ is a Nielsen coincidence class of $(f, g)$, there is a path $\lambda : I \to X$ from $x_0$ to $x_1$ such that $f \circ \lambda \simeq g \circ \lambda$. By the path lifting property of the covering map $h$, $\lambda$ can be lifted into $W$. Each lifting $\lambda'$ of $\lambda$ connects a point of $h^{-1}(x_0)$ to a point of $h^{-1}(x_1)$, and since $f \circ h \circ \lambda' \simeq g \circ h \circ \lambda'$, both ends belong to the same Nielsen
coincidence class in Coin\((f \circ h, g \circ h)\). In this way we obtain a one-to-one correspondence
\(\phi : h^{-1}(x_0) \to h^{-1}(x_1)\) which sends \(h^{-1}(x_0) \cap C'\) onto \(h^{-1}(x_1) \cap C'\).

When \(W, X, Y\) are orientable closed smooth manifolds of equal dimension, then as before we may assume that Coin\((f, g)\) is a finite set, hence \(C\) is also finite. The index of \(C'\) is then clearly a multiple of that of \(C\).

To prove (iv), let \(x_0, x_1 \in C, w, w_i \in h^{-1}(x_i), i = 0, 1,\) we will show that \(w_0, w_1\) are in the same Nielsen coincidence class of \((f \circ h, g \circ h)\). Let \(\lambda : I \to X\) be a path from \(x_0\) to \(x_1\) such that \(f \circ \lambda \simeq g \circ \lambda\). Since \(h\) is a primitive map, there is a path \(\lambda' : I \to W\) such that \(h \circ \lambda' \simeq \lambda\). Hence \(f \circ h \circ \lambda' \simeq g \circ h \circ \lambda'\), so \(w_0, w_1\) are in the same Nielsen coincidence class of \((f \circ h, g \circ h)\). When \(W, X, Y\) are orientable closed smooth manifolds of equal dimension, then as before we may assume that Coin\((f, g)\) is a finite set. By [3], up to homotopy of \(h\) we can also assume that there is an open disk \(U \in X\) such that \(h^{-1}(U)\) has \(d = \deg(h)\) components, and each component is mapped by \(h\) homeomorphically onto \(U\). Up to isotopy in \(X\), we can further assume that the finite set Coin\((f, g)\) lies in \(U\). So the index formula follows.

Now we turn to coincidence of maps between orientable closed surfaces. Let \((f, f') : S_h \to S_g\) be a pair of maps, and let \(A, B\) be the matrices of \(f_* , f'_* : H_1(S_h) \to H_1(S_g)\) respectively. We decompose \(A, B\) into \(2 \times 2\) blocks:

\[
A = \begin{pmatrix}
A_{1,1} & A_{1,2} & \ldots & A_{1,g} \\
A_{2,1} & A_{2,2} & \ldots & A_{2,g} \\
\vdots & \vdots & \ddots & \vdots \\
A_{h,1} & A_{h,2} & \ldots & A_{h,g}
\end{pmatrix}, \quad B = \begin{pmatrix}
B_{1,1} & B_{1,2} & \ldots & B_{1,g} \\
B_{2,1} & B_{2,2} & \ldots & B_{2,g} \\
\vdots & \vdots & \ddots & \vdots \\
B_{h,1} & B_{h,2} & \ldots & B_{h,g}
\end{pmatrix},
\]

where \(A_{i,j}, B_{i,j}\) are the blocks corresponding to the pair \(\{a_{2i-1}, a_{2j}\}\) in \(S_h\) and the pair \(\{a_{2j-1}, a_{2j}\}\) in \(S_g\).

A convenient formula for the Lefschetz coincidence number is given in [6].

**Theorem 3.** The Lefschetz coincidence number of the pair \((f, f')\) is

\[
L(f, f') = \sum_{i,j} \det(A_{i,j} - B_{i,j}) - (g - 1)(\deg(f) + \deg(f')).
\]

**Proof.** This is a restatement of [6, Theorem 1.1], taking into account the well known fact that

\[
\sum_{i=1}^h \det A_{i,j} = \deg(f)
\]

for any \(1 \leq j \leq g\).
3. A covering map

Suppose $g \geq 2$ and $n \geq 1$. Let $h = n(g - 1) + 1$. Then there are covering maps $S_h \to S_g$ of degree $n$. A cyclic covering is described below.

Let $F_{2g}$ be the free group with base $\{a_i\}_{1 \leq i \leq 2g}$. Let $R_{2g}$ be the word

$$R_{2g} := \prod_{i=1}^{g} [a_{2i-1}, a_{2i}] = [a_1, a_2] \cdots [a_{2g-1}, a_{2g}].$$

The fundamental group $\pi_1(S_g)$ has a presentation

$$\pi_1(S_g) = F_{2g}/\langle R_{2g} \rangle,$$

where $\langle R_{2g} \rangle$ denotes the normal subgroup generated by $R_{2g}$.

