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Nonblockers in hyperspaces

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ABSTRACT

Using nonblockers in hyperspaces (Illanes and Krupski (2011) [3]), we characterize some classes of locally connected continua: the simple closed curve, the arc, trees, and dendrites. We prove that the simple closed curve is the unique locally connected continuum for which the set of nonblockers of singletons is a continuum. We show that the set of nonblockers is also a continuum for the circle of pseudo-arcs.

1. Introduction

For a continuum X (nonempty, compact, connected metric space) the hyperspace of all nonempty closed subsets of X, with the Hausdorff metric, is denoted by 2^X [5]. Given a continuum X we say that an element B of $2^X \setminus \{X\}$ *does not block the singletons* of X provided that, for each $x \in X \setminus B$, there exists a map, $\alpha : [0,1] \to 2^X$, such that $\alpha(0) = \{x\}$, $\alpha(1) = X$ and $\alpha(t) \cap B = \emptyset$ for each $0 \leqslant t < 1$ [3]. The set of all elements of the hyperspace that does not block the singletons is denoted by $\mathcal{NB}(F_1(X))$. Using this set we present characterizations of some classes of locally connected continua: the simple closed curve, the arc, trees, and dendrites (Section 3). We prove that the simple closed curve is the unique locally connected continuum X for which $\mathcal{NB}(F_1(X))$ is a subcontinuum of 2^X (Section 4). Moreover, we study the behavior of nonblockers under open monotone maps and, as a consequence, we prove that the set of nonblockers of the circle of pseudo-arcs is also a continuum (Section 5). First, we present necessary preliminaries (Section 2).

2. Definitions and basic facts

A *continuum* is a nonempty, compact, connected metric space. A *subcontinuum* is a continuum contained in a space. An *arc* is a space homeomorphic to the closed interval $[0,1]$; a *simple closed curve* is a space homeomorphic to the circle

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$S^1 = \{(x, y) \in \mathbb{R}^2\colon\ x^2 + y^2 = 1\}$. A *tree* is a continuum which contains no simple closed curves and is the union of finitely many arcs, any two of which are either disjoint or intersect only in one of their end points. A *dendrite* is a locally connected continuum which contains no simple closed curves. A *circle of pseudo-arcs* is a circle-like continuum that has a continuous decomposition into pseudo-arcs such that the decomposition space is a circle [7]. Bing and Jones have proved that the circle of pseudo-arcs exists and it is topologically unique [2]. A subcontinuum A of a continuum X is said to be *terminal* if each subcontinuum B of X that intersects A satisfies either $A \subset B$ or $B \subset A$. Given a subset A of a space X, the interior, the closure and the boundary of A in X are denoted respectively by $int(A)$, $\overline{A}$, and $Bd(A)$. The cardinality of the set A is denoted by $|A|$. The symbol $\mathbb{N}$ denotes the set of positive integers.

A point x in a continuum X is called a *cut point of* X if $X \setminus \{x\}$ is not connected, otherwise x is called a *non-cut point* of X. The point $x \in X$ is called an *end point of* X provided that whenever U is an open subset of X such that $x \in U$, there exists an open subset V of X such that $x \in V \subset U$ and $|Bd(V)| = 1$. The symbol $E(X)$ denotes the set of all end points of X.

A map (i.e. continuous function) f from a continuum X onto a continuum Y is said to be *open* provided that $f(U)$ is an open subset of Y for each open subset U of X. The map f is *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$.

Given a continuum X, the *hyperspace of all nonempty closed subsets* of X equipped with the Hausdorff metric, or equivalently with the Vietoris topology, is denoted by 2^X [5, (0.2), (0.13)]. The *hyperspace of all subcontinua* of X, i.e. the elements of 2^X which are connected, is denoted by $C(X)$. For each $n \in \mathbb{N}$ we put $F_n(X) = \{A \in 2^X\colon\ |A| \leqslant n\}$. Given $A, B \in 2^X$ such that $A \subset B \neq A$, an *order arc from A to B* is a map $\alpha : [0, 1] \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$ and $s < t$ implies that $\alpha(s) \subset \alpha(t) \neq \alpha(s)$ [5, (1.8)].

