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PBW-bases of the twisted generic composition algebras of affine valued quivers

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Abstract

PBW type bases of the twisted generic composition algebras of the affine valued quivers are constructed by using the Frobenius morphism.

Keywords: Affine valued quivers; Ringel–Hall algebras; PBW-basis; AR-quivers; Frobenius morphism

1. Introduction

It is well known that Ringel–Hall algebras of hereditary algebras provide a successful model for the realization of quantum groups. One of the important features of Ringel–Hall algebra approach is that it makes it possible to study quantum groups by using the machinery of representation theory of algebras, in particular, homological techniques and Auslander–Reiten theory.

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In [19], Ringel constructed a PBW-basis for the Ringel–Hall algebra of any representation-finite hereditary algebra. Note that in the case of finite type, the composition algebra and the Ringel–Hall algebra are coincide.

In [14,26], the authors constructed integral PBW-basis for the (twisted) generic composition algebra of Kronecker quiver and affine quivers, respectively.

In both cases, the Auslander–Reiten quiver, AR-quiver for short, of the corresponding graph plays very important role.

In [4], the authors built a direct bridge between the quiver and valued quiver by combining the idea of folding graphs with the idea of Frobenius morphisms and consequently gave an explicit construction of the AR-quiver of the valued quiver from the AR-quiver of the corresponding quiver.

The aim of this paper is to construct a PBW-basis for the twisted generic composition algebra of any affine valued quiver by using its AR-quiver given by the method in [4].

2. Preliminaries

In this section, we recall some definitions and basic results of Ringel–Hall algebras and Frobenius morphisms and F-stable modules from [4,7,17]. We refer to [1,8,22] for the unexplained terminology and results used in this paper.

Throughout, $\mathbb{F}_q$ denotes a finite field with q elements. For any $r \geqslant 1$, let $\mathbb{F}_{q^r}$ denotes the unique extension field of $\mathbb{F}_q$ of degree r contained in the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. All modules considered are left modules of finite dimension over the base field. If M is a module, $[M]$ denotes the class of modules isomorphic to M, called the isoclass of M.

2.1. Valued quiver and its representation

2.1.1. A valued graph (Γ, d) is a finite set Γ (of vertices) together with non-negative integers d_{ij} for all $i, j \in \Gamma$ such that $d_{ii} = 0$ and there exist positive integers $\{\varepsilon_i\}_{i\in\Gamma}$ satisfying

$$d_{ij}\varepsilon_j = d_{ji}\varepsilon_i \quad \text{for all } i, j \in \Gamma.$$

Further, the valued quiver (Γ, d) defines a symmetrizable generalized Cartan matrix $C_\Gamma = (a_{ij})_{i,j\in\Gamma}$ by $a_{ii} = 2$ and $a_{ij} = -d_{ji}$ for $i \neq j$. In fact, all symmetrizable generalized Cartan matrices can be obtained in this way.

An orientation Ω of a valued graph (Γ, d) is given by prescribing for each edge $\{i, j\}$ of (Γ, d) an order (indicated by an arrow $i \to j$). We call (Γ, d, Ω), or simply Ω, a valued quiver. For $i \in \Gamma$, we can define a new orientation $\sigma_i\Omega$ of (Γ, d) by reversing the direction of arrows along all edges containing i.

2.1.2. Let (Γ, d, Ω) be a valued quiver. We assume that (Γ, d, Ω) is connected and without oriented cycles in an obvious sense. Let $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be a reduced $\mathbb{F}_q$-species of type Ω, that is, for all $i, j \in \Gamma$, ${}_iM_j$ is an F_i-F_j-bimodule, where F_i and F_j are finite extensions of $\mathbb{F}_q$ and $\dim({}_iM_j)_{F_j} = d_{ij}$, $\dim_{\mathbb{F}_q} F_i = \varepsilon_i$. A representation $(V_i, {}_j\varphi_i)$

of $\mathcal{S}$ is given by vector space $(V_i)_{F_i}$ and F_j-linear mapping ${}_j\varphi_i : V_i \otimes {}_iM_j \to V_j$ for any $i \to j$. Such a representation is called finite-dimensional if $\sum \dim_{\mathbb{F}_q} V_i < \infty$. A morphism $\alpha : V \to W$ from a representation $V = (V_i, {}_j\varphi_i)$ to $W = (W_i, {}_j\psi_i)$ is defined as a set $\alpha = (\alpha_i)$ of F_i-linear mappings $\alpha_i : V_i \to W_i$, $i \in \Gamma$, satisfying

$$ {}_j\psi_i(\alpha_i \otimes 1) = \alpha_j\, {}_j\varphi_i \quad \text{for all edges } i \to j. $$

We denote by rep-$\mathcal{S}$ the category of finite-dimensional representations of $\mathcal{S}$. Note that the category rep-$\mathcal{S}$ is equivalent to the module category of finite-dimensional modules over a finite-dimensional hereditary $\mathbb{F}_q$-algebra A, where A is the tensor algebra of $\mathcal{S}$. Furthermore, any basic finite-dimensional hereditary $\mathbb{F}_q$-algebra can be obtained in this way.

2.1.3. Let $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be an $\mathbb{F}_q$-species, $\varepsilon_i = \dim_{\mathbb{F}_q} F_i$, and $d_{ij} = \dim({}_iM_j)_{F_j}$. For a representation $V = (V_i, {}_j\varphi_i) \in$ rep-$\mathcal{S}$, we define the dimension vector of V to be $\mathbf{dim}\, V = (\dim_{F_i} V_i)_{i\in\Gamma}$. If $V, W \in$ rep-$\mathcal{S}$, assume that

$$ \alpha = \mathbf{dim}\, V = (a_1, \ldots, a_n) \quad \text{and} \quad \beta = \mathbf{dim}\, W = (b_1, \ldots, b_n), $$

and we define

$$ \langle \alpha, \beta \rangle = \sum_{i\in\Gamma} \varepsilon_i a_i b_i - \sum_{i\to j} d_{ij}\varepsilon_j a_i b_j. $$

One sees that (cf. [20])

$$ \langle \alpha, \beta \rangle = \dim_{\mathbb{F}_q} \operatorname{Hom}_A(V, W) - \dim_{\mathbb{F}_q} \operatorname{Ext}^1_A(V, W). $$

Set

$$ (\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle. $$

It is well known that both $\langle -,- \rangle$ and $(-,-)$ are well defined on the Grothendieck group $G_0(A)$ of rep-$\mathcal{S}$. The bilinear form $\langle -,- \rangle$ and $(-,-)$ are called the Euler form and symmetric Euler form, respectively. In fact, the Grothendieck group with the symmetric Euler form is a Cartan datum and any Cartan datum can be realized in this way (see [21]). Let $\varepsilon(\alpha) = \langle \alpha, \alpha \rangle$. We see that $\varepsilon(i) = \varepsilon_i$.

2.1.4. Denote by $\mathbb{Q}^\Gamma$ the vector space of all $x = (x_i)_{i\in\Gamma}$ over the rational numbers. In particular, for each $i \in \Gamma$, e_i denotes the vector with $x_i = 1$ and $x_j = 0$ for all $j \neq i$. Also, for each $i \in \Gamma$, we define the linear transformation $s_i : \mathbb{Q}^\Gamma \to \mathbb{Q}^\Gamma$ by $s_i(x) = y$ where $y_j = x_j$ for $j \neq i$ and

$$ y_i = -x_i + \sum_{j\in\Gamma} d_{ji} x_j. $$

The symbol $W = W_\Gamma$ will denote the *Weyl group*, i.e., the group of all linear transformations of $\mathbb{Q}^\Gamma$ generated by the fundamental reflections s_i, $i \in \Gamma$.

2.1.5. Let (Γ, d, Ω) be a valued quiver (connected and without oriented cycles) and $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be an $\mathbb{F}_q$-species of type Ω. Let $i \in \Gamma$ be a sink or source of (Γ, Ω). We define $\sigma_i(\mathcal{S})$ to be the $\mathbb{F}_q$-species obtained from $\mathcal{S}$ by replacing ${}_rM_s$ by its $\mathbb{F}_q$-dual for $r = i$ or $s = i$; then $\sigma_i\mathcal{S}$ is a reduced $\mathbb{F}_q$-species of type $\sigma_i(\Omega)$.

2.1.6. For each sink or source $i \in \Gamma$, one may define Bernstein–Gelfand–Ponomarev reflection functors $\sigma_i^\pm$: rep-$\mathcal{S} \to$ rep-$\sigma_i\mathcal{S}$ (see [2,5]).

If i is a vertex of Γ, let rep-$\mathcal{S}\langle i\rangle$ be the subcategory of rep-$\mathcal{S}$ of all representations which do not have S_i as a direct summand, where S_i is the simple representation with $\dim S_i = e_i$. If i is a sink or source, then rep-$\mathcal{S}\langle i\rangle$ is closed under direct summands and extensions. We point out that if i is a sink, then σ_i^+: rep-$\mathcal{S}\langle i\rangle \to$ rep-$\sigma_i\mathcal{S}\langle i\rangle$ is an exact equivalence and induces isomorphisms on both Hom and Ext. The assertion for σ_i^-: rep-$\mathcal{S}\langle i\rangle \to$ rep-$\sigma_i\mathcal{S}\langle i\rangle$ is the same if i is a source.

2.1.7. Let A be a finite-dimensional hereditary $\mathbb{F}_q$-algebra, $\mathcal{P}$ the set of isomorphism classes of finite-dimensional A-modules, and $I \subset \mathcal{P}$ the set of isomorphism classes of simple A-modules. We choose a representative $V_\alpha \in \alpha$ for any $\alpha \in \mathcal{P}$. By abuse of notation, we write

$$\langle\alpha, \beta\rangle = \langle\mathbf{dim}\, V_\alpha, \mathbf{dim}\, V_\beta\rangle$$

and

$$(\alpha, \beta) = (\mathbf{dim}\, V_\alpha, \mathbf{dim}\, V_\beta) \quad \text{for all } \alpha, \beta \in \mathcal{P}.$$

So the Euler form $\langle -,-\rangle$ and its symmetrization $(-,-)$ are defined on $\mathbb{Z}[I]$.

2.1.8. For $\alpha, \beta, \lambda \in \mathcal{P}$, let $g_{\alpha\beta}^\lambda$ be the number of submodules B of V_α such that $B \simeq V_\beta$ and $V_\lambda/B \simeq V_\alpha$.

2.1.9. Let $v = \sqrt{q}$ (hence $q = v^2$), and $\mathbb{Q}(v)$ be the rational function field of v. The Ringel–Hall algebra $\mathfrak{h}(A)$ is by definition the free $\mathbb{Q}(v)$-module with basis $\{u_\alpha \mid \alpha \in \mathcal{P}\}$ and the multiplication is given by

$$u_\alpha u_\beta = \sum_{\lambda\in\mathcal{P}} g_{\alpha\beta}^\lambda u_\lambda$$

for all $\alpha, \beta \in \mathcal{P}$.

The twisted Ringel–Hall algebra $\mathfrak{h}^*(A)$ is defined by setting $\mathfrak{h}^*(A) = \mathfrak{h}(A)$ as $\mathbb{Q}(v)$-vector space, but the multiplication is defined by

$$u_\alpha * u_\beta = v^{\langle\alpha,\beta\rangle} \sum_{\lambda\in\mathcal{P}} g_{\alpha\beta}^\lambda u_\lambda$$

for all $\alpha, \beta \in \mathcal{P}$.