A homomorphism $\theta = \theta_{g,n} : F_{2h} \to F_{2g}$ is defined by

$$\theta_{g,n} : \begin{cases} a_1 \mapsto a_1^n, \\ a_2 \mapsto a_2, \\ a_i \mapsto a_1^q a_i a_1^{-q} & \text{if } i = j + q(2g - 2), \text{ where } 3 \leq j \leq 2g \text{ and } 0 \leq q \leq n - 1. \end{cases} \tag{1}$$

Using the identity $[\alpha^n, \beta] = \prod_{q=n-1}^{0} (\alpha^n \lbrack \alpha, \beta \rbrack \alpha^{-q})$, we get

$$\theta_{g,n}(R_{2h}) = [a_1^n, a_2] \prod_{q=0}^{n-1} \left( a_1^q \prod_{j=2}^{g} [a_{2j-1}, a_{2j}] \cdot a_1^{-q} \right)$$

$$= \left( \prod_{q=n-1}^{0} a_1^q [a_1, a_2] a_1^{-q} \right) \left( \prod_{q=0}^{n-1} a_1^q \cdot ([a_1, a_2]^{-1} R_{2g}) \cdot a_1^{-q} \right).$$

Under the projection $F_{2g} \to \pi_1(S_g)$, the element $R_{2g}$ goes to 1, so $\theta(R_{2h})$ also goes to 1. Thus, $\theta$ induces a homomorphism $\theta : \pi_1(S_h) \to \pi_1(S_g)$, hence a map which we still denote by $\theta : S_h \to S_g$.

The homomorphism $\theta_* : H_1(S_h) \to H_1(S_g)$ has matrix

$$A = \begin{pmatrix} 1 \\ I_{2g-2} \\ \vdots \\ I_{2g-2} \end{pmatrix} \text{ n blocks}$$

where $I_k$ denotes the $k \times k$ identity matrix. It is clear that $\deg(f) = n$. Since $\chi(S_h) = n \chi(S_g)$, so by [17, Corollary 3.3.9], $\theta$ is a covering map up to homotopy.
4. The cases $B(g, g, 1, 1)$ and $B(g, g, 0, 1)$

In this section we deal with the case when $g = h$ and $d_2 = 1$. This means that the second map $f_2 : S_g \to S_g$ has degree $\pm 1$, hence, up to homotopy, can be assumed to be a homeomorphism. By routine argument we know that the inclusion $\text{Coin}(f_1, f_2) \to \text{Fix}((f_2^{-1} \circ f_1))$ is a homeomorphism which preserves the indices of the Nielsen classes, and $|\deg(f_1)| = |\deg(f_2^{-1} \circ f_1)|$. We conclude that when $g = h$ and $d_2 = 1$ the coincidence question is equivalent to the fixed point question. So this section will deal with indices of Nielsen fixed point classes.

In [9] the following result is proved:

**Theorem 4.** Let $F$ be a compact connected surface other than $S^2$ or $\mathbb{RP}^2$, and let $f : F \to F$ be a self-mapping. Then

$$
\sum_{\text{index}(\sigma) \geq 1} (\text{index}(\sigma) - 1) \leq 0, \quad \text{and} \quad \sum_{\text{index}(\sigma) \leq -1} (\text{index}(\sigma) + 1) \geq 2\chi(F),
$$

where $\sigma$ denotes a Nielsen fixed point class of $f$.

Let us recall that in order to prove the above result in [9] the same formula is shown to be true for a one dimensional complex. Now we will show that this lower bound is the best possible among all homeomorphisms $f$, i.e., $B(g, g, 1, 1) = 4g - 3$. Also we will show that for all maps $f$ of degree zero the best lower bound is $2g - 1$, i.e., $B(g, g, 0, 1) = 2g - 1$. In order to show the first equality, we construct an example (Example 1) of a map $f : S_2 \to S_2$ such that one Nielsen class of $f$ has index $-5 = 2\chi(S_2) - 1$ and all others have index $-1$. We also show how to generalize this example to a map $f_g : S_g \to S_g, g \geq 2$. See Example 2.

For the calculation of $B(g, g, 0, 1)$, in Proposition 5 we first show that $2g - 1$ is an upper bound for $B(g, g, 0, 1)$. Then we construct an example (Example 4) which has a Nielsen class of index $1 - 2g$.

