Axiomatic characterizations of the symmetric coalitional binomial semivalues

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Abstract

The symmetric coalitional binomial semivalues extend the notion of binomial semivalue to games with a coalition structure, in such a way that they generalize the symmetric coalitional Banzhaf value. By considering the property of balanced contributions within unions, two axiomatic characterizations for each one of these values are provided.

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1. Introduction

The symmetric coalitional binomial semivalues, which form a parametric family of coalitional values, are especially suited for the study of cooperative games where the players may show some (common) tendency as to the size of the coalitions they would agree to form. It is assumed that this tendency is defined by some parameter that ranges the interval $[0, 1]$. These values have been recently applied\cite{9} to the analysis of an interesting coalition formation political problem.

Example 1.1. The problem arose in the Catalonia Parliament at the beginning of Legislature 2003–2006, prematurely finished. Curiously, the analysis remains still valid for Legislature 2006–2010 since, in spite of the modification of the seat distribution issued from the elections held in November 1, 2006, the strategic possibilities are exactly the same.

In Catalonia, politics is based on two main axes: the classical left-to-right axis and a crossed one going from Spanish centralism to Catalanism (Catalan nationalism). The positions of the parties in such a two-dimensional ideological
The position of parties in a two-dimensional ideological space could be represented as in Fig. 1, where we attach to each party the number of seats obtained in 2003. The links between ERC, PSC and ICV on one hand, and ERC and CiU on the other, define affinities and hence the politically likely minimal winning coalitions.

Thus, in 2003 Esquerra Republicana de Catalunya (ERC), a radical nationalist and left-wing party, was faced with the dilemma of choosing among either a Catalanist majority coalition with Convergència i Unió (CiU) or a left-wing majority coalition with the Partit dels Socialistes de Catalunya (PSC) and Iniciativa per Catalunya–Verds (ICV), which was finally formed in 2003 and has been repeated in 2006.

A classical approach would consist in using either (a) the Shapley value [18] and the Owen value [14], (b) the Banzhaf value [13] and the Banzhaf–Owen value [15] or, alternatively, (c) the Banzhaf value [13] and the symmetric coalitional Banzhaf value [4], in order to evaluate the strategic possibilities of each party in both setups. The results are given in Table 1, where (NO) means no coalition formation, (C) means that CiU + ERC forms and (LW) means that PSC + ERC + ICV forms.

According to the Shapley and Owen values used in (a), ERC would strictly prefer joining CiU instead of PSC and ICV. The same conclusion is obtained according to the Banzhaf and Banzhaf–Owen values used in (b). In both cases, the results fail to provide a mathematical explanation of ERC’s actual decision (to join PSC and ICV). Instead, according to the Banzhaf and symmetric coalitional Banzhaf values used in (c), ERC would strictly prefer joining PSC and ICV instead of CiU.

As is shown in Section 2, for \( p = \frac{1}{2} \) the Banzhaf value is the \( p \)-binomial semivalue and the symmetric coalitional Banzhaf value is the symmetric coalitional \( p \)-binomial semivalue. Therefore, more generally, by using binomial
Two players—left-wing tripartite government but would have got more political power in joining CiU depending on the tendency of the theoretical analysis is that ERC, the crucial agent in this scenario, was not necessarily forced to participate in the semivalues—and symmetric coalitional binomial semivalues whenever a coalition structure exists—the conclusion of the interval $(5 - \sqrt{5})/10, (5 + \sqrt{5})/10$ and hence ERC should have preferred CiU for 44.72% of possibilities. The details can be found in [9].

This distinction, and hence the increase of strategic options for ERC, cannot be discovered by merely using the traditional (rigid) coalitional values: it follows only from the possibility to let a parameter vary, which is just one of the main features of the symmetric coalitional binomial semivalues.

A second main aspect of these values, still referring to a voting setup, is that they use a binomial semivalue (a generalization of the Banzhaf value) as a power measure in the quotient game (once a coalition structure is formed) but share then within each union the power so obtained by applying the Shapley value to an internal game that concerns only the players of that union. This looks highly interesting since, once an alliance is formed—and, especially, if it supports a coalition government—cabinet ministries, parliamentary and institutional positions, budget management and other political responsibilities have to be distributed among the members of the coalition efficiently, whence in a way as close as possible to the one suggested by the Shapley value. This two-step procedure (first power, then cake) offers a balanced approach for dealing with coalition bargaining.