We denote by $\dim : \mathbb{Z}^n \to \mathbb{Z}$ the linear form given by $\dim \mathbf{e}_i = \varepsilon_i$. For $\alpha \in \mathcal{P}$, let $\dim \alpha = \dim(\mathbf{dim}\, V_\alpha)$.

For convenience of use, in twisted Ringel–Hall algebra $\mathfrak{h}^*(A)$, we write $\langle u_\alpha \rangle = v^{-\dim\alpha + \varepsilon(\alpha)} u_\alpha$ for each $\alpha \in \mathcal{P}$ (noting that $\langle u_i \rangle = u_i$ for all $i \in I$). Then, it is easy to see that the multiplication of $\mathfrak{h}^*(A)$ can be replaced by

$$\langle u_\alpha \rangle * \langle u_\beta \rangle = v^{-\langle \alpha, \beta \rangle} \sum_{\lambda \in \mathcal{P}} g_{\alpha\beta}^{\lambda} \langle u_\lambda \rangle \quad \text{for all } \alpha, \beta \in \mathcal{P}.$$

Let A be the tensor algebra of an $\mathbb{F}_q$-species $\mathcal{S}$. We can identify the categories mod-A and rep-$\mathcal{S}$; therefore, $\mathfrak{h}(A)$ can be viewed as being defined for rep-$\mathcal{S}$. Also, we denote by $\sigma_i A$ the tensor algebra of $\sigma_i \mathcal{S}$. We define $\mathfrak{h}(A)\langle i \rangle$ to be the $\mathbb{Q}(v)$-subspace of $\mathfrak{h}(A)$ generated by u_α with $V_\alpha \in \text{rep-}\mathcal{S}\langle i \rangle$. If i is a sink or source, since rep-$\mathcal{S}\langle i \rangle$ is closed under extensions and so $\mathfrak{h}(A)\langle i \rangle$ is subalgebra of $\mathfrak{h}(A)$. Because

$$\sigma_i^+ : \text{rep-}\mathcal{S}\langle i \rangle \to \text{rep-}\sigma_i \mathcal{S}\langle i \rangle$$

is an exact equivalence and induces isomorphisms on both Hom- and Ext-spaces, it is not difficult to see the following result of Ringel [19].

Proposition 2.1. *Let i be a sink. The functor σ_i^+ yields a $\mathbb{Q}(v)$-algebra isomorphism*

$$\sigma_i : \mathfrak{h}(A)\langle i \rangle \to \mathfrak{h}(\sigma_i A)\langle i \rangle$$

with $\sigma_i(u_\alpha) = u_{\sigma_i^+ \alpha}$ for any $V_\alpha \in \text{rep-}\mathcal{S}\langle i \rangle$.

Of course, we have a dual statement for a source i.

2.1.10. In the study of quantum groups and the Ringel–Hall algebras, the following notations and relations are often used:

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}} = v^{n-1} + v^{n-3} + \cdots + v^{-n+1},$$

$$[n]! = \prod_{r=1}^{n} [r], \qquad \begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!},$$

$$|n] = \frac{q^n - 1}{q - 1} \left(q^{n-1} + \cdots + q + 1\right) = v^{n-1}[n],$$

$$|n]! = \prod_{t=1}^{n} |t] = v^{\binom{n}{2}} [n]!,$$

$$\left| \begin{matrix} n \\ t \end{matrix} \right] = \frac{|n]!}{|t]!|n-t]!} = v^{t(n-t)} \begin{bmatrix} n \\ t \end{bmatrix}.$$

The following equations are the basic ones.

Lemma 2.2. *For $n > 0$, we have*

$$\sum_{t=0}^{n}(-1)^t v^{t(t-1)}\begin{vmatrix} n \\ t \end{vmatrix} = 0 \quad \textit{and} \quad \sum_{t=0}^{n}(-1)^t v^{t(n-1)}\begin{vmatrix} n \\ t \end{vmatrix} = 0.$$

If $f(v)$ is a rational function of v, then by $f(v)_\alpha$ we mean $f(v^{\varepsilon(\alpha)})$.

2.1.11. Let (Γ, d, Ω) be a valued quiver. We assume that (Γ, d, Ω) is connected and without oriented cycles in an obvious sense. Let $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be a reduced $\mathbb{F}_q$-species of type Ω and A be the tensor algebra of $\mathcal{S}$, $\{S_i \mid i \in I\}$ is a complete set of pairwise non-isomorphic simple A-modules. We denote by $\mathfrak{c}(A)$ the $\mathbb{Q}(v)$-subalgebra of $\mathfrak{h}(A)$ generated by u_i, $i \in I$, which is called the composition algebra and whose twisted generic version is denoted by $\mathfrak{c}^*(A)$.

On the other hand, we have a symmetrizable Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(C_\Gamma)$ associated with C_Γ. Let $U_q^+(\mathfrak{g})$ be the positive part of the quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$. Then $U_q^+(\mathfrak{g})$ is a $\mathbb{Q}(v)$-algebra generated by the standard basis E_i, $i \in I$. The following well-known result of Green and Ringel (see [9] or [17]) lays down a base for our investigation.

Theorem 2.3. *There exists an isomorphism $\eta : U_q^+(\mathfrak{g}) \to \mathfrak{c}^*(A)$ of $\mathbb{Q}(v)$-algebras such that $\eta(E_i) = u_i$, $i \in I$.*

2.2. Frobenius morphisms and F-stable modules

2.2.1. Let now $k = \overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. A Frobenius map on a k-vector space V is an $\mathbb{F}_q$-linear isomorphism $F_V : V \to V$ satisfying $F_V(\lambda v) = \lambda^q F_V(v)$ for all $v \in V$ and $\lambda \in k$.

Let A be a finite-dimensional k-algebra with identity 1. A map $F_A : A \to A$ is called a Frobenius morphism on A if it is a Frobenius map on the k-space A and it is also an $\mathbb{F}_q$-algebra isomorphism sending 1 to 1.

Given a Frobenius morphism F_A on A, let

$$A^F := A^{F_A} = \{a \in A \mid F_A(a) = a\}$$

be the set of F_A-fixed points. Then A^F is an $\mathbb{F}_q$-subalgebra of A, and $A = A^F \otimes k$. The Frobenius morphism F_A on A^F is given by $F_A(a \otimes \lambda) = a \otimes \lambda^q$ for all $a \in A^F$, $\lambda \in k$.

Let M be a finite-dimensional A-module. We call M an F-stable A-module if there is a Frobenius map $F_M : M \to M$ such that

$$F_M(am) = F_A(a)F_M(m) \quad \text{for all } a \in A,\ m \in M.$$

We denote by mod^F-A the category of finite-dimensional F-stable A-modules (M, F_M). The morphisms from (M, F_M) to (N, F_N) are A-module homomorphisms compatible with Frobenius maps F_M and F_N. Note that mod^F-A is an abelian $\mathbb{F}_q$-category. We have following result.

Theorem 2.4. [4] *The abelian category* mod^F-A *is equivalent to the category* mod-A^F *of finite-dimensional* A^F*-modules.*

2.2.2. Let $Q=(Q_0,Q_1)$ be a quiver without loops, where Q_0 and Q_1 denote respectively the set of vertices and the set of arrows of Q. For each arrow ρ in Q_1, we denote by $s(\rho)$ and $t(\rho)$ its initial and terminal points, respectively.

Let σ be an automorphism of Q, that is, σ is a permutation on the vertices of Q and on the arrows of Q such that $\sigma(s(\rho))=s(\sigma(\rho))$ and $\sigma(t(\rho))=t(\sigma(\rho))$ for any $\rho\in Q_1$. We further assume that σ is admissible, that is, there are no arrows connected vertices in the same orbit of σ in Q_0. We call the pair (Q,σ) an admissible quiver, or simply an ad-quiver.

Let $A:=kQ$ be the path algebra of Q over $k=\bar{\mathbb{F}}_q$ which has identity $1=\sum_{i\in Q_0}e_i$, where e_i is the idempotent (or the length 0 path) corresponding to the vertex i. Then σ induces a Frobenius morphism

$$F_{Q,\sigma}=F_{Q,\sigma;q}:A\to A,\quad \sum_s x_s p_s\mapsto \sum_s x_s^q\sigma(p_s), \tag{1}$$

where $\sum_s x_s p_s$ is a k-linear combination of paths p_s, and $\sigma(p_s)=\sigma(\rho_t)\cdots\sigma(\rho_1)$ if $p_s=\rho_t\cdots\rho_1$ for arrows $\rho_1,\dots,\rho_t$ in Q_1.

2.2.3. A representation $V=(V_i,\varphi_\rho)$ of ad-quiver (Q,σ) is called F-stable if there is a Frobenius map $F_V:\bigoplus_{i\in Q_0}V_i\to\bigoplus_{i\in Q_0}V_i$ satisfying $F_V(V_i)=V_{\sigma(i)}$ for all $i\in Q_0$ such that $F_V\varphi_\rho=\varphi_{\sigma(\rho)}F_V$ for each arrow $\rho\in Q_1$. An F-stable representation is called indecomposable if it is not isomorphic to a direct sum of two non-zero F-stable representations. Clearly, for each $\mathbf{i}\in\Gamma_0$, $S_{\mathbf{i}}:=\bigoplus_{i\in\mathbf{i}}S_i$ is a simple F-stable representation of Q, whose dimension vector is $e_{\mathbf{i}}$. Let $\mathrm{Rep}^F(Q,\sigma)$ be the category of all finite-dimensional F-stable representations of (Q,σ) together with morphisms in Rep Q which are compatible with Frobenius maps. Then $\mathrm{Rep}^F(Q,\sigma)$ is an abelian $\mathbb{F}_q$-category. It is easy to see that the equivalence between mod-A and Rep Q induces an equivalence between mod^F-A, where $F=F_{Q,\sigma}$, and $\mathrm{Rep}^F(Q,\sigma)$. Hence, by Theorem 2.4, $\mathrm{Rep}^F(Q,\sigma)$ and mod-A^F are equivalent.

Given a representation $V=(V_i,\varphi_\rho)$ of Q together with a Frobenius map F_V: $\bigoplus_{i\in Q_0}V_i\to\bigoplus_{i\in Q_0}V_i$ satisfying $F_V(V_i)=V_{\sigma(i)}$ for all $i\in Q_0$, we define the Frobenius twist $V^{[1]}=(W_i,\psi_\rho)$ by

$$W_{\sigma(i)}=V_i\quad\text{and}\quad \psi_{\sigma(\rho)}=\varphi_\rho^{[1]}\quad\text{for all } i\in Q_0 \text{ and } \rho\in Q_1.$$

Note that up to isomorphism, $V^{[1]}$ is independent of the choice of the Frobenius map F_V. Inductively, we define $V^{[s]}=(V^{[s-1]})^{[1]}$ for $s>1$. A representation V of Q is called F-periodic if $V^{[s]}\cong V$ for some $s\geqslant 1$. Such a minimal positive integer s is called F-period of V. By [4], every finite-dimensional representation of Q is F-periodic.