For a continuum X and elements A and B of 2^X we say that B *does not block* A provided that there exists a map, $\alpha : [0, 1] \to 2^X$, such that $\alpha(0) = A$, $\alpha(1) = X$ and $\alpha(t) \cap B = \emptyset$ for each $0 \leqslant t < 1$, [3, 0.1]. In this paper we restrict ourselves to study the *set of nonblockers of the singletons of a continuum* X, it means the set of all elements B of $2^X \setminus \{X\}$ such that B does not block $\{x\}$ for each $x \in X \setminus B$, which is denoted by $\mathcal{NB}(F_1(X))$.

Remark 2.1. Let X be a continuum, $B \in 2^X$ and $x \in X \setminus B$. If B does not block $\{x\}$, then $int(B) = \emptyset$ and $X \setminus H$ is connected, for each $H \subset B$ [3, Proposition 1.1(b), (c), (f)]. In particular, B contains no cut points of X.

The following result is a particular case of Proposition 1.3 in [3].

Proposition 2.2. *For a continuum X, $B \in 2^X$ and $x \in X \setminus B$, the following statements are equivalent*:

(a) *B does not block $\{x\}$*;
(b) *there exists an order arc, $\alpha : [0, 1] \to C(X)$, from $\{x\}$ to X such that $\alpha(t) \cap B = \emptyset$, for each $0 \leqslant t < 1$*;
(c) *there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of elements in $C(X)$, such that $x \in A_n \subset A_{n+1} \subset X \setminus B$, $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} A_n$ is a dense set in X*;
(d) *$\bigcup\{A \in C(X)\colon\ x \in A \subset X \setminus B\}$ is a dense set in X.*

Remark 2.3. If X is a continuum and $\alpha : [0, 1] \to 2^X$ is a map such that $\alpha(1) = X$, then $\bigcup\{\alpha(t)\colon\ 0 \leqslant t < 1\}$ is a dense set in X. Indeed, denote $A_n = \bigcup \alpha([0, 1 - \frac{1}{n}])$, $n \in \mathbb{N}$. Notice that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence in $C(X)$ [5, (1.43)], which converges to X in $C(X)$. Since $A_n \subset A_{n+1}$, this sequence also converges to $\overline{\bigcup_{n\in\mathbb{N}} A_n}$ [4, 4.16]. Thus, $X = \overline{\bigcup_{n\in\mathbb{N}} A_n}$, and the result follows.

3. Nonblockers in locally connected continua

In this section we show that nonblockers of singletons in a locally connected continuum are precisely the nonempty closed sets with empty interior, that do not separate the continuum; we use this fact to characterize certain classes of continua.

Theorem 3.1. *If X is a locally connected continuum and $B \in 2^X$, then $B \in \mathcal{NB}(F_1(X))$ if and only if $int(B) = \emptyset$ and $X \setminus B$ is connected.*

Proof. The necessity is justified in Remark 2.1. For sufficiency, we fix $B \in 2^X$ such that $int(B) = \emptyset$ and $X \setminus B$ is connected. We note that $X \setminus B$ is arcwise connected [6, 8.26]. Thus, for each $x \in X \setminus B$, $\bigcup\{A \in C(X)\colon\ x \in A \subset X \setminus B\} = X \setminus B$. It follows that $\bigcup\{A \in C(X)\colon\ x \in A \subset X \setminus B\}$ is a dense subset in X. By Proposition 2.2(d), we have the conclusion. □

Theorem 3.2. *A locally connected continuum X is a simple closed curve if and only if $\mathcal{NB}(F_1(X)) = F_1(X)$.*

Proof. Suppose that X is a simple closed curve. It is clear that $F_1(X) \subset \mathcal{NB}(F_1(X))$. To obtain the other inclusion, let $B \in 2^X \setminus F_1(X)$ and consider two different points, p and q, in B. Notice that $X \setminus \{p, q\}$ is not connected. So, B blocks $\{x\}$ for each $x \in X \setminus B$ (Remark 2.1), hence, $B \notin \mathcal{NB}(F_1(X))$. Thus, $\mathcal{NB}(F_1(X)) \subset F_1(X)$.

