On certain analytic functions associated with Ruscheweyh derivatives and bounded Mocanu variation

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Abstract

Using Ruscheweyh derivative and convolution operator, we introduce a new subclass of analytic functions defined in the unit disc. Some inclusion results, a radius problem and some other interesting properties of this class are investigated.

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1. Introduction

Let $A$ denote the class of functions $f: f(z) = z + \sum_{m=2}^{\infty} a_m z^m$, which are analytic in the unit disc $E = \{z: |z| < 1\}$. Let $P_k(\gamma)$ be the class of functions $p(z)$ defined in $E$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\text{Re}(p(z) - \gamma)}{1 - \gamma} \right| d\theta \leq k\pi,$$

(1.1)

where $z = re^{it}$, $k \geq 2$ and $0 \leq \gamma < 1$. When $\gamma = 0$, we obtain the class $P_k$ defined in [8] and for $k = 2$, $\gamma = 0$, we have the class $P$ of functions with positive real part. We can write (1.1) as

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\gamma)ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k.$$

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From (1.1), we can write, for \( p \in P_k(\gamma) \),

\[
p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z),
\]

where \( z \in E \) and \( p_1, p_2 \in P_2(\gamma) = P(\gamma) \), where \( P(\gamma) \) is the class of functions with positive real part greater than \( \gamma \).

Let \( S, K, S^* \) and \( C \) denote the subclasses of \( \mathcal{A} \) which are univalent, close-to-convex, starlike and convex in \( E \), respectively. The class \( \mathcal{A} \) is closed under the Hadamard product or convolution

\[
(f \ast g)(z) = \sum_{m=0}^{\infty} a_m b_{m} z^{m+1},
\]

\[
f(z) = \sum_{m=0}^{\infty} a_m z^{m+1}, \quad g(z) = \sum_{m=0}^{\infty} b_m z^{m+1}.
\]

Let \( K_a(z) = \frac{z}{1-z^a} \), \( \Re(a) > 0 \), where we have chosen a suitable branch so that \( K_a \in \mathcal{A} \). Let \( f \in \mathcal{A} \), \( f(z) = z + \sum_{m=2}^{\infty} a_m z^m \), with properties that \( a_m \neq 0 \) for all \( m \) and \( \lim_{m \to \infty} |a_m|^{1/m} = 1 \). Then we denote by \( f^{(-1)} \) the unique well-defined function in \( \mathcal{A} \) for which \( f^{(-1)} \ast f = K_1 \).

**Definition 1.1.** Let \( f \in \mathcal{A} \) and let, for \( a > 0, \alpha \geq 0 \), \( \frac{(K_{a+1} \ast f)(z)}{z} \neq 0 \) and \( \frac{(K_a \ast f)(z)}{z} \neq 0 \), \( z \in E \),

\[
J(\alpha, a, f(z)) = \left[ \alpha(a + 1) \left( \frac{(K_{a+2} \ast f)(z)}{(K_{a+1} \ast f)(z)} - \frac{a}{a + 1} \right) + a(1 - \alpha) \left( \frac{(K_{a+1} \ast f)(z)}{(K_a \ast f)(z)} - \frac{1}{a} \right) \right].
\]

Then

\[
f \in R_k(\alpha, a, \gamma) \quad \text{if and only if} \quad J(\alpha, a, f(z)) \in P_k(\gamma) \quad \text{for} \quad z \in E.
\]

As special cases of this definition, we note the following:

(i) For \( a = 1, \alpha = 0 \), we have \( R_k(0, 1, 0) = R_k \), the class of functions of bounded radius rotation, see [6–8].

(ii) \( R_2(\alpha, 1, 0) \) is the class of alpha-starlike functions and it is well known that \( R_2(\alpha, 1, 0) \subset S \).

(iii) \( R_k(1, 1, 0) = V_k \), the well-known class of functions of bounded boundary rotation which was first introduced by Paatero [3].

(iv) \( R_k(\alpha, 1, 0) \) is the class of functions with bounded Mocanu variation [2].

(v) \( R_2(1, 1, 0) = C, R_2(0, 1, 0) = S^* \) and therefore \( R_2(1, 1, 0) \subset R_2(0, 1, 0) \subset S \), for \( z \in E \).