Theorem 2.5. [4] *Let V be an indecomposable representation of Q with F-period s. Then*

$$\widetilde{V}:=V\oplus V^{[1]}\oplus\cdots\oplus V^{[s-1]}$$

is an indecomposable F-stable representation of Q. Moreover, each indecomposable F-stable representation of Q can be obtained in this way.

2.2.4. To an ad-quiver (Q,σ), we can associate a valued quiver $\Gamma = \Gamma_{(Q,\sigma)}$ as follows. The vertex and arrow sets Γ_0 and Γ_1 of Γ are, respectively, the sets of σ-orbits in Q_0 and Q_1. For each $k \in Q_0$ and $\rho \in Q_1 : i \to j$, we denote by $\mathbf{k}$ and $\mathbf{a}$ the σ-orbits of k and ρ, respectively, and define

$$\varepsilon_{\mathbf{k}} = \#\{\text{vertices in } \sigma\text{-orbit } \mathbf{k}\}, \qquad \varepsilon_{\mathbf{a}} = \#\{\text{arrows in } \sigma\text{-orbit } \mathbf{a}\},$$
$$d_{\mathbf{a}} = \varepsilon_{\mathbf{a}}/\varepsilon_{\mathbf{j}}, \quad \text{and} \quad d'_{\mathbf{a}} = \varepsilon_{\mathbf{a}}/\varepsilon_{\mathbf{i}}, \quad \text{where } \mathbf{a} : \mathbf{i} \to \mathbf{j} \text{ in } \Gamma_1.$$

The valuation of Γ is given by $(\{\varepsilon_{\mathbf{i}}\}_{\mathbf{i}\in\Gamma_0}, \{(d_{\mathbf{a}}, d'_{\mathbf{a}})\}_{\mathbf{a}\in\Gamma_1})$.

Using the Frobenius morphism $F = F_{Q,\sigma}$ on A defined by σ, we can attach naturally to Γ an $\mathbb{F}_q$-modulation to obtain an $\mathbb{F}_q$-species as follows: for each vertex $\mathbf{i} \in I$ and each arrow $\boldsymbol{\rho}$ in Γ, we fix $i_0 \in \mathbf{i}$, $\rho_0 \in \boldsymbol{\rho}$, and consider the F_A-stable subspaces of A

$$A_{\mathbf{i}} = \bigoplus_{i\in\mathbf{i}} k e_i = \bigoplus_{s=0}^{\varepsilon_{\mathbf{i}}-1} k e_{\sigma^s(i_0)} \quad \text{and} \quad A_{\boldsymbol{\rho}} = \bigoplus_{\rho\in\boldsymbol{\rho}} k\rho = \bigoplus_{t=0}^{\varepsilon_{\boldsymbol{\rho}}-1} k\sigma^t(\rho_0),$$

where e_i denotes the idempotent corresponding to the vertex i. Then

$$A_{\mathbf{i}}^F = \left\{ \sum_{s=0}^{\varepsilon_{\mathbf{i}}-1} x^{q^s} e_{\sigma^s(i_0)} \,\middle|\, x \in k,\ x^{q^{\varepsilon_{\mathbf{i}}}} = x \right\} \quad \text{and}$$
$$A_{\boldsymbol{\rho}}^F = \left\{ \sum_{t=0}^{\varepsilon_{\boldsymbol{\rho}}-1} x^{q^t} \sigma^t(\rho_0) \,\middle|\, x \in k,\ x^{q^{\varepsilon_{\boldsymbol{\rho}}}} = x \right\}.$$

Further, the algebra structure of A induces an $A_{\mathbf{j}}^F$-$A_{\mathbf{i}}^F$-bimodule structure on $A_{\boldsymbol{\rho}}^F$ for each arrow $\boldsymbol{\rho} : \mathbf{i} \to \mathbf{j}$ in Γ. Thus, we obtain an $\mathbb{F}_q$-modulation $\mathcal{M} = \mathcal{M}(Q,\sigma) := (\{A_{\mathbf{i}}^{\mathcal{F}}\}_{\mathbf{i}}, \{A_{\boldsymbol{\rho}}^{\mathcal{F}}\}_{\boldsymbol{\rho}})$ over the valued quiver Γ. We shall denote the $\mathbb{F}_q$-species defined above by

$$\mathfrak{M}_{Q,\sigma} = \mathfrak{M}_{Q,\sigma;q} = (\Gamma, \mathcal{M}). \tag{2}$$

Let $T(\mathfrak{M}_{Q,\sigma})$ be the tensor algebra of the species $\mathfrak{M}_{Q,\sigma}$. Thus, by definition,

$$T(\mathfrak{M}_{Q,\sigma}) = \bigoplus_{n\geqslant 0} M^{\otimes n},$$

where $M = \bigoplus_{\boldsymbol{\rho}\in\Gamma_1} A_{\boldsymbol{\rho}}^F$ is viewed as an R-R-bimodule with $R = \bigoplus_{\mathbf{i}\in I} A_{\mathbf{i}}^F$ and $\otimes = \otimes_R$. If, for each σ-orbit $\mathbf{p}$ of a path $\rho_n \cdots \rho_2\rho_1$ in Q, we set $A_{\mathbf{p}} = \bigoplus_{p\in\mathbf{p}} kp$, then

$$A_{\mathbf{p}}^F \cong A_{\boldsymbol{\rho}_n}^F \otimes_{\mathbb{F}_{n-1}} \cdots \otimes_{\mathbb{F}_2} A_{\boldsymbol{\rho}_2}^F \otimes_{\mathbb{F}_1} A_{\boldsymbol{\rho}_1}^F,$$

where $\boldsymbol{\rho}_t$ is the σ-orbit of ρ_t and $\mathbb{F}_t = A^F_{h\boldsymbol{\rho}_t}$. Since $A^F = \bigoplus_{\mathbf{p}} A^F_{\mathbf{p}}$, it follows that the fixed point algebra A^F is isomorphic to the tensor algebra $T(\mathfrak{M}_{Q,\sigma})$. Thus, A^F-modules can be identified with representations of the species $\mathfrak{M}_{Q,\sigma}$ (see [6]). The above observation together with Theorem 2.4 implies the following.

Proposition 2.6. [4] *Let (Q,σ) be an ad-quiver with path algebra $A = kQ$ and induced Frobenius morphism $F = F_{Q,\sigma}$. Let $\mathfrak{M}_{Q,\sigma}$ be the associated $\mathbb{F}_q$-species defined as above. Then, we have an algebra isomorphism $A^F \cong T(\mathfrak{M}_{Q,\sigma})$. Hence the categories* mod-A^F *and* mod-$T(\mathfrak{M}_{Q,\sigma})$ *are equivalent.*

3. AR-quivers for affine valued quivers

3.1. We are now going to recall from [4] that the Frobenius morphism F on a finite-dimensional algebra A induces an automorphism $\mathfrak{s}$ of the Auslander–Reiten quiver $\mathcal{Q}$ of A and that the induced species $\mathfrak{M}_{\mathcal{Q},\mathfrak{s}}$ is isomorphic to the Auslander–Reiten quiver of the fixed point algebra A^F.

Let A be a finite-dimensional algebra over field $k = \mathbb{F}_q$. For an A-module M, let D_M denote the k-algebra

$$D_M := \operatorname{End}_A(M)/\operatorname{Rad}\big(\operatorname{End}_A(M)\big).$$

This is a division algebra if M is indecomposable. By definition, the Auslander–Reiten quiver (or AR-quiver for short) of A is a (simple) k-species $\mathcal{Q}_A$ consisting of a valued graph $\Gamma = \Gamma_A$ and a k-modulation $\mathbb{M} = \mathbb{M}_A$ defined on Γ. Here, the vertices of Γ are isoclasses $[M]$ of indecomposable A-modules and the arrows $[M] \to [N]$ for indecomposable M and N are defined by the condition $\operatorname{Irr}_A(M,N) \neq 0$, where

$$\operatorname{Irr}_A(M,N) := \operatorname{Rad}_A(M,N)/\operatorname{Rad}^2_A(M,N)$$

is the space of irreducible homomorphisms from M to N. Each arrow $[M] \to [N]$ has the valuation (d_{MN}, d'_{MN}) with d_{MN} and d'_{MN} being the dimensions of $\operatorname{Irr}_A(M,N)$ considered as left D_N-space and right D_M-space, respectively. The k-modulation $\mathbb{M}$ is given by division algebras D_M for vertices $[M]$ and (non-zero) D_N-D_M-bimodules $\operatorname{Irr}_A(M,N)$ for arrows $[M] \to [N]$.

Since the algebra A is defined over the algebraically close field $k = \overline{\mathbb{F}}_q$, we may regard the AR-quiver $\mathcal{Q} = \mathcal{Q}_A$ of A as an ordinary quiver. We first observe that $\mathcal{Q}$ admits an admissible automorphism $\mathfrak{s}$. For each vertex $[M] \in \mathcal{Q}$, $\mathfrak{s}([M])$ is defined to be $[M^{[1]}]$. If M and N are indecomposable A-modules, then there are n_{st} arrows $\gamma^{(m)}_{s,t}$ from $[M^{[s]}]$ to $[N^{[t]}]$ in $\mathcal{Q}$, where $0 \leqslant s \leqslant p(M) - 1$, $0 \leqslant t \leqslant p(N) - 1$, $n_{st} = \dim_k \operatorname{Irr}_A(M^{[s]}, N^{[t]})$ and $1 \leqslant m \leqslant n_{st}$. Note that $n_{st} = n_{s+1,t+1}$ for all s, t, where subscripts are considered as integers modulo $p(M)$ and $p(N)$, respectively. We now define

$$\mathfrak{s}\big(\gamma^{(m)}_{s,t}\big) = \gamma^{(m)}_{s+1,t+1} \quad \text{for all } 0 \leqslant s \leqslant p(M) - 1 \text{ and } 0 \leqslant t \leqslant p(N) - 1.$$

Clearly, $\mathfrak{s}$ is an admissible quiver automorphism and $(\mathcal{Q}, \mathfrak{s})$ is an ad-quiver.

Associated to $(\mathcal{Q}, \mathfrak{s})$, we may define a species $\mathfrak{M}_{\mathcal{Q},\mathfrak{s}}$ as follows: let $\mathcal{A} = k\mathcal{Q}$ denote the path algebra of $\mathcal{Q}$ and $F = F_{\mathcal{Q},\mathfrak{s}}$ be the Frobenius morphism of $\mathcal{A}$ induced by the automorphism $\mathfrak{s}$. For each vertex $\mathbf{i}(M)$ (i.e., the $\mathfrak{s}$-orbit of $[M]$) and each arrow $\boldsymbol{\rho}$ (i.e., an $\mathfrak{s}$-orbit of arrows in $\mathcal{Q}$) in $\Gamma(\mathcal{Q}, \mathfrak{s})$, we define subspaces

$$\mathcal{A}_{\mathbf{i}(M)} = \bigoplus_{s=0}^{p(M)-1} ke_{[M^{[s]}]} \quad \text{and} \quad \mathcal{A}_{\boldsymbol{\rho}} = \bigoplus_{\rho\in\boldsymbol{\rho}} k\rho,$$

of $\mathcal{A}$, which are obviously F-stable. By definition, the $\mathbb{F}_q$-modulation $\mathbb{M}(\mathcal{Q}, \mathfrak{s})$ is given by $(\mathcal{A}_{\mathbf{i}(M)})^F$ and $(\mathcal{A}_{\boldsymbol{\rho}})^F$ for all vertices $\mathbf{i}(M)$ and arrows $\boldsymbol{\rho}$ in $\Gamma(\mathcal{Q}, \mathfrak{s})$.