For the converse, we fix two different points p and q in X. By hypothesis, $\{p, q\} \notin \mathcal{NB}(F_1(X))$. By Theorem 3.1, $X \setminus \{p, q\}$ is not connected. It follows that X is a simple closed curve [6, 9.31]. □

Question 3.3. Is a simple closed curve the only continuum X such that $\mathcal{NB}(F_1(X)) = F_1(X)$?

Theorem 3.4. *For a locally connected continuum X, the following statements are equivalent*:

(a) *X is an arc*;
(b) *there are two distinct points p and q in X such that $\mathcal{NB}(F_1(X)) = \{\{p\}, \{q\}, \{p, q\}\}$*;
(c) $|\mathcal{NB}(F_1(X))| = 3$.

Proof. It is clear that (a) implies (b), and (b) implies (c). We will prove that (c) implies (a). Suppose that X has three distinct points, say p, q and r, which are non-cut points of X. By Theorem 3.1 and by hypothesis in (c), we have that $\{\{p\}, \{q\}, \{r\}\} = \mathcal{NB}(F_1(X))$. So, if x and y are distinct points in X, then $\{x, y\} \notin \mathcal{NB}(F_1(X))$. Hence, $X \setminus \{x, y\}$ is not connected (Theorem 3.1). It follows that X is a simple closed curve [6, 9.31]. By Theorem 3.2, we obtain that $|X| = 3$, a contradiction. We have showed that X has at most two non-cut points. So, X is an arc [6, 6.17]. $\square$

Theorem 3.5. *A locally connected continuum X is a tree if and only if $\mathcal{NB}(F_1(X))$ is a finite set.*

Proof. Suppose that X is a tree. Let $B \in \mathcal{NB}(F_1(X))$. We note that B contains no cut points of X (Remark 2.1). It follows that $B \subset E(X)$ [6, 9.27]. Hence, $\mathcal{NB}(F_1(X))$ is a subset of the power set of $E(X)$. Since $E(X)$ is a finite set [6, 9.27 and 9.28], we conclude that $\mathcal{NB}(F_1(X))$ is finite.

Conversely, let p be a non-cut point of X. We have that $\{p\} \in \mathcal{NB}(F_1(X))$ (Theorem 3.1). Then, by hypothesis, we obtain that X has only finitely many non-cut points. Therefore, X is a tree [6, 9.28]. $\square$

Theorem 3.6. *A locally connected continuum X is a dendrite if and only if $\mathcal{NB}(F_1(X)) = \{B \in 2^X\colon\ B \subset E(X)\}$.*

Proof. Suppose that X is a dendrite. Let $B \in \mathcal{NB}(F_1(X))$. Notice that B contains no cut points of X (Remark 2.1). Hence, $B \subset E(X)$ [6, 10.7]. So, $\mathcal{NB}(F_1(X)) \subset \{B \in 2^X\colon\ B \subset E(X)\}$. Now, let $B \in 2^X$ such that $B \subset E(X)$. Thus, B contains no cut points of X [6, 10.7]. Since the set of all cut points of a dendrite is a dense set [6, 5.5 and 10.8], it follows that $int(B) = \emptyset$. Moreover, we have that $X \setminus B$ is a connected set [6, 6.27]. So, by Theorem 3.1, we obtain that $B \in \mathcal{NB}(F_1(X))$. Thus, $\{B \in 2^X\colon\ B \subset E(X)\} \subset \mathcal{NB}(F_1(X))$.

Conversely, suppose that X is a locally connected continuum such that $\mathcal{NB}(F_1(X)) = \{B \in 2^X\colon\ B \subset E(X)\}$. Fix a point $x \in X \setminus E(X)$. We have that $\{x\} \notin \mathcal{NB}(F_1(X))$. By Theorem 3.1, x is a cut point of X. We have proved that each point of X is either an end point of X or a cut point of X. This proves that X is a dendrite [6, 10.7]. $\square$

Remark 3.7. Let X be the familiar $\sin(\frac{1}{x})$-continuum with enlarged limit segment, that is $X = (\{0\} \times [-2, 1]) \cup \{(x, \sin(\frac{1}{x}))\colon 0 < x \leqslant 1\}$. Denote $p = (0, -2)$ and $q = (1, \sin(1))$. It is easy to see that $\mathcal{NB}(F_1(X)) = \{\{p\}, \{q\}, \{p, q\}\}$. Thus local connectedness is a necessary condition in Theorems 3.4, 3.5, and 3.6.