**Remark 1.1.** We define

\[
F_{a,b,c} = K_a^{(-1)} \ast \left( (1-c)K_{b+1} + cK_b \right).
\]

We can easily verify the following:

(i) \( f \in R_k(0, a, 0) \) if and only if \( (K_a \ast f) \in R_k(0, 1, \frac{1-a}{2}) \).

(ii) \( f \in R_k(1, a, 0) \) if and only if \( (K_{a+1} \ast f) \in R_k(0, 1, \frac{a-1}{2}) \), and \( f \in R_k(1, a, \gamma) \) if and only if \( (F_{a,a,0} \ast f) \in R_k(0, a, \gamma) \).

Let \( f \in \mathcal{A} \). Denote by \( D^\lambda : \mathcal{A} \to \mathcal{A} \) the operator defined by

\[
D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} \ast f(z), \quad (\lambda > -1).
\]

It is obvious that \( D^0 f(z) = f(z), D^1 f(z) = zf'(z) \) and

\[
D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \quad n \in N_0 = \{0, 1, 2, \ldots\}.
\]
The following identity is easily established.

\[ z(D^\lambda f)' = (\lambda + 1)D^{\lambda+1}f - \lambda D^\lambda f. \]  

(1.5)

The operator \( D^\lambda f \) is called the Ruscheweyh derivative of \( f \).

From (1.3)–(1.5), we can write \( J(\alpha, \lambda, f(z)) \), for \( \lambda > -1 \), as follows:

\[ J(\alpha, \lambda, f(z)) = \left( \alpha z \frac{(D^{\lambda+1} f(z))'}{D^{\lambda+1} f(z)} + (1 - \alpha) \frac{z(D^\lambda f(z))'}{D^\lambda f(z)} \right). \]  

(1.6)

Therefore \( f \in R_k(\alpha, \lambda, \gamma) \), \( \lambda > -1 \), if and only if \( J(\alpha, \lambda, f) \in P_k(\gamma) \) in \( E \).

2. Preliminary results

We need the following results to obtain our results.

**Lemma 2.1.** (See [9].) Let \( \beta < 1 \), \( f(z) \) be prestarlike function of order \( \beta \) and \( g \in S^*(\beta) \). Then, for any analytic function \( F(z) \) in \( E \), \( f \ast (g F) \in \mathcal{Co}(F(E)) \), where \( \mathcal{Co}(F(E)) \) stands for the closed convex hull of \( F(E) \).

For our next lemma, we first define the class \( B(\beta + i\alpha) \) of Bazilevic functions. A function \( f \in A \) is in \( B(\beta + i\alpha) \) if, for some \( h \in P, g \in S^*, \alpha \) real and \( \beta > 0 \), it can be represented as

\[ f(z) = \left( \beta + i\alpha \right) \int_0^z \xi^{\beta+i\alpha-1} \left( \frac{g(\xi)}{\xi} \right)^\beta h(\xi) d\xi \right]^{1/\beta+i\alpha}. \]

Here all powers are meant as principal values. It was shown [1] that \( f \in B(\beta + i\alpha) \) is univalent in \( E \).

**Lemma 2.2.** (See [11].) Let \( f \in A \) and \( \frac{f(z)f''(z)}{z} \neq 0 \) in \( E \). Then, \( f \in B(\beta + i\alpha) \), if and only if, for \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \) and \( 0 < r < 1 \), we have

\[ \int_{\theta_1}^{\theta_2} \left[ \text{Re} \left\{ 1 + z\frac{f''(z)}{f'(z)} + (\beta - 1) \frac{zf'(z)}{f(z)} \right\} - \alpha \text{Im} \frac{zf'(z)}{f(z)} \right] d\theta \geq -\pi, \]

where \( z = re^{i\theta}, \beta > 0 \) and \( \alpha \) real.