How far from the classical framework does this lead us? Which is the price we must pay, in terms of mathematical properties, for introducing a parameter in our evaluation of games and games with a coalition structure? In other words: which are the theoretical foundations of the symmetric coalitional binomial semivalues? The aim of this paper is that properties, for introducing a parameter in our evaluation of games and games with a coalition structure? In other words:

Therefore we state here two axiomatic characterizations for each symmetric coalitional binomial semivalue, both based on the interesting property of balanced contributions within unions. First we use it jointly with additivity, the dummy player property, symmetry in the quotient game and the coalitional $p$-binomial total power property. Next, we prove that the symmetric coalitional $p$-binomial semivalue is the unique coalitional value of the $p$-binomial semivalue that satisfies balanced contributions within unions and the quotient game property.

The organization of the paper is as follows. In Section 2, a minimum of preliminaries is provided. In Section 3 we recall the definition of the symmetric coalitional binomial semivalues, introduce the property of balanced contributions within unions and state and prove the characterization theorems.

2. Preliminaries

2.1. Games and semivalues

Let $N$ be a finite set of players and $2^N$ be the set of coalitions (subsets of $N$). A cooperative game on $N$ is a function $v: 2^N \rightarrow \mathbb{R}$, that assigns a real number $v(S)$ to each coalition $S \subseteq N$ with $v(\emptyset) = 0$. A game $v$ is monotonic if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. A player $i \in N$ is a dummy in $v$ if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. Two players $i, j \in N$ are symmetric in $v$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

Endowed with the natural operations for real-valued functions, the set of all cooperative games on $N$ is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the unanimity game $u_T$ is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Finally, every permutation $\theta$ of $N$ induces a linear automorphism of $\mathcal{G}_N$ given by $(\theta v)(S) = v(\theta^{-1} S)$ for all $S \subseteq N$ and all $v$.

By a value on $\mathcal{G}_N$ we will mean a map $f: \mathcal{G}_N \rightarrow \mathbb{R}^N$, that assigns to every game $v$ a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Following the axiomatic description given in [20], $\psi: \mathcal{G}_N \rightarrow \mathbb{R}^N$ is a semivalue (on $\mathcal{G}_N$) iff it satisfies the following properties:

(i) linearity: $\psi[v + v'] = \psi[v] + \psi[v']$ and $\psi[\lambda v] = \lambda \psi[v]$ if $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;
(ii) anonymity: $\psi_{\theta i}[v] = \psi_i[v]$ for all $\theta$ on $N$, $i \in N$ and $v \in \mathcal{G}_N$.
(iii) positivity: if \( v \) is monotonic, then \( \psi[v] \geq 0 \);
(iv) dummy player property: if \( i \in N \) is a dummy in game \( v \), then \( \psi_i[v] = v(\{i\}) \).

In [10] there is an interesting characterization of semivalues by means of weighting coefficients. Set \( n = |N| \). Then:
(a) for every weighting vector \( \{p_k\}_{k=0}^{n-1} \) such that \( \sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1 \) and \( p_k \geq 0 \) for all \( k \), the expression
\[
\psi_i[v] = \sum_{S \subseteq N \setminus \{i\}} p_s [v(S \cup \{i\}) - v(S)] \quad \text{for all} \ i \in N \text{ and all} \ v \in \mathcal{G}_N,
\]
where \( s = |S| \), defines a semivalue \( \psi \); (b) conversely, every semivalue can be obtained this way; (c) the correspondence given by \( \{p_k\}_{k=0}^{n-1} \mapsto \psi \) is bijective.

Well known examples of semivalues are the Shapley value \( \varphi \) [18], for which \( p_k = 1/n \binom{n-1}{k} \), and the Banzhaf value \( \beta \) [13], for which \( p_k = 2^{1-n} \). The Shapley value \( \varphi \) is the only efficient semivalue, in the sense that \( \sum_{i \in N} \varphi_i[v] = v(N) \) for every \( v \in \mathcal{G}_N \).

Notice that these two classical values are defined for each \( N \). The same happens with the binomial semivalues, introduced in [17] as follows. Let \( p \in [0, 1] \) and \( p_k = p^k(1-p)^{n-k-1} \) for \( k = 0, 1, \ldots, n-1 \) (by convention, we take \( 0^0 = 1 \) if \( p = 0 \) or \( p = 1 \)). Then \( \{p_k\}_{k=0}^{n-1} \) is a weighting vector and defines a semivalue on \( \mathcal{G}_N \) that will be denoted as \( \psi^p \) and called the \( p \)-binomial semivalue. Of course, \( \psi^{1/2} = \beta \).