4. A characterization of the circle with nonblockers

Here we show that the circle is the only locally connected continuum for which its set of nonblockers is a continuum. We state the following three lemmas to ease the proof of this fact.

Lemma 4.1. *If X is a locally connected continuum and $\mathcal{NB}(F_1(X))$ is a continuum, then $F_1(X) \subset \mathcal{NB}(F_1(X))$.*

Proof. Suppose that there exists $x \in X$ such that $\{x\}$ is not an element of $\mathcal{NB}(F_1(X))$. By Lemma 3.1, x is a cut point of X. Let U and V be nonempty, disjoint, open sets of X such that $X \setminus \{x\} = U \cup V$. There are points $p \in U$ and $q \in V$ such that p and q are non-cut points of X [6, 6.6]. By Theorem 3.1, $\{p\}, \{q\} \in \mathcal{NB}(F_1(X))$. Denote $A = \bigcup \mathcal{NB}(F_1(X))$. By hypothesis and [5, (1.43)], we have that A is a subcontinuum of X. Notice that $p, q \in A$. It follows that $x \in A$. So, there exists $B \in \mathcal{NB}(F_1(X))$ such that $x \in B$. This is a contradiction (Remark 2.1). $\square$

The following result can be proved as Exercise 8.45 in [6].

Lemma 4.2. *If X is a locally connected continuum and $\{p_1, \dots, p_m\}$ is a finite subset of X such that $X \setminus \{p_1, \dots, p_m\}$ is a connected set, then there exist pairwise disjoint open subsets of X, $U_1, \dots, U_m$, such that $p_i \in U_i$, $1 \leqslant i \leqslant m$, and $X \setminus \bigcup_{i=1}^m U_i$ is a connected set.*

In the following lemma, for a continuum X and an integer $k \geqslant 2$, we use the standard notation for the open basic sets of the Vietoris topology in $F_k(X)$, i.e., for a finite collection of open sets in X, $U_1, \dots, U_n$, we denote $\langle U_1, \dots, U_n\rangle_k = \{A \in F_k(X)\colon\ A \subset \bigcup_{i=1}^n U_i,\ \ A \cap U_i \neq \emptyset,\ \ i \in \{1, \dots, n\}\}$. We also use the following notation,

$$\mathcal{M}_k = \mathcal{NB}\big(F_1(X)\big) \cap \big(F_k(X) \setminus F_1(X)\big). \tag{1}$$

Lemma 4.3. *If X is a locally connected continuum, $k \geqslant 2$, and $F_{k-1}(X)$ is contained in $\mathcal{NB}(F_1(X))$, then $\mathcal{M}_k$ is an open set in $F_k(X)$.*

Proof. Let $B \in \mathcal{M}_k$. We have that $X \setminus B$ is connected (Remark 2.1), and there exist $m \in \{2, \dots, k\}$ and points $p_1, \dots, p_m$ of X such that $B = \{p_1, \dots, p_m\}$. We consider pairwise disjoint open sets in X, $U_1, \dots, U_m$, such that $p_i \in U_i$ and $X \setminus \bigcup_{i=1}^m U_i$ is connected (Lemma 4.2). It is clear that $B \in \langle U_1, \dots, U_m \rangle_k$. We will prove that $\langle U_1, \dots, U_m \rangle_k \subset \mathcal{M}_k$.

Let $A \in \langle U_1, \dots, U_m \rangle_k$. Denote $A = \{q_1, \dots, q_n\}$. Notice that $2 \leqslant m \leqslant n \leqslant k$, and so, $1 \leqslant |U_i \cap A| \leqslant k-1$. Denote $E = X \setminus \bigcup_{i=1}^m U_i$, $\mathcal{C}_i = \{L\colon\ L \text{ is a component of } U_i \setminus A\}$ and $\mathcal{C}_i^* = \bigcup \mathcal{C}_i$, $1 \leqslant i \leqslant m$. Notice that $\mathcal{C}_i^* = U_i \setminus A$.