**Lemma 2.3.** (See [10].) Let \( p \) be an analytic function in \( E \) with \( p(0) = 1 \) and \( \text{Re}\{p(z)\} > 0, z \in E \). Then, for \( s > 0 \) and \( \mu \neq -1 \) (complex),

\[ \text{Re}\left[ p(z) + \frac{szp'(z)}{p(z) + \mu} \right] > 0 \quad \text{for } |z| < r_0, \]

where \( r_0 \) is given by

\[ r_0 = \frac{|\mu + 1|}{\sqrt{A + (A^2 - |\mu^2 - 1|^2)^{1/2}}}, \]

(2.1)

\[ A = 2(s + 1)^2 + |\mu|^2 - 1, \]

(2.2)

and this radius is best possible.
3. Main results

**Theorem 3.1.** For $\alpha > 0$, $\gamma \in [\gamma_0, 1)$ with $\gamma_0 = \text{Max}\{\frac{1-\lambda}{2} - \alpha, -\lambda\}$,

$$R_k(\alpha, \lambda, \gamma) \subset R_k(0, \lambda, \gamma), \quad z \in E,$$

where

$$\gamma_1 = \left\{ \frac{(1 + \lambda)}{2F_1(\frac{2}{\alpha}(1 - \gamma), 1, \frac{1 + \lambda + \alpha}{\alpha}; \frac{z}{z - 1})}, -\lambda \right\}$$

(3.1)

and $2F_1$ is hypergeometric function. This result is sharp.

**Proof.** Set

$$\frac{z(D^k f(z))^}{D^k f(z)} = H(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z),$$

(3.2)

where $H$ is analytic in $E$ with $H(0) = 1$. We want to show that $H \in P_k(\gamma_1)$ for $z \in E$. Now, from (1.5), (1.6) and (3.2), we have

$$J(\alpha, \lambda, f(z)) = \left\{ H(z) + \frac{\alpha z H'(z)}{H(z) + \lambda} \right\} \in P_k(\gamma), \quad z \in E.$$

Define

$$\phi_{\alpha, \lambda}(z) = (1 - \alpha) \frac{1}{(1 - z)^{k+2}} + \alpha \frac{1}{(1 - z)^{k+1}}.$$ (3.3)

From (3.2) and (3.3), we have

$$H * \phi_{\alpha, \lambda} = H + \alpha \frac{z H'}{H + \lambda} = \left( \frac{k}{4} + \frac{1}{2} \right) [h_1 * \phi_{\alpha, \lambda}] - \left( \frac{k}{4} - \frac{1}{2} \right) [h_2 * \phi_{\alpha, \lambda}]

= \left( \frac{k}{4} + \frac{1}{2} \right) \left[ h_1 + \alpha \frac{z h'_1}{h_1 + \lambda} \right] - \left( \frac{k}{4} - \frac{1}{2} \right) \left[ h_2 + \alpha \frac{z h'_2}{h_2 + \lambda} \right],$$

where $(h_i + \alpha \frac{z h'_i}{h_i + \lambda}) \in P(\gamma_i), z \in E, i = 1, 2$.

We now use a result [4, Theorem 3.3e] to obtain that $h_i \in P(\gamma_1)$, where $\gamma_1$ is given by (3.1). Sharpness is given by the function $q(z)$ defined as follows.

$$q(z) = \frac{\alpha}{g(z)} - \lambda,$$

and

$$g(z) = \int_0^1 \left[ \frac{1 + z}{1 - t z} \right]^\frac{1}{2(1 - \gamma)} t^{(\frac{1 + \lambda}{\alpha} - 1)} dt

= \left( \frac{\alpha}{1 + \lambda} \right) 2F_1\left( \frac{2}{\alpha}(1 - \gamma), 1, \frac{1 + \lambda + \alpha}{\alpha}; \frac{z}{z - 1} \right).$$

Since $h_i \in P(\gamma_1)$, $i = 1, 2$, it follows from (3.2) that $H \in P_k(\gamma_1)$ and this completes the proof. \(\square\)

As a particular case, we note that, for $\lambda = 0$, $\alpha = 1$, $\gamma = 0$, $V_k \subset R_k(\frac{1}{2})$. For $k = 2$, we obtain a well-known result that every convex univalent function is starlike of order $\frac{1}{2}$.

Following the similar technique, we can easily show that, for $-1 < \lambda_1 < \lambda_2$,

$$R_k(0, \lambda_2, \gamma_1) \subset R_k(0, \lambda_1, \gamma_1) \subset R_k(0, 0, \gamma_1), \quad z \in E,$$

and by using Theorem 3.1, it follows that $f \in R_k(\alpha, \lambda, \gamma)$ is a function of bounded radius rotation. Thus we deduce that $f \in R_2(\alpha, \lambda, \gamma)$ is starlike and hence univalent in $E$.