Let $\tau = \mathrm{D}\,\mathrm{Tr} = \mathrm{D}\,\mathrm{Ext}_A^1(-, A)$ and $\tau^{-1} = \mathrm{Tr}\,\mathrm{D} = \mathrm{Ext}_A^1(\mathrm{D}(A), -)$, where $\mathrm{D} = \mathrm{Hom}_{\mathbb{F}_q}(\mathrm{D}(A), -)$, be the Auslander–Reiten translates (cf. [1]). An indecomposable A-module M is said to be preprojective (respectively preinjective) provided that there exists a positive integer m such that $\tau^m(M) = 0$ (respectively $\tau^{-m}(M) = 0$), and to be regular otherwise. An arbitrary A-module X is said to be preprojective (respectively regular, preinjective) provided that every indecomposable direct summand of X is so.

If P, R and I are preprojective, regular and preinjective modules, respectively, then there holds the nice properties

$$\mathrm{Hom}_A(R, P) = \mathrm{Hom}_A(I, P) = \mathrm{Hom}_A(I, R) = 0$$

and

$$\mathrm{Ext}_A^1(P, R) = \mathrm{Ext}_A^1(P, I) = \mathrm{Ext}_A^1(R, I) = 0.$$

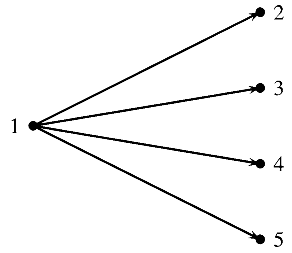
By [7], the Auslander–Reiten quiver of A has one preprojective component, which consists of all indecomposable preprojective modules and one preinjective component consists of all indecomposable preinjective modules; all other components turn out to be "tubes" which are of the form $T = \mathbb{Z}\mathbb{A}_\infty/m$, where m is called the rank of T. If $m = 1$, then T is called a homogenous tube, and if otherwise, it is a non-homogenous tube. The ranks of non-homogenous tubes of A is completely determined by the type of valued quiver (except for type $\widetilde{A}_n$).

Theorem 3.1. [4] *The species $\mathfrak{M}_{\mathcal{Q},\mathfrak{s}}$ associated to the AR-quiver $(\mathcal{Q}, \mathfrak{s})$ of A defined above is isomorphic to the AR-quiver $\mathcal{Q}_{A^F}$ of A^F. Moreover, the Auslander–Reiten translation of A naturally induces that of the fixed-point algebra A^F.*

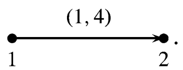
3.2. In this paper, we will consider the affine valued graphs $\widetilde{A}_{11}$, $\widetilde{A}_{12}$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{BC}_n$, $\widetilde{CD}_n$, $\widetilde{DD}_n$, $\widetilde{F}_{41}$, $\widetilde{F}_{42}$, $\widetilde{G}_{21}$ and $\widetilde{G}_{22}$.

In the following, we will construct the AR-quivers for these valued graphs with an admissible orientation from the AR-quiver for corresponding ordinary quivers case by case.

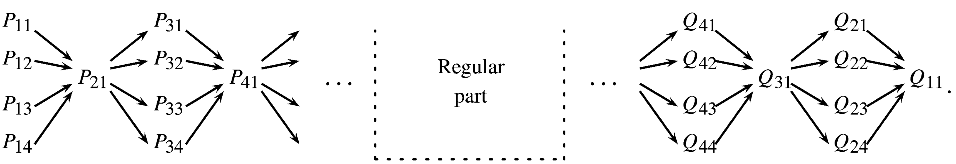
(a) If we define $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(4) = 5$, $\sigma(5) = 2$ and $\sigma(1) = 1$ for the quiver $(\widetilde{D}_4, \Omega)$



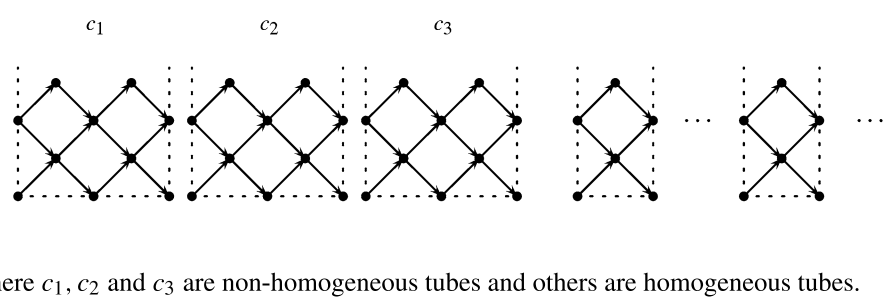
then $((\widetilde{D}_4, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{A}_{11}, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:



We know that the AR-quiver of $(\widetilde{D}_4, \Omega)$ has the following form:



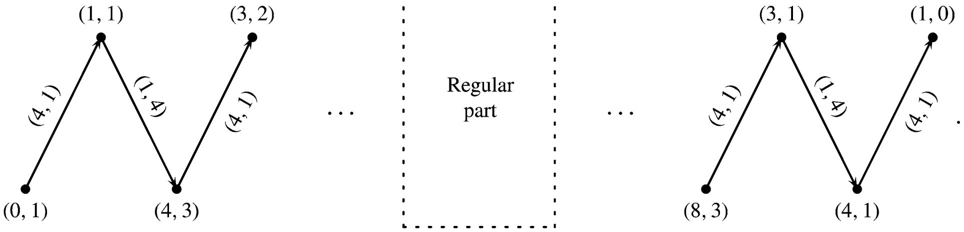
The regular part is of the form:



where c_1, c_2 and c_3 are non-homogeneous tubes and others are homogeneous tubes.

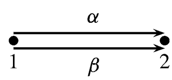
It is easy to see that the F-period of P_{ij} is 4 and 1 for odd i and even i, respectively.

Similarly, the F-period of Q_{ij} is 4 and 1 for even i and odd i, respectively. So from the construction in 3.1, we get that the AR-quiver of $(\widetilde{A}_{11}, \widetilde{\Omega})$ has the form:

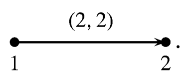


For the regular part, because the F-period of the modules in c_1 and c_2 is 4 and that of the modules in c_3 is 2, so the regular part of the AR-quiver of $(\widetilde{A}_{11}, \widetilde{\Omega})$ consists of homogeneous tubes only.

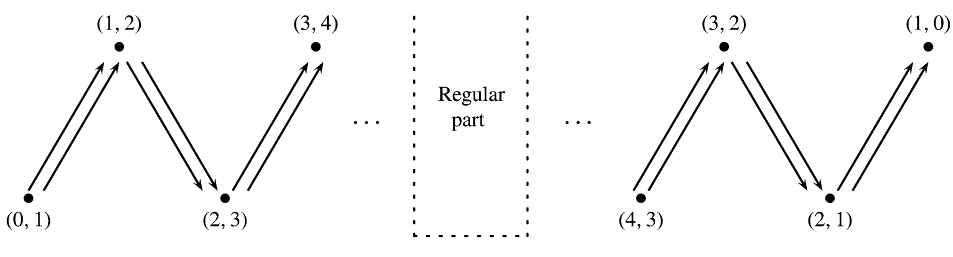
(b) If we define $\sigma(1) = 1, \sigma(2) = 2$ and $\sigma(\alpha) = \beta, \sigma(\beta) = \alpha$ for the quiver $(\widetilde{A}_1, \Omega)$



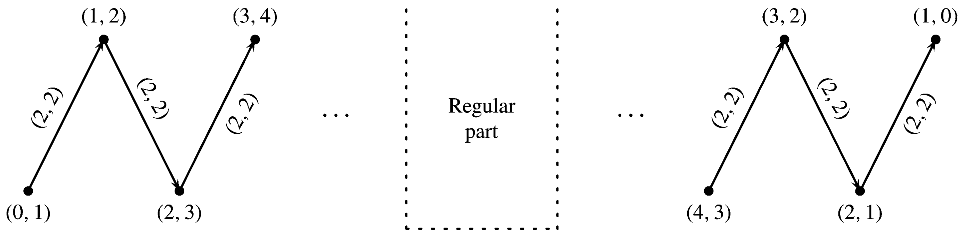
then $((\widetilde{A}_1, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{A}_{12}, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:



The AR-quiver of $((\widetilde{A}_1, \Omega), \sigma)$ is of the form:



where the regular part consists of homogeneous tubes only. Because the automorphism $\mathfrak{s}$ fixes the points and permutes the two arrows, and the number of elements of $\mathfrak{s}$-orbit of each point is one and the number of elements of $\mathfrak{s}$-orbit of each arrow is two, so from the construction in 3.1, we get the AR-quiver of $(\widetilde{A}_{12}, \widetilde{\Omega})$



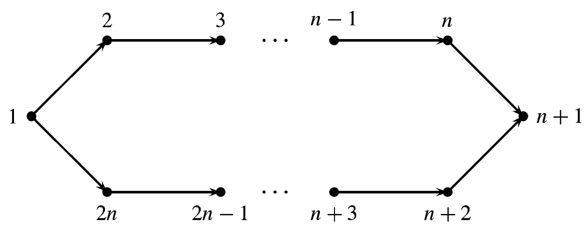
where the regular part consists of homogeneous tubes only.

Similarly, we can get the AR-quivers of all other tame valued quivers with the given admissible orientation. In the remaining cases we omit the AR-quivers, but for later use we express the non-homogeneous tubes in the regular part explicitly.

(c) If we define

$$\begin{array}{lllll} \sigma(1) = 1, & \sigma(2) = 2n, & \sigma(3) = 2n-1, & \ldots, & \sigma(n) = n+2, \\ \sigma(n+1) = n+1, & \sigma(2n) = 2, & \sigma(2n-1) = 3, & \ldots, & \sigma(n+2) = n \end{array}$$

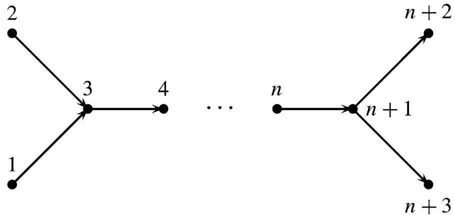
for the quiver $(\widetilde{A}_{2n-1}, \Omega)$



then $((\widetilde{A}_{2n-1}, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{B}_n, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:

The regular part contains one non-homogeneous tube c of rank n and only the even multiples of δ, i.e., $2\delta, 4\delta, \ldots$ occur in c, where δ is the minimal positive imaginary root.

(d) If we define $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$, $\sigma(4) = 4$, $\ldots$, $\sigma(n) = n$, $\sigma(n+1) = n+1$, $\sigma(n+2) = n+3$ and $\sigma(n+3) = n+2$ for the quiver $(\widetilde{D}_{n+2}, \Omega)$



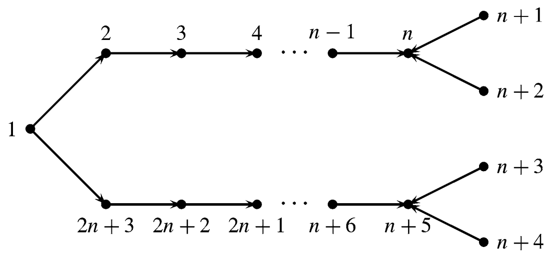
then $((\widetilde{D}_{n+2}, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{C}_n, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:

The regular part contains one non-homogeneous tube c of rank n and all positive imaginary roots occur in c.