**Example 1.** This is an example of a pseudo-Anosov map on the surface $S_2$.

Consider the primitive branched cover $p : S_2 \to T$ given in [7], which has $\deg(p) = 3$ and has only one branched point $x_0$ for which $p^{-1}(x_0)$ is a single point $y_0$. In a neighborhood of the point $y_0$ the map $p$ looks like the complex function $z \mapsto z^3$. Let $T' = T - \{x_0\}$ and $S_2' = S_2 - \{y_0\}$. Then $p$ restricts to a 3-fold covering map $p' : S_2' \to T'$.

Using the generators $z_1, z_2, z_3, z_4 \in \pi_1(S_2)$ given by Fig. 1 in [7], and the standard generators $a, b \in \pi_1(T')$, we have:

$$p'_a(z_1) = aba^{-1}, \quad p'_a(z_2) = b^{-1}aba^{-2}, \quad p'_a(z_3) = b^{-3}ab, \quad p'_a(z_4) = b^2.$$

We would like to construct a pseudo-Anosov map of $S_2$ by lifting a suitable Anosov map of $T$ into the branched cover $S_2$.

Take the Anosov diffeomorphism $f : T \to T$ which induces in homology the matrix

$$f_* = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}.$$
We have $\deg(f) = 1$ and $L(f) = -1$. The map $f$ has only one fixed point, which we can assume to be $x_0$, and it has index $-1$. The map $f$ restricted on $T'$ is a homeomorphism $f' : T' \to T'$. The induced homomorphism $f'_q : \pi_1(T') \to \pi_1(T')$ is given by (see [13]):

$$f'_q(a) = aba^2, \quad f'_q(b) = a^{-1}.$$

We claim that the iterate $q = f^3 = f \circ f \circ f$ does admit a lift $\tilde{q}$:

$$\begin{array}{c c c}
S'_2 & \xrightarrow{\tilde{q}} & S'_2 \\
p' \downarrow & & \downarrow p' \\
T' & \xrightarrow{q'} & T'
\end{array}$$

Clearly it suffices to prove that the map $q' = f'^3 : T' \to T'$ admits a lift $\tilde{q}'$:

$$\begin{array}{c c c}
S'_2 & \xrightarrow{\tilde{q}'} & S'_2 \\
p' \downarrow & & \downarrow p' \\
T' & \xrightarrow{q'} & T'
\end{array}$$

By the theory of covering spaces, this again reduces to the problem of finding a lifting homomorphism $\varphi$ for the diagram

$$\begin{array}{c c c}
\pi_1(S'_2) & \xrightarrow{\varphi} & \pi_1(S'_2) \\
p'_s \downarrow & & \downarrow p'_s \\
\pi_1(T') & \xrightarrow{q'_s} & \pi_1(T')
\end{array}$$

where $q'_s = f'^3$.

By direct computation,

$$q'_s p'_s(z_1) = aba^{-1}b^{-1}a^{-2}b^{-1}a^{-3}b^{-1}a^{-2}b^{-1}a^{-1} = (aba^{-1})(b^{-1}a^{-1}b)^2(b^{-2})(b^{-1}a^{-1}b)$$

$$= (a^{-1}aba^{-2})(a^{-1}b^{-1}a^{-1})(a^{-1}b^{-1}a^{-1}).$$

$$q'_s p'_s(z_2) = aba^2ba^3ba^{-1}b^{-1}a^{-2}b^{-1}a^{-3}b^{-1}a^{-2}b^{-1}a^{-1}a^{-3}b^{-1}a^{-2}b^{-1}a^{-1}$$

$$= (aba^2)(a^{-2})(a^{-1}b^{-1}a^{-1})(a^{-1}b^{-1}a^{-1})(a^{-1}b^{-1}a^{-1}).$$

$$q'_s p'_s(z_3) = aba^2ba^3ba^2ba^2ba^2ba^2 = (aba^2)(a^{-2})(ba^3ba^2)^3,$$

$$q'_s p'_s(z_4) = a^{-2}b^{-1}a^{-3}b^{-1}a^{-2}b^{-1}a^{-3}b^{-1}a^{-2}b^{-1}a^{-1}$$

$$= (a^{-2})(b^{-1}a^{-2}b^{-1}a^{-2})(a^{-1}b^{-1}a^{-1}).$$

We now observe that there exists a lift $\varphi$ defined as follows:
As we see, \( L(\tilde q) = 1 \) is a pair of induced transverse measured foliations on \( S_f \) of the index, is finite. (See [8, Theorem 7.2.9].) Therefore we have a lift homomorphism by \( \tilde q \) of the Nielsen class that