We now have the following.
Theorem 3.2. Let \( f \in R_k(\alpha, \lambda, \gamma) \). Then \( D^\lambda f \) univalent in \( E \), if \( k \leq \frac{2(\alpha - \gamma + 1)}{(1 - \gamma)}, \) \( \alpha > 0, \) \( 0 \leq \gamma < 1 \).

Proof. Since \( f \in R_k(\alpha, \lambda, \gamma) \), we have
\[
\int_0^{2\pi} \left| \text{Re} \left\{ \frac{J(\alpha, \lambda, f) - \gamma}{1 - \gamma} \right\} \right| d\theta \leq k\pi
\]
and
\[
\int_0^{2\pi} \left\{ \text{Re} \left\{ \frac{J(\alpha, \lambda, f) - \gamma}{1 - \gamma} \right\} \right\} d\theta = 2\pi
\]
together imply
\[
\int_{\theta_1}^{\theta_2} \left\{ \text{Re} J(\alpha, \lambda, f(z)) \right\} d\theta \geq -\left\{ \frac{(1 - \gamma)k + 2(\gamma - 1)}{2} \right\} \pi,
\]
where \( z = re^{i\theta}, \) \( 0 \leq r < 1, \) \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \).

This is equivalent to
\[
\int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{z(D^{\lambda + 1} f)' + \left( \frac{1}{\alpha} - 1 \right) z(D^{\lambda} f)'}{D^{\lambda + 1} f + \frac{1}{\alpha} - 1} \right\} d\theta > -\left\{ \frac{(1 - \gamma)k + 2(\gamma - 1)}{2\alpha} \right\} \pi.
\]
Now using Lemma 2.2, we obtain the required result. \( \square \)

Theorem 3.3. Let \(-1 < \lambda_1 < \lambda_2\). Then
\( R_k(\alpha, \lambda_2, 0) \subset R_k(\alpha, \lambda_1, 0) \).

Proof. Define
\[
\phi(z) = z + \sum_{m=2}^{\infty} \frac{(\lambda_1 + 1)(\lambda_1 + 2) \cdots (\lambda_1 + m - 1)}{(\lambda_2 + 1)(\lambda_2 + 2) \cdots (\lambda_2 + m - 1)} z^m, \quad z \in E.
\]

Then \( \phi(z) \in A \) and, for \( z \in E \),
\[
\frac{z}{(1 - z)^{\lambda_2 + 1}} \ast \phi(z) = \frac{z}{(1 - z)^{\lambda_1 + 1}} (-1 < \lambda_1 < \lambda_2).
\]

This implies
\[
\frac{z}{(1 - z)^{\lambda_2 + 1}} \ast \phi(z) \in \mathcal{S}^{*} \left( \frac{1 - \lambda_2}{2} \right) \subset \mathcal{S}^{*} \left( \frac{1 - \lambda_1}{2} \right)
\]
and therefore \( \phi(z) \) is prestarlike of order \( \left( \frac{1 - \lambda_2}{2} \right) \).

Now let
\[
\frac{z(D^{\lambda_1} f)'}{D^{\lambda_1} f} = H = \left( \frac{k}{4} + \frac{1}{4} \right) h_1 - \left( \frac{k}{4} - \frac{1}{2} \right) h_2 = \left( \frac{k}{4} + \frac{1}{2} \right) z(D^{\lambda_1} f_1)' - \left( \frac{k}{4} - \frac{1}{2} \right) z(D^{\lambda_1} f_2)',
\]
and with
\[
\phi_{\alpha, \lambda_1}(z) = \alpha \frac{1}{(1 - z)^{\lambda_1 + 1}} + (1 - \alpha) \frac{1}{(1 - z)^{\lambda_1}},
\]
we have
\[ H \ast \phi_{\alpha, \lambda_1} = H + \alpha \frac{z H'}{H + \lambda_1} \]
\[ = \alpha \frac{z(D^{\lambda_1 + 1} f)' - D^{\lambda_1 + 1} f}{D^{\lambda_1 + 1} f} + (1 - \alpha) \frac{z(D^{\lambda_1} f)' - D^{\lambda_1} f}{D^{\lambda_1} f} \]
\[ = \left( \frac{k}{4} + \frac{1}{2} \right) \left( h_1 + \alpha \frac{z h_1'}{h_1 + \lambda_1} \right) - \left( \frac{k}{4} - \frac{1}{2} \right) \left( h_2 + \alpha \frac{z h_2'}{h_2 + \lambda_1} \right) \]
\[ = \left( \frac{k}{4} + \frac{1}{2} \right) \left( \alpha \frac{z(D^{\lambda_1 + 1} f_1)' - (1 - \alpha) z(D^{\lambda_1} f_1)'}{D^{\lambda_1 + 1} f_1} \right) \]
\[ - \left( \frac{k}{4} - \frac{1}{2} \right) \left( \alpha \frac{z(D^{\lambda_1 + 1} f_2)' - (1 - \alpha) z(D^{\lambda_1} f_2)'}{D^{\lambda_1 + 1} f_2} \right). \tag{3.5} \]