(e) If we define

$$
\begin{aligned}
\sigma(1) = 1,\ \ \sigma(2) &= 2n+3,\ \ldots,\ \ \sigma(n-1) = n+6,\ \ \sigma(n) = n+5,\ \ \sigma(n+1) = n+3,\\
\sigma(2n+3) &= 2,\ \ldots,\ \ \sigma(n+6) = n-1,\ \ \sigma(n+5) = n,\ \ \sigma(n+3) = n+2,\\
&\sigma(n+2) = n+4,\\
&\sigma(n+4) = n+1
\end{aligned}
$$

for the quiver $(\widetilde{D}_{2n+2}, \Omega)$



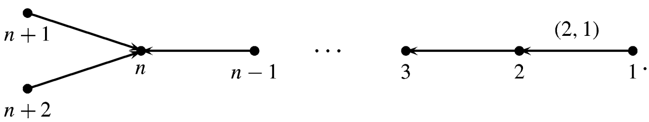
then $((\widetilde{D}_{2n+2}, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{BC}_n, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:

$\widetilde{BC}_n$: 

The regular part contains one non-homogeneous tube c of rank n and all positive imaginary roots occur in c.

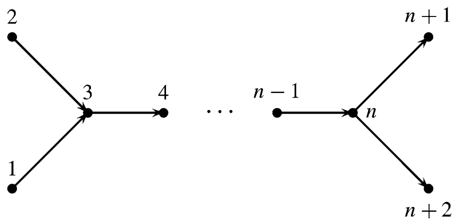
(f) If we define $\sigma(1) = 1, \sigma(2) = 2n+3, \sigma(2n+3) = 2, \sigma(3) = 2n+2, \sigma(2n+2) = 3, \sigma(4) = 2n+1, \sigma(2n+1) = 4, \ldots, \sigma(n-1) = n+6, \sigma(n+6) = n-1, \sigma(n) =$

$n+5, \sigma(n+5)=n, \sigma(n+1)=n+3, \sigma(n+2)=n+4, \sigma(n+3)=n+1, \sigma(n+4)=n+2$ for the quiver $(\widetilde{D}_{2n+2}, \Omega)$ in case (e), then $((\widetilde{D}_{2n+2}, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{CD}_{n+1}, \widetilde{\Omega})$ where $\widetilde{\Omega}$ is the following admissible orientation:

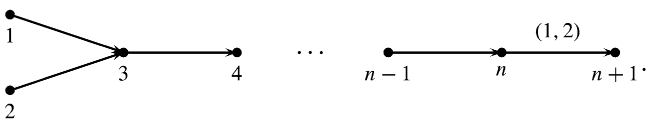


The regular part contains two non-homogeneous tubes c_1 and c_2 of rank $n-1$ and 2, respectively, and all positive imaginary roots occur in both c_1 and c_2.

(g) If we define $\sigma(1)=1$, $\sigma(2)=2$, $\sigma(3)=3, \ldots, \sigma(n-1)=n-1$, $\sigma(n)=n$, $\sigma(n+1)=n+2$, $\sigma(n+2)=n+1$, for the quiver $(\widetilde{D}_{n+1}, \Omega)$

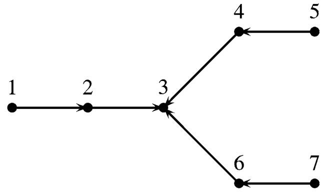


then $((\widetilde{D}_{n+1}, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{DD}_n, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:

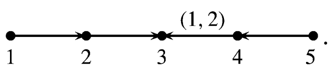


The regular part contains two non-homogeneous tubes c_1 and c_2 of rank n and 2, respectively. All positive imaginary roots occur in c_1 but in c_2 only positive even multiples of δ occur.

(h) If we define $\sigma(1)=1, \sigma(2)=2, \sigma(3)=3, \sigma(4)=6, \sigma(5)=7, \sigma(6)=4, \sigma(7)=5$ for the quiver $(\widetilde{E}_6, \Omega)$

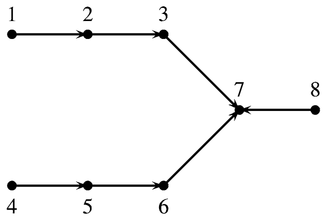


then $((\widetilde{E}_6, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{F}_{41}, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:

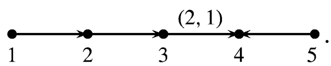


The regular part contains two non-homogeneous tubes c_1 and c_2 of rank 3 and 2, respectively. In c_1, only the positive even multiples of the minimal positive imaginary roots occur and in c_2, all positive imaginary roots occur.

(i) If we define $\sigma(1) = 4$, $\sigma(2) = 5$, $\sigma(3) = 6$, $\sigma(4) = 1$, $\sigma(5) = 2$, $\sigma(6) = 3$, $\sigma(7) = 7$, $\sigma(8) = 8$ for the quiver $(\widetilde{E}_7, \Omega)$

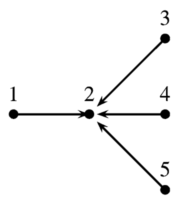


then $((\widetilde{E}_7, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{F}_{42}, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:

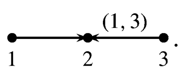


The regular part contains two non-homogeneous tubes c_1 and c_2 of rank 3 and 2, respectively. All positive imaginary roots occur in both c_1 and c_2.

(j) If we define $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 4$, $\sigma(4) = 5$, $\sigma(5) = 3$ for the quiver $(\widetilde{D}_4, \Omega)$

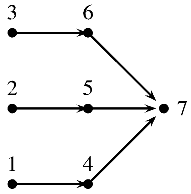


then $((\widetilde{D}_4, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{G}_{21}, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation

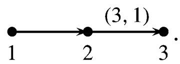


The regular part contains one non-homogeneous tube c of rank 2 and in c only $3\delta, 6\delta, \ldots$ occur.

(k) If we define $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$, $\sigma(4) = 5$, $\sigma(5) = 6$, $\sigma(6) = 5$, $\sigma(7) = 7$ for the quiver $(\widetilde{E}_6, \Omega)$



then $((\widetilde{E}_6, \Omega), \sigma)$ is an ad-quiver with automorphism σ. By the construction in 2.2.4, we get the valued quiver $(\widetilde{G}_{22}, \widetilde{\Omega})$, where $\widetilde{\Omega}$ is the following admissible orientation:



The regular part contains one non-homogeneous tube c of rank 2 and all positive imaginary roots occur in c.

4. PBW-bases of the twisted generic composition algebras of affine valued quivers

Let (Γ, d, Ω) be a valued quiver and $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be the $\mathbb{F}_q$-species of type Ω. We denote by $\mathfrak{h}(A)$ and $\mathfrak{c}(A)$ the Ringel–Hall algebra and the generic composition algebra of the tensor algebra A of $\mathcal{S}$, respectively. We denote by $\mathcal{P}(A)$ and $\mathcal{I}(A)$ the subalgebra of $\mathfrak{h}(A)$ generated by the isomorphism classes of preprojective modules and the preinjective modules, respectively. Because of Theorem 2.3, we will discuss the twisted Ringel–Hall algebra $\mathfrak{h}^*(A)$ and twisted generic composition algebra $\mathfrak{c}^*(A)$ in the following.

In this section we will construct a PBW-basis of the twisted generic composition algebra of affine valued quivers by using their AR-quivers we obtained in the previous section.

4.1. The cases $\widetilde{A}_{11}$ and $\widetilde{A}_{12}$

4.1.1. Given a positive integer n, a partition p of n is a finite sequence $(n_1, \ldots, n_t)$ of positive integers such that

$$n_1 \geqslant \cdots \geqslant n_t \geqslant 1 \quad \text{and} \quad n_1 + \cdots + n_t = n.$$

We call t the length of p, which is denoted by $l(p) = t$. Denote by $\mathbb{P}(n)$ the set of all partitions of n, and by $p(n)$ the number of elements in $\mathbb{P}(n)$. Set $p(0) = 1$; and for $p \in \mathbb{P}(0)$ set $l(p) = 0$. For partitions λ and μ, we define their cup product $\lambda \sqcup \mu$ to be the partition formed by arranging all the parts of λ and μ in descending order. Moreover, if λ and μ are partitions of some n, then we set $\lambda < \mu$ if the first time that $\lambda_i \neq \mu_i$ implies that $\lambda_i > \mu_i$ (i.e., the reverse lexicographic ordering).

Remark 4.1.2. We also know from [25] that for any homogeneous tube T, the Ringel–Hall subalgebra $\mathfrak{h}^*(T)$ generated by the isomorphism classes of indecomposables in T is commutative and it is well known that there is no Hom and Ext between two different tubes, so the Ringel–Hall subalgebra generated by the isomorphism classes of regular indecomposables is also commutative.

4.1.3. For a given $\mathbf{d} \in \mathbb{Z}^n$, where n is the number of vertices of the affine valued quiver (Γ, d, Ω), we define the following element in $\mathfrak{h}^*(A)$ following [26]

$$r_{\mathbf{d}} = \sum_{[M]} u_{[M]} \quad \text{where } M \text{ ranges over the regular modules with } \mathbf{dim}\, M = \mathbf{d}.$$

Note that this is a finite sum since $\mathbb{F}_q$ is a finite field, and that M can be decomposable. If there is no regular modules M with $\mathbf{dim}\, M = \mathbf{d}$, then we define $r_{\mathbf{d}} = 0$. Then $r_{\mathbf{d}} \in \mathfrak{c}^*(A)$ (see [26]).

For a positive integer n and a partition $p = (n_1, \ldots, n_t)$, we define

$$r_p = r_{n_1\delta} * r_{n_2\delta} * \cdots * r_{n_t\delta} \in \mathfrak{c}^*(A),$$

where δ is the minimal positive imaginary root. From the above remark, we know that r_p does not depend on the ordering of the factors.

4.1.4. From the AR-quiver of $\widetilde{A}_{11}$ and $\widetilde{A}_{12}$, we can define a total ordering for the isomorphism classes of preprojective part as follows

$$[P_i] \leqslant [P_j] \quad \text{if and only if} \quad P_i \text{ is on the left side of } P_j.$$

Similarly, we can define a same total ordering for the preinjective part.

Now, we state our main theorem about $\widetilde{A}_{11}$ and $\widetilde{A}_{12}$.