\[
\begin{align*}
\varphi(z_1) &= z_1 \cdot z_3^{-2} \cdot z_4^{-1} \cdot z_3 \cdot (z_3^{-1} z_2 z_1^{-1})^2, \\
\varphi(z_2) &= (z_1 z_2^{-1} z_3^2 \cdot z_2^{-1} z_3 \cdot z_4 z_3^3 \cdot (z_3^{-1} z_2 z_1^{-1})^2 \cdot z_3^{-1} z_2 \\
& \quad \cdot z_1^{-1} \cdot z_3^{-1} z_2 \cdot z_3^{-1} \cdot z_4 \cdot z_3^{-1} z_2 \cdot (z_3^{-1} z_2 z_1^{-1})^2, \\
\varphi(z_3) &= (z_1 z_2^{-1} z_3^2 \cdot z_2^{-1} z_3 \cdot (z_4 z_3^3 \cdot z_2^{-1} z_3)^3, \\
\varphi(z_4) &= z_3^{-1} z_2 \cdot \left( z_3^{-3} z_4 z_3^{-1} z_2^2 \right) \cdot (z_3^{-1} z_2 z_1^{-1})^2.
\end{align*}
\]

Thus we have found a lifting \( \tilde q : S_2 \to S_2 \) which maps \( y_0 \) to \( y_0 \). The induced homomorphism by \( \tilde q \) on \( H_1 \) is given by the matrix

\[
\begin{pmatrix}
-1 & 3 & -5 & -1 \\
-4 & 9 & -12 & -2 \\
2 & -6 & 15 & 3 \\
-2 & 5 & -11 & -2
\end{pmatrix}
\]

As we see, \( L(\tilde q) = 1 \) is an Anosov map, by [4] there is a pair of transverse measured foliations preserved by \( f \) hence by its iterates. Through the branched covering \( p \), there is a pair of induced transverse measured foliations on \( S_2 \), where the 6-prong point \( y_0 \) is the only singularity, and \( \tilde q \) is a pseudo-Anosov map with respect to this pair of foliations. By [10, Section 2] each Nielsen class is a single point. At the singularity \( y_0 \), we may assume that \( \tilde q \) keeps the prongs invariant. (If not, replace it with a further iterate.) Then the index of the Nielsen class \( \{ y_0 \} \) is \(-5\). Since \( 2 \chi(S_2) - 1 = -5 \), we have that \( 2 \chi(S_2) - 1 \) is the best lower bound for Nielsen fixed point classes and the result follows.

**Example 2.** Similarly we can construct a pseudo-Anosov map \( S_{g'} \to S_g \) with a fixed point class of index \( 3 - 4g = 2 \chi(S_g) - 1 \).

Let \( p_g : S_{g'} \to T \) be the branched cover of degree \( 2g - 1 \), constructed in [7], which has only one branched point \( x_0 \), and \( p_g^{-1}(x_0) \) is one point \( y_0 \). Let \( T' = T - \{ x_0 \} \) and \( S'_{g'} = S_g - \{ y_0 \} \). The unbranched cover \( S'_{g'} \to T' \) corresponds to a subgroup \( H \) of \( \pi_1(T') \) of index \( 2g - 1 \).

Consider \( f : T \to T \) as in Example 1, and keep the same convention on notation.

If \( f'(H) \subset H \), then we have a lift \( \tilde f : S_{g'} \to S_g \) of \( f \), and the rest of the proof follows the same steps as in Example 1.

If \( f'(H) \not\subset H \), some iterate of \( f \), i.e., \( q = f^k \), will have the property that \( q'(H) \subset H \). This follows from the fact that the number of subgroups of finite index, for any given value of the index, is finite. (See [8, Theorem 7.2.9].) Therefore we have a lift \( \tilde q : S_g \to S_g \) of \( q \). Now we can do the same to \( \tilde q \).

Now we will show that \( 2g - 1 \) is an upper bound for \( B(g, g, 0, 1) \).

**Proposition 5.** \( B(g, g, 0, 1) \leq 2g - 1 \).
Proof. Let \( f : S_g \to S_g \) be a map of degree zero. Then by [11], or [3] for a more modern version, the map factors through the one skeleton. By [16] or [12, Proposition 7.3] the image of the fundamental group, \( f_\pi(\pi_1(S_g)) \) is a free subgroup of rank at most \( g \). So, up to homotopy, the map \( f \) in fact factors through the bouquet \( K \) of at most \( g \) circles as the composite of a map \( f' : S_g \to K \) with a map \( j : K \to S_g \). Now by the commutativity property of the fixed point index, the Nielsen classes of \( f \) have the same indices as that of the composite of the map \( j \) with \( f' \). By [9] the indices of the Nielsen classes of \( f' \circ j \) have \( 1 - 2g \) as a lower bound and the result follows. \( \square \)

Example 3. Consider the bouquet of \( 2g \) circles \( S^1_i, i = 1, \ldots, 2g \). Map each circle for \( i \) odd into itself by a map of degree 2 and the others to the base point. This map certainly extends to a self map of \( S_g \) and it has degree zero. The only fixed point is the base point. Therefore there is only one Nielsen class whose index equal to the Lefschetz number \( 1 - 2g \). It follows that \( B(g, g, 0, 1) = 2g - 1 \).