**Definition 2.1 (Puente [17]).** Let \( p \in [0, 1] \). A value \( f \) on \( \mathcal{G}_N \) satisfies the \( p \)-binomial total power property if
\[
\sum_{i \in N} f_i[v] = \frac{1}{2^{n-1}} \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} p_s [v(S \cup \{i\}) - v(S)] \quad \text{for all} \ v \in \mathcal{G}_N.
\]

**Proposition 2.2 (Carreras and Puente [9]).** Let \( p \in [0, 1] \). The unique semivalue on \( \mathcal{G}_N \) that satisfies the \( p \)-binomial total power property is the \( p \)-binomial semivalue \( \psi^p \). Hence, \( \sum_{i \in N} \psi_i[v] = \sum_{i \in N} \psi_i^p[v] \) for all \( v \in \mathcal{G}_N \) implies \( \psi = \psi^p \).

In particular, by setting \( p = \frac{1}{2} \) we obtain that the Banzhaf value \( \beta \) is the only semivalue that satisfies the classical total power property:
\[
\sum_{i \in N} f_i[v] = \frac{1}{2^{n-1}} \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} [v(S \cup \{i\}) - v(S)] \quad \text{for all} \ v \in \mathcal{G}_N. \quad \square
\]

Which is the reason for letting \( p \) range from 0 to 1? Notice that a reasonable regularity assumption on players’ behavior is that the probability to form coalitions follows a monotonic (increasing or decreasing) behavior. Then, it is not difficult to see that the only semivalues such that \( p_{k+1} = \lambda p_k \) for all \( k \) (maybe the simplest form of monotonicity) are precisely the \( p \)-binomial semivalues, in which case \( \lambda = p/(1-p) \) for each \( p \in [0, 1] \). For example, \( p = 0.1 \) means that the players are very reticent to form coalitions, whereas \( p = 0.8 \) means that great coalitions are more likely. The neutral case \( p = 0.5 \) corresponds to the Banzhaf value.

**Remark 2.3.** (a) Within the class of all semivalues, Proposition 2.2 provides a characterizing property of each binomial semivalue, whereas monotonicity of the weighting coefficients is a characterizing property of all binomial semivalues as a family.

(b) As pointed out by one of the referees, Proposition 2.2 is a particular case of a more general statement concerning all semivalues (and several other values).

(c) The classical total power property of the Banzhaf value mentioned in Proposition 2.2 can be generalized to all semivalues. Indeed, for every semivalue \( \psi \), defined by a weighting vector \( \{p_k\}_{k=0}^{n-1} \), we have
\[
\sum_{i \in N} \psi_i[v] = \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} p_s [v(S \cup \{i\}) - v(S)] \quad \text{for all} \ v \in \mathcal{G}_N,
\]
which states that the total amount shared among the players is a weighted sum of all marginal contributions (swings in the simple case) arising in the game, each one of them being affected by its probability to occur. This agrees with the
standard view that each player $i$ has only a probability $p_s$ to join coalition $S \subseteq N \setminus \{i\}$, where $s = |S|$. The $p$-binomial total power property is nothing but a particular case of this general statement. Just the same statement that in the case of the Shapley value nicely gives rise to efficiency due to the special weighting coefficients that define it.

2.2. Games with a coalition structure

Let us consider a finite set, say, $N = \{1, 2, \ldots, n\}$. We will denote by $P(N)$ the set of all partitions of $N$. Each $P \in P(N)$ is called a coalition structure or system of unions on $N$. The so-called trivial coalition structures are $P^N = \{\{1\}, \{2, \ldots, |N|\}\}$ and $P^N = \{N\}$. A cooperative game with a coalition structure is a pair $[v; P]$, where $v \in \mathcal{G}_N$ and $P \in P(N)$ for a given $N$. We denote by $\mathcal{G}^n_N$ the set of all cooperative games with a coalition structure on $N$.

If $[v; P] \in \mathcal{G}^n_N$ and $P = \{P_1, P_2, \ldots, P_m\}$, the quotient game $v^P$ is the cooperative game played by the unions, or, rather, by the set $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^P(R) = v \left( \bigcup_{P \in R} P^r \right)$$

for all $R \subseteq M$.

Unions $P_r, P_s$ are said to be symmetric in $[v; P]$ if $r, s$ are symmetric players in $v^P$.