Theorem 4.1. *Let (Γ_1, d, Ω_1) be $\widetilde{A}_{11}$ or $\widetilde{A}_{12}$, where the orientation Ω_1 as in* 3.2. *Then the twisted generic composition algebra $\mathfrak{c}^*(A_1)$ has a PBW-basis consisting of elements of the form $u_{[P]} * r_p * u_{[I]}$, where A_1 is the tensor algebra of $\mathcal{S}_1$, $\mathcal{S}_1$ the corresponding $\mathbb{F}_q$-species of type Ω_1, and*

(1) $u_{[P]} = u_{[P_1]} * \cdots * u_{[P_s]}$ *with* $0 \leqslant [P_1] \leqslant \cdots \leqslant [P_s]$ *indecomposable preprojectives*;
(2) $p \in \mathbb{P}(n),\ n \in \mathbb{N}_0$;
(3) $u_{[I]} = u_{[I_1]} * \cdots * u_{[I_t]}$ *with* $0 \leqslant [I_1] \leqslant \cdots \leqslant [I_t]$ *indecomposable preinjectives.*

Proof. From [24] we know that the preprojectives, preinjectives and $r_{m\delta}$, for all $m \in \mathbb{N}_0$, belong to $\mathfrak{c}^*(A_1)$. So the space $V(A_1)$, spanned by the elements $u_{[P]} * r_p * u_{[I]}$, is a subspace of $\mathfrak{c}^*(A_1)$. Clearly $V(A_1)$ also inherits the gradation of $\mathfrak{c}^*(A_1)$. More precisely, $V(A_1) = \bigoplus_\alpha V(A_1)_\alpha$ where $V(A_1)_\alpha = \mathfrak{c}^*(A_1)_\alpha \cap V(A_1)$. Moreover, the elements $u_{[P]} * r_p * u_{[I]}$ in the theorem are linearly independent over $\mathbb{Q}(v)$. So it is enough to prove that the dimension of each graded part is same.

Let $\mathfrak{g}$ be the affine Kac–Moody Lie algebra corresponding to the underlying graph of $\mathcal{S}_1$. The positive roots of $\mathfrak{g}$ are of two types [11]: the real roots (having multiplicity one) and the imaginary roots (the multiples of δ).

We also recall that the dimension vectors of the indecomposable modules for A_1 are precisely the positive roots of $\mathfrak{g}$. (This is proved in [7] for the affine case; see also [11–13] for the general result.)

It follows from the PBW-basis of $\mathcal{U}^+(\mathfrak{g})$ that the dimension of the graded part $\mathfrak{c}^*(A_1)_\alpha$ is precisely the number of ways of expressing α as a sum of positive roots (with multiplicity). For example, $\dim \mathfrak{c}^*(A_1)_\delta$ is 1 plus the number of ways of expressing δ as a sum of positive real roots.

For an arbitrary preprojective module $P \in \mathcal{P}(A_1)$, we can write

$$P = P_1^{a_1} \oplus \cdots \oplus P_t^{a_t},$$

where $P_1, \ldots, P_t$ are indecomposable preprojective modules such that $[P_1] \leqslant \cdots \leqslant [P_t]$, then by [17]

$$u_{[P]} = \frac{1}{|a_1]!} \cdots \frac{1}{|a_t]!} u_{[P_1]}^{a_1} * \cdots * u_{[P_t]}^{a_t}.$$

It follows that the number of preprojective modules (up to isomorphism) with dimension vector α is precisely the number of ways of expressing α as a sum of dimension vectors of indecomposable preprojective modules (with multiplicity). An analogous result holds for the preinjectives.

For any dimension vector α, we consider an arbitrary expression of α as a sum of positive roots

$$\alpha = \alpha_1(pp) + \cdots + \alpha_s(pp) + n_1\delta + \cdots + n_t\delta + \alpha_1(pi) + \cdots + \alpha_r(pi),$$

where $s, t, r \geqslant 0$, $n_1 \geqslant \cdots \geqslant n_t \geqslant 0$, and $\alpha_1(pp), \ldots, \alpha_s(pp)$ are the real positive roots which are dimension vectors of indecomposable preprojective modules, $n_1\delta, \ldots, n_t\delta$ are the positive imaginary roots which are the dimension vectors of indecomposable regular modules and $\alpha_1(pi), \ldots, \alpha_r(pi)$ are the positive real roots which are dimension vectors of indecomposable preinjective modules.

We take

$$P = P_1 \oplus \cdots \oplus P_s, \quad [P_1] \leqslant \cdots \leqslant [P_s],$$
$$I = I_1 \oplus \cdots \oplus I_r, \quad [I_1] \leqslant \cdots \leqslant [I_r],$$
$$r_p = r_{n_1\delta} * r_{n_2\delta} * \cdots * r_{n_t\delta},$$

where $\mathbf{dim}\, P_1 = \alpha_1(pp), \ldots, \mathbf{dim}\, P_s = \alpha_s(pp)$ and $\mathbf{dim}\, I_1 = \alpha_1(pi), \ldots, \mathbf{dim}\, I_r = \alpha_r(pi)$. Then the element $u_{[P]} * r_p * u_{[I]}$ is in $V(A_1)_\alpha$. This implies that $\dim V(A_1)_\alpha \geqslant \dim \mathfrak{c}^*(A_1)_\alpha$. So for each dimension vector α, the dimension of $\mathfrak{c}^*(A_1)_\alpha$ and the dimension of $V(A_1)_\alpha$ are coincide. $\square$

4.2. The preprojective and preinjective cases

4.2.1. Let (Γ, d, Ω) be a tame valued quiver and $\Delta = (I, (\text{-},\text{-}))$ be the corresponding Cartan datum in the sense of Lusztig, $\mathfrak{g}$ the corresponding symmetrizable Kac–Moody algebra. We have the Drinfeld–Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$ attached to the Cartan datum Δ. Its Chevalley generators: E_i, F_i and K_α with $\alpha \in \mathbb{Z}[I]$. Lusztig in [15] has introduced the symmetries $T''_{i,1} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ for $i \in I$, which are automorphisms of $U_q(\mathfrak{g})$ and satisfy braid group relations:

$$T''_{i,1}(E_i) = -F_i K_i^{\varepsilon_i}, \qquad T''_{i,1}(F_i) = -K_i^{\varepsilon_i} E_i,$$

$$T''_{i,1}(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r\varepsilon_i} E_i^{(s)} E_j E_i^{(r)} \quad \text{for } j \neq i \text{ in } I,$$

$$T''_{i,1}(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{r\varepsilon_i} F_i^{(r)} F_j F_i^{(s)} \quad \text{for } j \neq i \text{ in } I,$$

$$T''_{i,1}(K_\beta) = K_{s_i(\beta)},$$

where $a_{ij} = (i, j)$ for $i, j \in I$, $\beta \in \mathbb{Z}I$, $E_i^{(r)} = E_i^r/[r]_{\varepsilon_i}$, $(\varepsilon_i)_i$ is the minimal symmetrization, and $s_i(\beta) = \beta - (\beta, i)i$.

For each $i \in I$, one may define

$$U_q^+(\mathfrak{g})[i] = \{x \in U_q^+(\mathfrak{g}) \mid T''_{i,1}(x) \in U_q^+(\mathfrak{g})\}.$$

Then $T''_{i,1} : U_q^+(\mathfrak{g})[i] \to U_q^+(\mathfrak{g})[i]$ is an automorphism.

If $i \in I$ is sink, then the $\mathbb{Q}(v)$-algebra isomorphism σ_i in Proposition 2.1 induces homomorphism

$$\sigma_i : \mathfrak{c}^*(A)\langle i\rangle \to \mathfrak{c}^*(\sigma_i A)\langle i\rangle,$$

where $\mathfrak{c}^*(A)\langle i\rangle = \{x \in \mathfrak{c}^*(A) \mid \sigma_i(x) \in (\sigma_i A)\}$. It is well known that $\sigma_i = T''_{i,1}$ under the identification $\mathfrak{c}^*(A) = U_q^+(\mathfrak{g})$ (see [23]).

Dually, if i is a source, we have analogous results.

4.2.2. We call an indecomposable A-module M to be exceptional if it has no self extension, i.e., $\mathrm{Ext}^1(M, M) = 0$. Then it has been proved (see [3]) that $\langle u_{[sM]}\rangle \in \mathfrak{c}^*(A)$ for any $s \geqslant 1$ if M is exceptional.

4.2.3. We denote by *Prep* and *Prei* the set of isomorphism classes of indecomposable representations in the preprojective and preinjective components of mod A, respectively. In particular, the set

$$\{\langle u_{[sM]}\rangle \mid M \text{ is indecomposable in } \textit{Prep} \text{ or } \textit{Prei} \text{ and } s \geqslant 1\}$$

lies in $\mathfrak{c}^*(A)$.

The situation we meet in this section is essentially the same as in the case of finite type.

Let $i_m, \ldots, i_1$ be an admissible sink sequence of (Γ, d, Ω), that is, i_m is a sink of (Γ, d, Ω) and for any $1 \leqslant t \leqslant m$, the vertex i_t is a sink for the orientation $\sigma_{i_{t+1}} \cdots \sigma_{i_m} \Omega$. Let M be an indecomposable module in *Prep*. Then there exists an admissible sink sequence of (Γ, d, Ω) such that

$$M = \sigma_{i_1}^{\pm} \cdots \sigma_{i_m}^{\pm}(S_{i_{m+1}}),$$

where $S_{i_{m+1}}$ is a simple module in $\operatorname{mod} \sigma_{i_m} \cdots \sigma_{i_1} A$. We have following (see [19])

Lemma 4.2. *Let M be indecomposable preinjective module. Then*

$$\langle u_M \rangle = T''_{i_1,1} \cdots T''_{i_m,1}(E_{i_{m+1}}),$$

where $M = \sigma_{i_1}^{+} \cdots \sigma_{i_m}^{+}(S_{i_{m+1}})$, for an admissible sink sequence $i_m, \ldots, i_1$ of (Γ, d, Ω).

Since *Prei* is directed, we may give a total ordering of *Prei* as follows. Let

$$\{\ldots, \beta_3, \beta_2, \beta_1\}$$

be all positive real roots appearing in *Prei*, and

$$\{\ldots, M(\beta_3), M(\beta_2), M(\beta_1)\}$$

be all indecomposables in *Prei* with $\mathbf{dim}\, M(\beta_i) = \beta_i$. We require that a total ordering $\preccurlyeq$ in *Prei* satisfies the following

$$\operatorname{Hom}\bigl(M(\beta_i), M(\beta_j)\bigr) \neq 0 \quad \text{implies} \quad \beta_i \prec \beta_j \quad \text{and} \quad i \geqslant j.$$

Then such an ordering has the property

$$\begin{aligned} \langle \beta_i, \beta_j \rangle > 0 \quad &\text{implies} \quad \beta_i \preccurlyeq \beta_j \quad \text{and} \quad i \geqslant j, \\ \langle \beta_i, \beta_j \rangle < 0 \quad &\text{implies} \quad \beta_j \prec \beta_i \quad \text{and} \quad i < j \end{aligned}$$

and

$$\operatorname{Ext}\bigl(M(\beta_i), M(\beta_j)\bigr) = 0 \quad \text{for } i \geqslant j.$$

There is no harm to denote by $Prei = \{\ldots, \beta_3, \beta_2, \beta_1\}$.