5. The root case

In this section we will assume that we have a map \( f : S_h \to S_g \) of degree different from zero. Otherwise the Nielsen number is zero by Brooks [1], therefore \( BR(g, h, 0) = 0 \). If the target is the sphere \( S^2 \), then it is easy to see that \( BR(0, h, d) = d \). First let us consider the case \( f : S_h \to T \) where the target is the torus.

Proposition 6. \( BR(1, 1, d) = 1 \) and \( BR(1, h, d) = d \) for \( h > 1 \).

Proof. The first part \( BR(1, 1, d) = 1 \) is well known. Since the index of a root class is the degree of the map divided by the index of the image of the fundamental group of \( S_h \) in \( \pi_1(T) \), it suffices to construct maps \( f : S_g \to T \) which are primitive (i.e., \( f_\pi : \pi_1(S_h) \to \pi_1(T) \) is surjective) and has degree \( d \). For \( h = 2 \) consider for example a map \( f \) inducing on \( H_1 \) the matrix

\[
\begin{pmatrix}
  d - 1 & 0 \\
  0 & 1 \\
  1 & 0 \\
  0 & 1
\end{pmatrix}
\]

We have only one root class and the index is \( d \). By composing this map with a pinch map \( S_h \to S_2 \) we get the result for arbitrary genus \( h \). \( \square \)

Now let the target be the surface \( S_g \) for \( g \geq 2 \). Let \( f : S_h \to S_g \) be a map.

Proposition 7. There are primitive maps \( f : S_h \to S_g \) for all values of \( |\deg(f)| < (h - 1)/(g - 1) \).
Proof. A specific construction is provided later. See the \( \psi_d \) in the proof of Theorem 10. ✷

Theorem 8. For \( g > 1 \) the bound \( \text{BR}(g, h, d) \) is either 1 or \( d \) depending on whether \( d = (h - 1)/(g - 1) \) or \( d < (h - 1)/(g - 1) \).

Proof. For any map \( f : S_h \rightarrow S_g \) we know by [11] that

\[
|\text{deg}(f)| \leq \frac{h - 1}{g - 1}.
\]

If \( |\text{deg}(f)| = (h - 1)/(g - 1) \), then \( f \) is homotopic to an unbranched covering. So \( |\text{deg}(f)| \) is equal to the subgroup index \([\pi_1(S_g) : f\pi_1(S_h)]\) and hence \( \text{BR}(g, h, d) = 1 \).

Now suppose that \( d < (h - 1)/(g - 1) \). By Proposition 7 we have a primitive map \( f \) of degree \( d \). Hence \( f \) has only one root class. It follows that \( \text{BR}(g, h, d) = d \). ✷

6. Some unboundedness results

We will consider first the cases \( B(1, h, d_1, d_2) \) for \( h > 1 \). (It is well known that \( B(1, 1, d_1, d_2) = 1 \) even if \( d_1 = d_2 = 0 \).)

Proposition 9. \( B(1, h, d_1, d_2) = \infty \) for \( h > 1 \).

Proof. For a fixed integer \( r \) we will consider two maps \( f_1, f_2 : S_2 \rightarrow T \) of degrees \( d_1, d_2 \) respectively. In \( H_1 \) they induce the matrices

\[
\begin{pmatrix}
  r + 1 & r \\
  1 & 1 \\
  0 & 1 \\
  1 - d_1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  1 & 0 \\
  0 & 1 \\
  d_2 - 1 & 0 \\
  0 & 1
\end{pmatrix}.
\]

By Theorem 3 we get that the Lefschetz number \( L(f_1, f_2) \) is equal to \( d_1 + d_2 - 2 - r \) which is certainly unbounded when \( r \) goes to infinity. But there is only one Reidemeister class, independent of the value \( r \). So the result follows. The general case for the domain \( S_h \) is obtained from this last one just by composing with a pinching map \( S_h \rightarrow S_2 \). ✷

Remark 1. The torus is a Lie group. The coincidence question of a pair of maps \( (f_1, f_2) : S_h \rightarrow T \) is equivalent to a root question of the map \( f_1/f_2 : S_h \rightarrow T \) defined by \( (f_1/f_2)(x) = (f_1(x)) \cdot (f_2(x))^{-1} \). But the degree of \( f_1/f_2 \) is not determined by that of \( f_1 \) and \( f_2 \), i.e., the degree of \( f_1/f_2 \) is not constant when \( f_1, f_2 \) run over maps of degree \( d_1, d_2 \) respectively.