We will prove that $E \cup \mathcal{C}_i^*$ is a connected set for each $i \in \{1, \dots, m\}$. Fix $i \in \{1, \dots, m\}$ and $L \in \mathcal{C}_i$. First we will prove that $\bar{L} \cap E \neq \emptyset$; suppose on the contrary that $\bar{L} \cap E = \emptyset$. Notice that $X \setminus (U_i \cap A) = E \cup \mathcal{C}_i^* \cup (\bigcup\{U_j\colon\ 1 \leqslant j \leqslant m,\ \ j \neq i\}) = L \cup E \cup (\mathcal{C}_i^* \setminus L) \cup (\bigcup\{U_j\colon\ 1 \leqslant j \leqslant m,\ \ j \neq i\})$. Since X is locally connected, the elements of $\mathcal{C}_i$ are open sets of X, thus L and $(\mathcal{C}_i^* \setminus L) \cup (\bigcup\{U_j\colon\ 1 \leqslant j \leqslant m,\ \ j \neq i\})$ are disjoint open sets. It follows that L and $E \cup (\mathcal{C}_i^* \setminus L) \cup (\bigcup\{U_j\colon\ 1 \leqslant j \leqslant m,\ \ j \neq i\})$ are separate sets. So, $X \setminus (U_i \cap A)$ is not connected. Then, by Theorem 3.1, $U_i \cap A \notin \mathcal{NB}(F_1(X))$. This is a contradiction to the hypothesis, since $U_i \cap A \in F_{k-1}(X)$. Therefore, we have proved that $\bar{L} \cap E \neq \emptyset$. Hence, $E \cup L$ is a connected set for each $L \in \mathcal{C}_i$. Since $E \cup \mathcal{C}_i^* = \bigcup\{E \cup L\colon\ L \in \mathcal{C}_i\}$, it follows that $E \cup \mathcal{C}_i^*$ is connected.

Now, we have that $E \cup (\bigcup_{i=1}^m \mathcal{C}_i^*)$ is connected. Then, since $E \cup (\bigcup_{i=1}^m \mathcal{C}_i^*) = E \cup (\bigcup_{i=1}^m (U_i \setminus A)) = X \setminus A$, we have that $X \setminus A$ is connected. Therefore $A \in \mathcal{NB}(F_1(X))$ (Theorem 3.1). We conclude that $A \in \mathcal{M}_k$. So, we have proved that $B \in \langle U_1, \dots, U_m \rangle_k \subset \mathcal{M}_k$. □

Theorem 4.4. *For a locally connected continuum X, the following statements are equivalent*:

(a) *X is a simple closed curve*;
(b) $\mathcal{NB}(F_1(X)) = F_1(X)$;
(c) *$\mathcal{NB}(F_1(X))$ is a continuum.*

Proof. Theorem 3.2 shows that (a) implies (b). It is clear that (b) implies (c). We will prove that (c) implies (a). We consider the set $\mathcal{M}_k$ as defined in (1). We assert that there exists an integer $k \geqslant 2$ such that $\mathcal{M}_k = \emptyset$.

To prove this, suppose that $\mathcal{M}_k \neq \emptyset$, for each $k \geqslant 2$. By Lemma 4.1, $F_1(X) \subset \mathcal{NB}(F_1(X))$. Now, as inductive hypothesis, assume that $F_k(X) \subset \mathcal{NB}(F_1(X))$ for an integer $k \geqslant 2$. Thus, by Lemma 4.3, $\mathcal{M}_{k+1}$ is an open set in $F_{k+1}(X)$, and so it is open in $F_{k+1}(X) \setminus F_1(X)$. By hypothesis in (c) and definition (1), we also have that $\mathcal{M}_{k+1}$ is a closed set in $F_{k+1}(X) \setminus F_1(X)$. On the other hand we know that $F_{k+1}(X) \setminus F_1(X)$ is a connected space [1, 4.2.3]. Since $\mathcal{M}_{k+1} \neq \emptyset$, it follows that $\mathcal{M}_{k+1} = F_{k+1}(X) \setminus F_1(X)$. Then, by (1), $F_{k+1}(X) \setminus F_1(X) \subset \mathcal{NB}(F_1(X))$, and so $F_{k+1}(X) \subset \mathcal{NB}(F_1(X))$. Hence $\bigcup_{k \in \mathbb{N}} F_k(X) \subset \mathcal{NB}(F_1(X))$. Since $\bigcup_{k \in \mathbb{N}} F_k(X)$ is a dense set in 2^X [5, (0.66.6)], we obtain that $\mathcal{NB}(F_1(X)) = 2^X$, which is a contradiction. Thus, the assertion is proved.