Similarly, *Prep* is directed, we may give a total ordering of *Prep* as follows. Let

$$\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$$

be all positive real roots appearing in *Prep*, and

$$\{M(\alpha_1), M(\alpha_2), M(\alpha_3), \ldots\}$$

be all indecomposables in *Prep* with $\mathbf{dim}\, M(\alpha_i) = \alpha_i$. We require that a total ordering $\preccurlyeq$ in *Prep* satisfies the following

$$\mathrm{Hom}\big(M(\alpha_i), M(\alpha_j)\big) \neq 0 \quad \text{implies} \quad \alpha_i \prec \alpha_j \quad \text{and} \quad i \leqslant j.$$

Then such an ordering has the property

$$\langle \alpha_i, \alpha_j \rangle > 0 \quad \text{implies} \quad \alpha_i \preccurlyeq \alpha_j \quad \text{and} \quad i \leqslant j,$$
$$\langle \alpha_i, \alpha_j \rangle < 0 \quad \text{implies} \quad \alpha_j \prec \alpha_i \quad \text{and} \quad j < i$$

and

$$\mathrm{Ext}\big(M(\alpha_i), M(\alpha_j)\big) = 0 \quad \text{for } i \leqslant j.$$

There is no harm to denote by $Prep = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$.

Let $\mathbf{b} : Prei \to \mathbb{N}$ be a support finite function, we denote it by $\mathbf{b} \in \mathbb{N}^{Prei}$. Then

$$M(\mathbf{b}) = \bigoplus_{\beta_i \in Prei} \mathbf{b}(\beta_i) M(\beta_i)$$

is a preinjective module and any preinjective module is of this form, up to isomorphism. We set

$$\langle u_{[M(\mathbf{b})]} \rangle = v^{-\dim M(\mathbf{b}) + \dim \mathrm{End}(M(\mathbf{b}))} u_{[M(\mathbf{b})]}.$$

By Ringel [19], we have

Lemma 4.3. *For any* $\mathbf{b} \in \mathbb{N}^{Prei}$,

$$\langle u_{M(\mathbf{b})} \rangle = \langle u_{[\mathbf{b}(\beta_{i_m}) M(\beta_{i_m})]} \rangle * \cdots * \langle u_{[\mathbf{b}(\beta_{i_1}) M(\beta_{i_1})]} \rangle,$$

where $\{\beta_{i_m} \prec \beta_{i_{m-1}} \prec \cdots \prec \beta_{i_1}\}$ *are those* $\beta \in Prei$ *such that* $\mathbf{b}(\beta) \neq 0$.

Because all indecomposable preinjectives are exceptional, $\langle u_{[M(\mathbf{b})]} \rangle \in \mathfrak{c}^*(A)$ for all $\mathbf{b} \in \mathbb{N}^{Prei}$. Therefore, we are ready to define $\mathfrak{c}^*(Prei)$ to be the $\mathbb{Q}(v)$-submodule of $\mathfrak{c}^*(A)$ generated by

$$\big\{ \langle u_{[M(\mathbf{b})]} \rangle \mid \mathbf{b} \in \mathbb{N}^{Prei} \big\}.$$

We have

Lemma 4.4. *The* $\mathbb{Q}(v)$*-submodule* $\mathfrak{c}^*(Prei)$ *is an subalgebra of* $\mathfrak{c}^*(A)$ *and*

$$\big\{ \langle u_{[M(\mathbf{b})]} \rangle \mid \mathbf{b} \in \mathbb{N}^{Prei} \big\}$$

is a $\mathbb{Q}(v)$*-basis of* $\mathfrak{c}^*(Prei)$.

For a proof, see [14]. We have similar results for *Prep*.

Lemma 4.5. *For any* $\mathbf{a} \in \mathbb{N}^{Prep}$, *we have*

$$\langle u_{[M(\mathbf{a})]}\rangle = \langle u_{[\mathbf{a}(\alpha_{i_1})M(\alpha_{i_1})]}\rangle * \cdots * \langle u_{[\mathbf{a}(\alpha_{i_m})M(\alpha_{i_m})]}\rangle,$$

where $\{\alpha_{i_1}, \ldots, \alpha_{i_m}$ *with* $\alpha_{i_1} \prec \alpha_{i_2} \prec \cdots \prec \alpha_{i_m}\}$ *are those* $\alpha \in Prep$ *such that* $\mathbf{a}(\alpha) \neq 0$.

Lemma 4.6. *Let* $\mathfrak{c}^*(Prep)$ *be the* $\mathbb{Q}(v)$*-submodule of* $\mathfrak{c}^*(A)$ *generated by*

$$\{\langle u_{[M(\mathbf{a})]}\rangle \mid \mathbf{a} \in \mathbb{N}^{Prep}\}.$$

Then $\mathfrak{c}^*(Prep)$ *is an subalgebra of* $\mathfrak{c}^*(A)$ *and* $\{\langle u_{[M(\mathbf{a})]}\rangle \mid \mathbf{a} \in \mathbb{N}^{Prep}\}$ *is a* $\mathbb{Q}(v)$*-basis of* $\mathfrak{c}^*(Prep)$.

4.3. The PBW-basis for affine valued quivers

4.3.1. From [7] we know that there is a full exact embedding $T : \text{mod-}A_1 \hookrightarrow \text{mod-}A$. This gives rise to an injection of algebras, still denoted by $T : \mathfrak{h}^*(A_1) \hookrightarrow \mathfrak{h}^*(A)$. The injection T maps $\mathfrak{c}^*(A_1)$ into $\mathfrak{c}^*(A)$. We have defined the elements r_p ($p \in \mathbb{P}(n)$, $n \in \mathbb{N}_0$) in $\mathfrak{h}^*(A_1)$ and we know that they are in $\mathfrak{c}^*(A_1)$, so

$$E_{p\delta} =: E_{p_1\delta} * \cdots * E_{p_t\delta} =: T(r_p) = T(r_{p_1\delta}) * \cdots * T(r_{p_t\delta}) \in \mathfrak{c}^*(A)$$

for all $p = (p_1, \ldots, p_t) \in \mathbb{P}(n)$, $n \in \mathbb{N}_0$.

4.3.2. We may list all non-homogeneous tubes $\mathcal{T}_1(r_1), \mathcal{T}_2(r_2), \ldots, \mathcal{T}_s(r_s)$ in mod-A where r_i is the length of $\mathcal{T}_i(r_i)$ (in fact $s \leqslant 3$). Let $S_1, \ldots, S_{r_i}$ be all the simple objects in $\mathcal{T}(r_i)$ (called quasi-simple modules in mod-A) and $S_i[l]$ the (unique) indecomposable module in $\mathcal{T}(r_i)$ with top S_i and length l.

Let Π denote the set of all n-tuples of partitions. Then for each element $\pi = (\pi(1), \pi(2), \ldots, \pi(n)) \in \Pi$, we define a module in the non-homogenous tube $\mathcal{T}_k(r_k)$, $k \in \{1, 2, \ldots, s\}$,

$$M(\pi) = \bigoplus_{\substack{i \in \Delta_0 \\ j \geqslant 1}} S_i\big[\widetilde{\pi}_j^{(i)}\big],$$

where $\widetilde{\pi}^{(i)} = (\widetilde{\pi}_1^{(i)}, \widetilde{\pi}_2^{(i)}, \ldots)$ is the partition dual to $\pi^{(i)}$ and Δ_0 is the set of vertices on the mouth of $\mathcal{T}_k(r_k)$. In this way, we obtain a bijection between Π and the set of isomorphism classes of nilpotent representations in $\mathcal{T}_k(r_k)$.

An n-tuple $\pi = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)})$ of partition is called aperiodic (in the sense of Lusztig [15]), or separated (in the sense of Ringel [18]), if for each $l \geqslant 1$ there is some $i = i(l) \in \Delta_0$ such that $\widetilde{\pi}_j^{(i)} \neq l$ for all $j \geqslant 1$. By Π^a, we denote the set of aperiodic n-tuples of partitions. A module in $\mathcal{T}_k(r_k)$ is called aperiodic if $M \simeq M(\pi)$ for some $\pi \in \Pi^a$.

It is well known that (see [7])

$$\sum_{i=1}^{s}(r_i - 1) = n - 1,$$

where $n+1$ is the number of vertices.

4.3.3. For each non-homogeneous tube $\mathcal{T}_i(r_i)$, we have the generic composition algebra $\mathfrak{c}^*(\mathcal{T}_i(r_i))$ and the set Π_i^a of aperiodic r_i-tuples of partitions such that for any $\pi \in \Pi_i^a$, $M(\pi)$ is a aperiodic module in $\mathcal{T}_i(r_i)$. Note that in all cases in 3.1, the valuations for each arrow in tubes are $(1,1)$. So we can identify each non-homogeneous tube $\mathcal{T}_i(r_i)$ with the regular part of $\widetilde{A}_{r_i}$ with the cyclic orientation. Hence, from [14], for each $\mathcal{T}_i(r_i)$, we have a PBW-basis $\{E_\pi \mid \pi \in \Pi_i^a\}$ of $\mathfrak{c}^*(\mathcal{T}_i(r_i))$, where

$$E_\pi = \langle u_{[M(\pi)]}\rangle + \sum_{\lambda\in\Pi_i^a,\, \lambda\prec\pi} \eta_\lambda^\pi \langle u_{[M(\lambda)]}\rangle.$$

For later use, we recall a lemma from [14].

Lemma 4.7. *Let $\{S_j \mid 1 \leqslant j \leqslant r_i\}$ be a complete set of non-isomorphic quasi-simple modules of a non-homogeneous tube $\mathcal{T}$ such that $S_j = \tau^{(j-1)}S_1$ and $\mathfrak{h}^*(\mathcal{T})$ is the twisted Ringel–Hall algebra of $\mathcal{T}$ over $\mathbb{Q}(v)$, where τ is the AR-translation.*

(1) *If $r_i \nmid l$, $1 \leqslant j \leqslant r_i$, then*

$$u_{S_j[l]} \equiv \sum_{\lambda\preccurlyeq\pi,\, \lambda\in\Pi^a} a_\lambda E_\lambda \quad \big(\mathrm{mod}(v-1)\mathfrak{h}^*(\mathcal{T})\big),$$

where $a_\lambda \in \mathbb{Q}$ and $S_j[l] \simeq M(\pi)$.

(2) *If $r_i \mid l$, $1 \leqslant j \leqslant r_i - 1$, then*

$$u_{S_j[l]} - u_{S_{j+1}[l]} \equiv \sum_{\lambda\preccurlyeq\pi\,(\mathrm{or}\ \pi'),\, \lambda\in\Pi^a} a_\lambda E_\lambda \quad \big(\mathrm{mod}(v-1)\mathfrak{h}^*(\mathcal{T})\big),$$

where $a_\lambda \in \mathbb{Q}$ and $S_j[l] \simeq M(\pi)$, $S_{j+1}[l] \simeq M(\pi')$.

Note that there is natural imbedding $\mathfrak{c}^*/(v-1)\mathfrak{c}^*$ into $\mathfrak{h}^*/(v-1)\mathfrak{h}^*$, so we may replace $\mathfrak{h}^*$ in Lemma 4.7 by $\mathfrak{c}^*$.

Now we define a set $\mathcal{M}$ by the following rule. Any $\mathbf{c} \in \mathcal{M}$ is given by the data:

(1) a support-finite function $\mathbf{a_c}: Prep \to \mathbb{N}$,
(2) a support-finite function $\mathbf{b_c}: Prei \to \mathbb{N}$,
(3) an element $\pi_{i\mathbf{c}} \in \Pi_i^a$ for each $\mathcal{T}_i$, $1 \leqslant i \leqslant s$,
(4) a partition $\omega_{\mathbf{c}} = (\omega_1, \omega_2, \ldots, \omega_t)$ for some $t \geqslant 1$, where $\omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_t$ are in $\mathbb{N} \setminus \{0\}$.