So Proposition 9 does not contradict the result \( \text{BR}(1, h, d) = d \) of Proposition 6.

Theorem 10. When \( 2 \leq g \leq h \) and \( 0 \leq d_1, d_2 < (h - 1)/(g - 1) \), hence neither of \( f_1, f_2 \) is homotopic to a covering map, then

\[
B(g, h, d_1, d_2) = \infty.
\]
Before going into the proof, we define some special maps. We shall define homomorphisms \( \phi_d, \psi_d : F_{2h} \to F_{2g} \), where \( d \geq 0 \) is an integer with \( h \geq d(g - 1) + 2 \). The function \( \theta_{g,n} \) refers to the formula (1). The integer \( r \) is a parameter.

The homomorphisms \( \phi_0, \psi_0 : F_{2h} \to F_{2g} \) are given by

\[
\phi_0 : \begin{cases} 
  a_i \mapsto 1 & \text{for all } i, \text{ except } \\
  a_{2h-1} \mapsto a_{2g}'. 
\end{cases}
\]

\[
\psi_0 : \begin{cases} 
  a_{2h-2} \mapsto a_{2g}, \\
  a_{2h} \mapsto a_{2g-1}. 
\end{cases}
\]

When \( d \geq 1 \) and \( h > d(g - 1) + 2 \), define \( \phi_d, \psi_d : F_{2h} \to F_{2g} \) by

\[
\phi_d : \begin{cases} 
  a_i \mapsto 1 & \text{for } i = 1, 2, \\
  a_i \mapsto \theta_{g,d}(a_{i-2}) & \text{if } 3 \leq i \leq d(2g - 2) + 4, \\
  a_i \mapsto 1 & \text{if } i > d(2g - 2) + 4, \text{ except } \\
  a_{2h-1} \mapsto a_{2g}'. 
\end{cases}
\]

\[
\psi_d : \begin{cases} 
  a_i \mapsto a_i & \text{for } 1 \leq i \leq 2g, \\
  a_i \mapsto \theta_{g,d-1}(a_{i-2g}) & \text{if } 2g + 1 \leq i \leq d(2g - 2) + 4, \\
  a_i \mapsto 1 & \text{if } i > d(2g - 2) + 4, \text{ except } \\
  a_{2h} \mapsto a_{2g-1}. 
\end{cases}
\]

When \( d \geq 1 \) and \( h = d(g - 1) + 2 \), they are

\[
\phi_d : \begin{cases} 
  a_i \mapsto 1 & \text{for } i = 1, 2, \\
  a_i \mapsto \theta_{g,d}(a_{i-2}) & \text{if } 3 \leq i \leq 2h - 2, \\
  a_{2h-1} \mapsto a_1^{d-1}a_{2g-1}a_{2g-1}^{1-d}, \\
  a_{2h} \mapsto a_1^{d-1}a_{2g}a_{1}^{d}. 
\end{cases}
\]

\[
\psi_d : \begin{cases} 
  a_i \mapsto a_i & \text{for } 1 \leq i \leq 2g, \\
  a_i \mapsto \theta_{g,d-1}(a_{i-2}) & \text{if } 2g + 1 \leq i \leq 2h - 2, \\
  a_{2h-1} \mapsto a_1^{d-2}a_{2g-1}a_{2g-1}^{1-d}, \\
  a_{2h} \mapsto a_1^{d-2}a_{2g}a_{2g-1}a_{1}^{2-d}. 
\end{cases}
\]

Clearly \( \phi_0(R_{2h}) = \psi_0(R_{2h}) = 1 \). Using the identities \( [\alpha \beta^n, \beta] = [\alpha, \beta \alpha^n] = [\alpha, \beta] \), we see that for \( d \geq 1 \),

\[
\phi_d(R_{2h}) = \theta_{g,d}(R_{2h}), \quad \psi_d(R_{2h}) = R_{2g} \cdot \theta_{g,d-1}(R_{2h}).
\]
both go to 1 in $\pi_1(S_g)$. Thus, $\phi_d, \psi_d$ induce homomorphisms $\pi_1(S_h) \to \pi_1(S_g)$, hence also maps $\phi_d, \psi_d : S_h \to S_g$, with $\deg(\phi_d) = \deg(\psi_d) = d$.