We use (3.4) to write
\[ \frac{z(D^{\lambda_1 + 1} f)'}{D^{\lambda_1 + 1} f} = \frac{\phi \ast z(D^{\lambda_1} f)'}{D^{\lambda_1} f} = \frac{\phi \ast \frac{z(D^{\lambda_2 + 1} f_1)'}{D^{\lambda_2 + 1} f_1}}{\phi \ast D^{\lambda_2 + 1} f_1}. \]

Thus we can write from (3.5),
\[ \alpha \frac{z(D^{\lambda_1 + 1} f)' - D^{\lambda_1 + 1} f}{D^{\lambda_1 + 1} f} + (1 - \alpha) \frac{z(D^{\lambda_1} f)' - D^{\lambda_1} f}{D^{\lambda_1} f} = \left( \frac{k}{4} + \frac{1}{2} \right) (\alpha A_1 + (1 - \alpha) B_1) - \left( \frac{k}{4} - \frac{1}{2} \right) (\alpha A_2 + (1 - \alpha) B_2), \]

where
\[ A_i = \frac{\phi \ast (\frac{z(D^{\lambda_2 + 1} f_1)'}{D^{\lambda_2 + 1} f_1})}{\phi \ast D^{\lambda_2 + 1} f_1}, \]
\[ B_i = \frac{\phi \ast (\frac{z(D^{\lambda_2} f_1)'}{D^{\lambda_2} f_1})}{\phi \ast D^{\lambda_2} f_1}, \quad i = 1, 2. \]

This implies that \[ \frac{z(D^{\lambda_2 + 1} f_1)'}{D^{\lambda_2 + 1} f_1} \] is in the convex hull of the image of \( E \) under \( z(D^{\lambda_2 + 1} f_1)'. \) Similarly \( \frac{\phi \ast z(D^{\lambda_2} f_1)'}{D^{\lambda_2} f_1} \)

is in the convex hull of the image of \( E \) under \( z(D^{\lambda_2} f_1)' \).

Since \( f \in R_k(\alpha, \lambda_2, 0) \), it follows, from Theorem 3.1,
\[ \frac{z(D^{\lambda_2 + 1} f_1)'}{D^{\lambda_2 + 1} f_1} \in P_k, \quad \frac{z(D^{\lambda_2} f_1)'}{D^{\lambda_2} f_1} \in P_k, \quad z \in E. \]

Therefore
\[ \frac{z(D^{\lambda_2 + 1} f_1)'}{D^{\lambda_2 + 1} f_1} \in P, \quad \frac{z(D^{\lambda_2} f_1)'}{D^{\lambda_2} f_1} \in P, \quad z \in E \text{ for } i = 1, 2. \]

Also \( \phi \) is prestarlike of order \( \left( \frac{1 - \lambda_2}{\lambda_2} \right) \). We use Lemma 2.1 to conclude that \( f \in R_k(\alpha, \lambda_1, 0) \) in \( E \). \( \square \)

**Theorem 3.4.** Let \( f \in R_k(0, \lambda, \gamma) \) for \( z \in E \). Then \( f \in R_k(\alpha, \lambda, \gamma) \) for \( |z| < r_0 \) where \( r_0 \) is given by (2.1) with \( \mu = \frac{\lambda + \gamma}{1 - \gamma} \) and \( s = \frac{\alpha}{1 - \gamma} \) and the value of \( r_0 \) is exact.