Now, let $k \geqslant 2$ be such that $\mathcal{M}_k = \emptyset$. Consider two distinct points x and y in X. We have that $\{x, y\} \in F_k(X) \setminus F_1(X)$. By definition of $\mathcal{M}_k$, it follows that $\{x, y\} \notin \mathcal{NB}(F_1(X))$. Then, by Theorem 3.1, $X \setminus \{x, y\}$ is not connected. Thus, X is a simple closed curve [6, 9.31]. □

Question 4.5. For which nonlocally connected continua X is $\mathcal{NB}(F_1(X))$ a continuum?

5. Nonblockers and open monotone maps

In this section we show results related to nonblockers and open monotone maps. As a consequence, we show that the circle of pseudo-arcs is an example for Question 4.5.

Lemma 5.1. *Let f be an open monotone map from a continuum X onto a continuum Y. If B is an element of 2^Y, then $B \in \mathcal{NB}(F_1(Y))$ if and only if $f^{-1}(B) \in \mathcal{NB}(F_1(X))$.*

Proof. For necessity, we fix a point $x \in X \setminus f^{-1}(B)$ and consider an order arc, $\alpha : [0, 1] \to C(Y)$, from $\{f(x)\}$ to Y such that $\alpha(t) \cap B = \emptyset$ if $0 \leqslant t < 1$ (Proposition 2.2(b)). For each $n \in \mathbb{N}$, denote $A_n = f^{-1}(\alpha(1 - \frac{1}{n}))$. Since f is a monotone map, A_n is a subcontinuum of X [8, (2.2), p. 138]. It is clear that $x \in A_n \subset A_{n+1} \subset X \setminus f^{-1}(B)$, for each $n \in \mathbb{N}$. Next, we will prove that $\bigcup_{n \in \mathbb{N}} A_n$ is a dense set in X. Let U be a nonempty open set in X. Since f is an open map, $f(U)$ is an open set in Y. Notice that $\bigcup\{\alpha(t)\colon\ 0 \leqslant t < 1\}$ is a dense set in Y (Remark 2.3). So, there exists $t \in [0, 1)$ such that $f(U) \cap \alpha(t) \neq \emptyset$. Let $k \in \mathbb{N}$ such that $t < 1 - \frac{1}{k}$. We have that $\alpha(t) \subset \alpha(1 - \frac{1}{k})$, and so $f(U) \cap \alpha(1 - \frac{1}{k}) \neq \emptyset$. It follows that $U \cap A_k \neq \emptyset$. Thus $\bigcup_{n \in \mathbb{N}} A_n$ is dense in X. By Proposition 2.2(c), we conclude that $f^{-1}(B) \in \mathcal{NB}(F_1(X))$.

For sufficiency, let $y \in Y \setminus B$ and $x \in f^{-1}(y)$. Notice that $x \in X \setminus f^{-1}(B)$. Consider a map, $\alpha : [0, 1] \to C(X)$, such that $\alpha(0) = \{x\}$, $\alpha(1) = X$ and $\alpha(t) \cap f^{-1}(B) = \emptyset$ if $0 \leqslant t < 1$. Define $\beta : [0, 1] \to C(Y)$ by $\beta(t) = f(\alpha(t))$ for each $t \in [0, 1]$. We note that β is a continuous function [4, 13.3]. It is clear that $\beta(0) = \{y\}$, $\beta(1) = Y$ and $\beta(t) \cap B = \emptyset$ if $0 \leqslant t < 1$. Thus $B \in \mathcal{NB}(F_1(Y))$. □

Theorem 5.2. *Let f be an open monotone map from a continuum X onto a continuum Y. If $f^{-1}(y)$ is a terminal subcontinuum of X for each $y \in Y$, then $\mathcal{NB}(F_1(X)) = \{f^{-1}(B)\colon B \in \mathcal{NB}(F_1(Y))\}$.*

Proof. Let $A \in \mathcal{NB}(F_1(X))$. We assert that if $y \in Y$ and $f^{-1}(y) \cap A \neq \emptyset$, then $f^{-1}(y) \subset A$.