The above construction has some useful features.

$$\phi_d(F_{2k}) \subset F_{2k-2}; \quad \phi_d^g(F_{2g}) = 1. \quad (2)$$

$$\psi_d(w) = w \quad \text{when } w \in F_{2g} \text{ and } d \geq 1. \quad (3)$$

Let $A$ and $B$ denote the matrices of $\phi_d^*, \psi_d^* : H_1(S_h) \to H_1(S_g)$ respectively. Their $2 \times 2$ blocks have very limited possibilities.

$$A_{i,j}, B_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 \leq n \leq d, \quad \text{if } (i,j) \neq (h,g).$$

On the other hand,

$$A_{h,g} = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix},$$

$$B_{h,g} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

**Proof of Theorem 10.** Assume that $0 \leq d_1 \leq d_2 < (h-1)/(g-1)$. Take $f_1 = \phi_{d_1}$ and $f_2 = \psi_{d_2}$, and let $C$ be a coincidence class of the pair $(f_1, f_2)$. To show that $\text{ind}(C)$ can be arbitrarily large, it suffices to prove the following two claims.

**Claim 1.** $L(\phi_{d_1}, \psi_{d_2}) \to \infty$ when $r \to \infty$.

This is in fact an immediate consequence of Theorem 3 and the above analysis of the possible forms of $A_{i,j}$ and $B_{i,j}$.

**Claim 2.** $C$ is the unique Nielsen coincidence class of $(\phi_{d_1}, \psi_{d_2})$, so that $\text{ind}(C) = L(\phi_{d_1}, \psi_{d_2})$.

Proof of Claim 2 divides into two cases. First the case $d_2 > 0$. Suppose the coordinate of a coincidence point $x \in S_h$ is the Reidemeister class $[w]_R$, where $w$ is a word in $F_{2g}$. Since

$$w \sim_R \phi_{d_1}(w) \cdot w \cdot \psi_{d_2}(w^{-1}) = \phi_{d_1}(w)$$

by (3), we have

$$w \sim_R \phi_{d_1}(w) \sim_R \ldots \sim_R \phi_{d_1}^{g}(w) = 1$$

by (2). Thus, all coincidences have the same Reidemeister class $[1]_R$ as coordinate. Hence there is only one Nielsen coincidence class.

Consider the case $d_1 = d_2 = 0$. Let $W$ be the last handle of $S_g$, which is a punctured torus. The fundamental group $\pi_1(W)$ is the free group with basis $\{a_{2g-1}, a_{2g}\}$, and the inclusion $W \subset S_g$ induces the inclusion $\pi_1(W) \subset \pi_1(S_g)$. Both maps $\phi_0$ and $\psi_0$ factor
through $W$. So, by Lemma 1(ii) and (iv), it suffices to show that the maps $\phi_0, \psi_0 : S_h \to W$ have only one Nielsen coincidence class.

Now suppose the coordinate of a coincidence point $x \in S_h$ is the Reidemeister class $[w]_R$, where $w = w(a_{2h-1}, a_{2h}) \in \pi_1(W)$ is a word in $a_{2h-1}, a_{2h}$. Let $u$ denote the corresponding word in $a_{2h}, a_{2h-2}$, i.e., $u = w(a_{2h}, a_{2h-2}) \in F_{2h}$. Then by definition $\phi_0(u) = 1$ and $\psi_0(u) = w$, hence $w \sim_R \phi_0(u) \cdot w \cdot \psi_0(u^{-1}) = 1$.

Thus $\phi_0, \psi_0 : S_h \to W$ have a unique Nielsen coincidence class.

This completes the proof of the theorem. $\blacksquare$

7. A boundedness result

There remains the reasonable possibility that $B(g, h, d_1, d_2)$ is finite when one leg of the pair $(f_1, f_2)$ is homotopic to a covering map. We explore this possibility in this section.

Suppose $h > g > 1$ and $p : S_h \to S_g$ is a covering map. Then $h - 1$ is a multiple of $g - 1$, we denote $m = (h - 1)/(g - 1)$. Let $f : S_h \to S_g$ be a map, and let $C$ be an arbitrary Nielsen coincidence class of the pair $(f, p)$. Look at its index $\text{ind}(C) = \text{ind}(C; f, p)$. Are such indices bounded?

We seek to derive boundedness results from that in fixed point theory. Take a point $x_0 \in C$ as the base point in $S_h$, and take $x'_0 := f(x_0) = p(x_0)$ as the base point in $S_g$. For the sake of brevity, we shall write $G = \pi_1(S_g)$ and $H = \pi_1(S_h)$. Thus $p : H \to G$ is injective.

Proposition 11. Assume that there exists a subgroup $K \subset H$ such that

$$[H : K] = n < \infty \quad \text{and} \quad f_* (K) \subset p_* (K).$$

(*)

Then $|\text{ind}(C)| \leq |2n\chi(S_h) - 1|$.