**Proof.** Let \( H(z) \) be defined by (3.2) and since \( f \in R_k(0, \lambda, \gamma) \), it follows that \( H \in P_k(\gamma) \) for \( z \in E \). Proceeding as in Theorem 3.1, we have
\[ J(\alpha, \lambda, f(z)) = H(z) + \alpha \frac{z H'}{H(z) + \lambda} \]
\[ = \left( \frac{k}{4} + \frac{1}{2} \right) \left[ h_1(z) + \alpha \frac{z h_1'}{h_1(z) + \lambda} \right] - \left( \frac{k}{4} - \frac{1}{2} \right) \left[ h_2(z) + \alpha \frac{z h_2'}{h_2(z) + \lambda} \right]. \]
Since $h_i \in P(\gamma), i = 1, 2$, we can write

$$h_i(z) = (1 - \gamma) p_i(z) + \gamma, \quad p_i \in P, \; i = 1, 2.$$  

Thus

$$\frac{1}{1 - \gamma} \left[ J(\alpha, \lambda, f(z)) - \gamma \right] = \left( \frac{k}{4} + \frac{1}{2} \right) \left[ p_1(z) + \frac{zp'_1(z)}{(1 - \gamma) p_1(z) + \lambda + \gamma} \right]$$

$$- \left( \frac{k}{4} - \frac{1}{2} \right) \left[ p_2(z) + \frac{zp'_2(z)}{(1 - \gamma) p_2(z) + \lambda + \gamma} \right]$$

$$= \left( \frac{k}{4} + \frac{1}{2} \right) \left[ p_1(z) + \frac{\alpha \gamma}{(1 - \gamma) p_1(z) + \lambda + \gamma} \right]$$

$$- \left( \frac{k}{4} - \frac{1}{2} \right) \left[ p_2(z) + \frac{\alpha \gamma}{p_2(z) + \lambda + \gamma} \right].$$

Using Lemma 2.3 with $\mu = \frac{\lambda + \gamma}{1 - \gamma} \neq -1$ and $s = \frac{\alpha}{1 - \gamma} > 0$, we see that $f \in R_k(\alpha, \lambda, \gamma)$ for $|z| < r_0$ where

$$r_0 = \frac{|\mu + 1|}{\sqrt{A + (A^2 - |\mu^2 - 1|^2) \frac{1}{2}}}$$

$$A = 2(s + 1)^2 + |\mu|^2 - 1,$$

and this radius is exact. \qed

As a special case, we note that, for $k = 2, \alpha = 1, \lambda = 0, \gamma = 0,$

$$r_0 = \frac{1}{\sqrt{7 + 48}} \simeq 0.268 \simeq 2 - \sqrt{3}.$$

**Theorem 3.5.** For $0 \leq \alpha_2 < \alpha_1,$

$$R_k(\alpha_1, \lambda, \gamma) \subset R_k(\alpha_2, \lambda, \gamma).$$

**Proof.** For $\alpha_2 = 0$ the proof is immediate from Theorem 3.1. Therefore we let $\alpha_2 > 0$ and $f \in R_k(\alpha_1, \lambda, \gamma)$. There exists $H_1, H_2 \in P_k(\gamma)$ such that

$$H_1(z) = \left[ \alpha_1 \frac{z(D^{k+1} f)'}{D^{k+1} f} + (1 - \alpha_1) \frac{z(D^k f)'}{D^k f} \right],$$

$$H_2(z) = \frac{z(D^k f)'}{D^k f}.$$  

Hence

$$\left[ \alpha_2 \frac{z(D^{k+1} f)'}{D^{k+1} f} + (1 - \alpha_2) \frac{z(D^k f)'}{D^k f} \right] = \frac{\alpha_2}{\alpha_1} H_1 + \left( 1 - \frac{\alpha_2}{\alpha_1} \right) H_2, \quad H_1, H_2 \in P_k(\gamma).$$  

(3.6)

Since $P_k(\gamma)$ is a convex set, see [5], it follows that the right-hand side of (3.6) belongs to $P_k(\gamma)$ and this establishes the required result. \qed

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