Then for each $\mathbf{c} \in \mathcal{M}$, we may define

$$E_{\mathbf{c}} = \langle u_{[M(\mathbf{a_c})]} \rangle * E_{\pi_{1\mathbf{c}}} * E_{\pi_{2\mathbf{c}}} * \cdots * E_{\pi_{s\mathbf{c}}} * E_{\omega_{\mathbf{c}}\delta} * \langle u_{[M(\mathbf{b_c})]} \rangle,$$

where $\langle u_{[M(\mathbf{a_c})]} \rangle$ and $\langle u_{[M(\mathbf{b_c})]} \rangle$ are defined just before Lemma 4.3, $E_{\pi_{i\mathbf{c}}}$ for $1 \leqslant i \leqslant s$ are defined in 3.3.3 and $E_{\omega_{\mathbf{c}}\delta}$ is defined in 4.3.1. Obviously, $\{E_{\mathbf{c}} \mid \mathbf{c} \in \mathcal{M}\}$ lies in $\mathfrak{c}^*(A)$ and linearly independent over $\mathbb{Q}(v)$.

Theorem 4.8. *The set* $\{E_{\mathbf{c}} \mid \mathbf{c} \in \mathcal{M}\}$ *is a PBW-basis of* $\mathfrak{c}^*(A)$.

Proof. Let V be the subspace of $\mathfrak{c}^*(A)$ spanned by the basis elements $E_{\mathbf{c}}$. Then V is contained in $\mathfrak{c}^*(A)$ and inherits the gradation of $\mathfrak{c}^*(A)$. More precisely, $V = \bigoplus_\alpha V_\alpha$, where $V_\alpha = V \cap \mathfrak{c}^*(A)_\alpha$. From the observation just before the theorem, it is enough to prove that $\dim \mathfrak{c}^*(A)_\alpha$ and $\dim V_\alpha$ are equal for all dimension vectors α. For this, first we investigate the corresponding dimensions at Lie algebra level.

For each tame valued quiver (Γ, d, Ω), we denote by $\mathfrak{g}^+(A)$ the positive part of corresponding symmetrizable Kac–Moody Lie algebra, where A is the tensor algebra of $\mathbb{F}_q$-species $\mathcal{S}$ of type Ω. Then we have the root space decomposition

$$\mathfrak{g}^+(A) = \bigoplus_\alpha \mathfrak{g}(A)_\alpha,$$

where α ranges over all positive roots of $\mathfrak{g}(A)$.

Let $\mathfrak{g}_1(A)$ be the Lie algebra arising from module category of A (see [16]). It is well known from Ringel that the Lie subalgebra $\mathfrak{g}_1^+(A) \subset \mathfrak{c}^*(A)/(v-1)\mathfrak{c}^*(A)$ generated by $u_{[S_i]}$, $i \in I$ over $\mathbb{Q}$, is just the positive part of the corresponding symmetrizable affine Kac–Moody Lie algebra $\mathfrak{g}(A)$ over $\mathbb{Q}$, and $\mathfrak{c}^*(A)/(v-1)\mathfrak{c}^*(A)$ is the universal enveloping algebra of $\mathfrak{g}_1^+(A)$. So the dimensions of each corresponding root spaces of $\mathfrak{g}^+(A)$ and $\mathfrak{g}_1^+(A)$ are coincide.

In the following we will construct a suitable basis for each root spaces of $\mathfrak{g}_1^+(A)$.

If α is positive real root, then $\dim \mathfrak{g}(A)_\alpha = 1$ and there is unique indecomposable module M with dimension vector α and we will take $u_{[M]}$ as a basis for $\mathfrak{g}_1(A)_\alpha$.

If α is positive imaginary root, say $\alpha = m\delta$, $m > 0$, where δ is the minimal positive imaginary root of $\mathfrak{g}(A)$. Then we examine this case by case.

(i) Type $\widetilde{B}_n$. Then by [10]

$$\dim \mathfrak{g}(A)_\alpha = \begin{cases} n, & \text{if } m \text{ is even,} \\ 1, & \text{if } m \text{ is odd.} \end{cases}$$

By 3.2(c), for each even $m = 2l$, we can take $n-1$ elements of the form $u_{[S_j[ln]]} - u_{[S_{j+1}[ln]]}$. Together with the element $T(r_{m\delta})$, we have n linearly independent elements in $\mathfrak{g}_1(A)_{m\delta}$ for even m and one element for odd m, respectively.

(ii) Type $\widetilde{C}_n, \widetilde{BC}_n$ or $\widetilde{CD}_n$. Then by [10] $\dim \mathfrak{g}(A)_\alpha = n$. By 3.2(d), (e) and (f) we can take $n-1$ elements of the form $u_{[S_j[l]]} - u_{[S_{j+1}[l]]}$, $(n \mid l)$ for each m. Together with the element $T(r_{m\delta})$, we have n linearly independent elements in $\mathfrak{g}_1(A)_{m\delta}$ for each m.

(iii) Type $\widetilde{DD}_n$. Then by [10]

$$\dim \mathfrak{g}(A)_\alpha = \begin{cases} n, & \text{if } m \text{ is even,} \\ n-1, & \text{if } m \text{ is odd.} \end{cases}$$

By 3.2(g), for each even $m = 2l$, we can take $n-1$ elements of the form $u_{[S_j[2l]]} - u_{[S_{j+1}[2l]]}$. For each odd $m = 2l-1$, we can take $n-1$ elements of the form $u_{[S_j[2l-1]]} - u_{[S_{j+1}[2l-1]]}$. So together with the element $T(r_{m\delta})$, we have n linearly independent elements for each even m and $n-1$ linearly independent elements of the form $u_{[S_j[2l]]} - u_{[S_{j+1}[2l]]}$ for each odd m, in $\mathfrak{g}_1(A)_{m\delta}$, respectively.

(iv) Type $\widetilde{F}_{41}$. Then by [10]

$$\dim \mathfrak{g}(A)_\alpha = \begin{cases} 4, & \text{if } m \text{ is even,} \\ 2, & \text{if } m \text{ is odd.} \end{cases}$$

By 3.2(h), for each even $m = 2l$, we can take two elements of the form $u_{[S_j[3l]]} - u_{[S_{j+1}[3l]]}$ and one element of the form $u_{[S_j[4l]]} - u_{[S_{j+1}[4l]]}$. For each odd $m = 2l-1$, we can take one element of the form $u_{[S_j[4l-2]]} - u_{[S_{j+1}[4l-2]]}$. Together with the element $T(r_{m\delta})$, we have 4 and 2 linearly independent elements in $\mathfrak{g}_1(A)_{m\delta}$ for even m and odd m, respectively.

(v) Type $\widetilde{F}_{42}$. Then by [10] $\dim \mathfrak{g}(A)_\alpha = 4$.

By 3.2(i), for each m, we can take two elements of the form $u_{[S_j[3l]]} - u_{[S_{j+1}[3l]]}$ and one element of the form $u_{[S_j[2l]]} - u_{[S_{j+1}[2l]]}$. Together with the element $T(r_{m\delta})$, we have 4 linearly independent elements in $\mathfrak{g}_1(A)_{m\delta}$ for each m.

(vi) Type $\widetilde{G}_{21}$. Then by [10]

$$\dim \mathfrak{g}(A)_\alpha = \begin{cases} 2, & \text{if } 3 \mid m, \\ 1, & \text{if } 3 \nmid m. \end{cases}$$

By 3.2(j), for each m with $3 \mid m$, we can take one element of the form $u_{[S_j[3l]]} - u_{[S_{j+1}[3l]]}$. Together with the element $T(r_{m\delta})$, we have 2 and 1 linearly independent elements in $\mathfrak{g}_1(A)_{m\delta}$ for $m, 3 \mid m$ and for $m, 3 \nmid m$, respectively.

(vii) Type $\widetilde{G}_{22}$. Then by [10] $\dim \mathfrak{g}(A)_\alpha = 2$.

By 3.2(k), for each m, we can take one element of the form $u_{[S_j[2l]]} - u_{[S_{j+1}[2l]]}$. Together with the element $T(r_{m\delta})$, we have 2 linearly independent elements in $\mathfrak{g}_1(A)_{m\delta}$ for each m.

It is well known from Lusztig that

$$\dim_{\mathbb{Q}(v)} \mathfrak{c}^*(A)_\alpha = \dim_{\mathbb{Q}}\big(\mathfrak{c}^*(A)/(v-1)\mathfrak{c}^*(A)\big)_\alpha$$

for each dimension vector α.

It follows from the PBW-basis of $U_q^+(\mathfrak{g}(A))$ that the dimension of each graded part $\mathfrak{c}^*(A)_\alpha$ is precisely the number of ways of expressing α as a sum of positive roots (with multiplicity).

For the root space decomposition

$$\mathfrak{g}_1(A) = \bigoplus_\alpha \mathfrak{g}_1(A)_\alpha$$

of $\mathfrak{g}_1(A)$, by $\sum_{i=1}^s (r_i - 1) = n - 1$, we could take a homogeneous basis $v_1, v_2, \ldots$ where v_j is the element of the form (i) u_{α_j} if α_j is a positive real root corresponding to preprojective or preinjective indecomposable module; (ii) $u_{[S_j[l]]}$ if α_j is positive real root corresponding to regular indecomposable module and $l \nmid$ the rank of corresponding tube; (iii) $u_{[S_j[l]]} - u_{[S_{j+1}[l]]}$ if α_j is positive imaginary root corresponding to regular indecomposable module in non-homogeneous tube and $l \mid$ the rank of corresponding tube; (iv) $E_{\omega\delta}$ if $\alpha_j = \omega\delta$ is the positive imaginary root corresponding to regular indecomposable module in homogeneous tube.

Let $\dim_{\mathbb{Q}}(\mathfrak{c}^*(A)/(v-1)\mathfrak{c}^*(A))_\alpha = t$, then we can take a basis for $\dim_{\mathbb{Q}}(\mathfrak{c}^*(A)/(v-1)\mathfrak{c}^*(A))_\alpha$ consisting of the monomials $x_i = v_{i_1} \cdots v_{i_s}$ of the basis elements $v_1, v_2, \ldots$ of $\mathfrak{g}_1(A)$, where $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_s$ and $i = 1, \ldots, t$.

From Lemma 4.7, we know that each x_i belong to V_α modulo $(v-1)\mathfrak{c}^*(A)$. In this way, we can find elements $y_1, \ldots, y_t$ in V_α such that $y_i = x_i + (v-1)z_i$, $i = 1, \ldots, t$, where each $z_i \in \mathfrak{c}^*(A)$. From Nakayama lemma we can see that the elements $y_1, \ldots, y_t$ are $\mathbb{Q}(v)$-linearly independent. Otherwise, $x_1, \ldots, x_t$ are $\mathbb{Q}(v)$-linearly dependent, a contradiction. So $\dim_{\mathbb{Q}(v)} V_\alpha \geqslant \dim_{\mathbb{Q}(v)} \mathfrak{c}^*(A)_\alpha$. Hence the dimensions of V_α and $\mathfrak{c}^*(A)_\alpha$ coincide. □

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