Proof. The finite cover of $S_h$ corresponding to $K \subset H$ is an orientable closed surface $S_k$ of genus $k$, where $\chi(S_k) = n\chi(S_h)$. Let

$$q : (S_k, x'_0) \to (S_h, x_0)$$

be the covering map with $q_* (\pi_1(S_k, x'_0)) = K$. Then there is a map $f' : S_k \to S_k$ making the following diagram commutative.

$$\begin{array}{ccc}
(S_k, x'_0) & \xrightarrow{f'} & (S_k, x'_0) \\
\downarrow{q} & & \downarrow{q} \\
(S_h, x_0) & \xrightarrow{p} & (S_g, x_0)
\end{array}$$
Let $C'$ be the Nielsen coincidence class of the pair $(f', \text{id})$ containing $x'_0$. Consider the diagram

$$
\begin{array}{ccc}
\text{Coin}(f', \text{id}) & \rightarrow & \text{Coin}(f, p) \\
\downarrow \text{incl} & & \uparrow q \\
\text{Coin}(p \circ q \circ f', p \circ q) & \longrightarrow & \text{Coin}(f \circ q, p \circ q)
\end{array}
$$

Since $p \circ q$ is a covering map, by Lemma 1(iv) $C'$ is a Nielsen coincidence class of $(p \circ q \circ f', p \circ q) = (f \circ q, p \circ q)$, and clearly

$$\text{ind}(C'; f', \text{id}) = \text{ind}(C'; f \circ q, p \circ q).$$

On the other hand, since $q$ is a covering map, by Lemma 2 (iii), $C'$ covers $C$ $n'$ times, $1 \leq n' \leq n$, and

$$\text{ind}(C'; f \circ q, p \circ q) = n' \text{ind}(C; f, p).$$

Hence

$$|\text{ind}(C; f, p)| \leq |\text{ind}(C'; f', \text{id})|.$$
(f, p) and \(\text{ind}(C; f, p) = \text{ind}(C; f', \text{id}) = 3 - 4h\). Note that in this example \(f\) is strongly equivalent to \(p\).

**Remark 2.** The assumption of Proposition 11 is not always satisfied even for two covering maps that are equivalent to each other.

**Example 5.** Let \(p = \theta_2 : S_3 \rightarrow S_2\) be the covering map defined by the formula (1). With canonical generators \(\{a_1, \ldots, a_4\}\) for \(G = \pi_1(S_2)\) and \(\{a'_1, \ldots, a'_6\}\) for \(H = \pi_1(S_3)\), \(p\) and another map \(f : S_3 \rightarrow S_2\) are defined by

\[
\begin{align*}
p\pi : & \\
ap_1' & \mapsto a_4, \\
ap_2' & \mapsto a_3, \\
ap_3' & \mapsto a_2, \\
ap_4' & \mapsto a_1, \\
ap_5' & \mapsto a_1a_2a_1^{-1}, \\
ap_6' & \mapsto a_1a_3a_1^{-1};
\end{align*}
\]

\[
\begin{align*}
f\pi : & \\
a_1 & \mapsto a_4, \\
a_2 & \mapsto a_3, \\
a_3 & \mapsto a_2, \\
a_4 & \mapsto a_1, \\
a_5' & \mapsto a_5, \\
a_6' & \mapsto a_6; \\
a_1' & \mapsto a_1, \\
a_2' & \mapsto a_2, \\
a_3' & \mapsto a_3, \\
a_4' & \mapsto a_4,
\end{align*}
\]

Let \(t : S_2 \rightarrow S_2\) and \(t' : S_3 \rightarrow S_3\) be the involutions defined by

\[
\begin{align*}
t\pi : & \\
a_1 & \mapsto a_4, \\
a_2 & \mapsto a_3, \\
a_3 & \mapsto a_2, \\
a_4 & \mapsto a_1; \\
t'_\pi : & \\
a_1' & \mapsto a_1, \\
a_2' & \mapsto a_2, \\
a_3' & \mapsto a_3, \\
a_4' & \mapsto a_4,
\end{align*}
\]

Then clearly \(t \circ f = p \circ t'\). Thus both \(p\) and \(f\) are covering maps and they are equivalent.

We claim that \(p\pi, f\pi : H \rightarrow G\) do not satisfy the assumption of Proposition 11.

**Proof.** Suppose there were a subgroup \(K \subset H\) satisfying the condition (*). Since \([H : K] < \infty\), there would be a minimal integer \(k > 0\) such that \(a_1^k \in K\). Since \(f\pi(K) \subset p\pi(K)\), \(a_1^k = f\pi(a_1^k) \in p\pi(K)\). The total \(a_1\)-exponent of an element in \(p\pi(K) \subset G\) is always even, so \(k = 2k', 0 \leq k' < k\). Now \(p\pi(a_1^{k'}) = a_1^k \in p\pi(K)\), and \(p\pi\) is injective, hence \(a_1^{k'} \in K\). This contradicts the minimality of \(k\). \(\Box\)

**References**