To prove this, assume that there exists a point $x \in f^{-1}(y) \setminus A$. Consider a map $\alpha : [0,1] \to C(X)$ such that $\alpha(0) = \{x\}$, $\alpha(1) = X$ and $\alpha(t) \cap A = \emptyset$ if $0 \leqslant t < 1$. Since $\bigcup\{\alpha(t) : 0 \leqslant t < 1\}$ is a dense set in X (Remark 2.3), and $X \setminus f^{-1}(y)$ is a nonempty open set of X, there exists $r \in (0,1)$ such that $\alpha(r) \cap (X \setminus f^{-1}(y)) \neq \emptyset$. Notice that $\bigcup \alpha([0,r])$ is a subcontinuum of X [5, (1.43)]. Moreover, this subcontinuum intersects both $f^{-1}(y)$ and $X \setminus f^{-1}(y)$. So, since $f^{-1}(y)$ is a terminal subcontinuum of X, we have that $f^{-1}(y) \subset \bigcup \alpha([0,r])$. Now, since $A \cap (\bigcup \alpha([0,r])) = \emptyset$, it follows that $f^{-1}(y) \cap A = \emptyset$. This proves our assertion.

We note that $f^{-1}(f(A)) \subset A$. To see this let $x \in f^{-1}(f(A))$. So, there exists $a \in A$ such that $f(x) = f(a)$, thus $f^{-1}(f(x)) \cap A \neq \emptyset$. By assertion at the beginning, we obtain that $f^{-1}(f(x)) \subset A$, and so $x \in A$. This proves that $f^{-1}(f(A)) \subset A$. It follows that $A = f^{-1}(f(A))$. Denote $E = f(A)$. So, $A = f^{-1}(E)$ and $f^{-1}(E) \in \mathcal{NB}(F_1(X))$. Then, by *if part* of Lemma 5.1, we have that $E \in \mathcal{NB}(F_1(Y))$. Therefore $A \in \{f^{-1}(B)\colon B \in \mathcal{NB}(F_1(Y))\}$. So, $\mathcal{NB}(F_1(X)) \subset \{f^{-1}(B)\colon B \in \mathcal{NB}(F_1(Y))\}$. The converse inclusion follows from *only if part* of Lemma 5.1. □

Theorem 5.3. *Let f be an open monotone map from a continuum X onto a continuum Y. If $f^{-1}(y)$ is a terminal subcontinuum of X for each $y \in Y$, and $\mathcal{NB}(F_1(Y)) = F_1(Y)$, then $\mathcal{NB}(F_1(X))$ is a continuum homeomorphic to Y.*

Proof. Denote $\mathcal{D}_f = \{f^{-1}(y)\colon y \in Y\}$. By Theorem 5.2, $\mathcal{NB}(F_1(X)) = \mathcal{D}_f$. Notice that the decomposition space $\mathcal{D}_f$, with the quotient topology, is a continuum homeomorphic to Y [6, 3.21]. Moreover, since f is an open map, $\mathcal{D}_f$ is a continuous decomposition [6, 13.11], which implies that the quotient topology coincides with the Vietoris topology in $\mathcal{D}_f$ [6, 13.10]. This proves that $\mathcal{NB}(F_1(X))$ is a continuum homeomorphic to Y. □

The quotient map from the circle of pseudo-arcs onto the circle is an open monotone map such that the preimages of points are terminal continua. Hence, by considering Theorems 3.2 and 5.2, we obtain the following result.

Corollary 5.4. *If X is the circle of pseudo-arcs, then $\mathcal{NB}(F_1(X))$ is a simple closed curve.*

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