By a coalitional value on $\mathcal{G}^n_N$ we will mean a map $g: \mathcal{G}^n_N \rightarrow \mathbb{R}^N$, which assigns to every pair $[v; P]$ a vector $g[v; P]$ with components $g_i[v; P]$ for each $i \in N$.

Given a value $f$ on $\mathcal{G}_N$, a coalitional value of $f$ is a coalitional value on $\mathcal{G}^n_N$ such that $g[v; P^n] = f[v]$ for all $v \in \mathcal{G}_N$.

The notion of coalitional semivalue was axiomatically introduced in [1] as a map that assigns a payoff vector to every game with a coalition structure and satisfies some standard properties. Formula (13) of this reference is the analogue of our expression in Definition 3.1 below; the only difference is that, instead of the weighting coefficients of the binomial semivalue and the Shapley value, use is made in [1] of the weighting coefficients of two arbitrary semivalues. All coalitional semivalues are characterized this way in Theorem 7 [1]. It will be clear that our symmetric coalitional binomial semivalues fall within this category.

3. The symmetric coalitional $p$-binomial semivalues

The symmetric coalitional $p$-binomial semivalue represents a two-step bargaining procedure where, first, the unions are allocated in the quotient game the payoff given by the $p$-binomial semivalue $\psi^p$ and, then, this payoff is efficiently shared within each union according to the Shapley value $\phi$.

**Definition 3.1 (Carreras and Puente [9]).** Let $p \in [0, 1]$. For any fixed player set $N$, the symmetric coalitional $p$-binomial semivalue is the coalitional value $\Omega^p$ defined on $\mathcal{G}^n_N$ by

$$\Omega^p_i[v; P] = \sum_{R \subseteq M \setminus \{i\}} \sum_{T \subseteq P_i \setminus \{i\}} p^r(1-p)^{m-r-1} \frac{1}{p_k(P_{i-1}^r)} [v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$

for all $i \in N$ and $[v; P] \in \mathcal{G}^n_N$, where $P_k \in P$ is the union such that $i \in P_k$ and $Q = \bigcup_{P \in R} P^r$. In case $p = \frac{1}{2}$, we get $\Omega^{1/2} = \Pi$, the symmetric coalitional Banzhaf value introduced in [4]. Also note that the Shapley value (when $P = P^N$), all binomial semivalues (when $P = P^n$) and the Banzhaf value (when, moreover, $p = \frac{1}{2}$) will arise as particular cases (cf. Theorem 3.5 below).

**Definition 3.2.** A coalitional value $g$ on $\mathcal{G}^n_N$ satisfies the property of balanced contributions within unions if, for all $[v; P] \in \mathcal{G}^n_N$, all $P_k \in P$ and all $i, j \in P_k$,

$$g_i[v; P] - g_i[v; P_{-j}] = g_j[v; P] - g_j[v; P_{-i}],$$

where $P_{-i}$ is the coalition structure that results when player $i$ leaves the union s(he) belongs to, i.e.

$$P_{-i} = \{P_1, \ldots, P_{k-1}, P_k \setminus \{i\}, P_{k+1}, \ldots, P_m, \{i\}\},$$

and $P_{-j}$ is defined analogously. Notice that in $P_{-i}$ player $i$ does not leave the game, but only union $P_k$. 

This property states that the loss (or gain) of a player \( i \in P_k \) when a player \( j \in P_k \) decides to leave the union and remain alone is the same as the loss (or gain) of player \( j \) when player \( i \) decides to leave the union. It is reminiscent of Myerson’s fairness concept [12]. The balanced contributions principle has been also used e.g. in [7] when characterizing the level value.

Let us consider the following properties for a coalitional value \( g \) on \( \mathcal{G}_N^{cs} \):

- **additivity**: \( g[v + v'; P] = g[v; P] + g[v'; P] \) for all \( v, v' \) and \( P \);
- **dummy player property**: if \( i \) is a dummy in \( v \), then \( g_i[v; P] = v(\{i\}) \) for all \( P \);
- **coalitional \( p \)-binomial total power property**: for all \( [v; P] \in \mathcal{G}_N^{cs} \),
  \[
  \sum_{i \in N} g_i[v; P] = \sum_{k \in M} \sum_{R \subseteq M \setminus \{k\}} p^r (1 - p)^{m-r-1} [v^P(R \cup \{k\}) - v^P(R)];
  \]
- **symmetry in the quotient game**: if \( P_r, P_s \in P \) are symmetric in \( [v; P] \) then
  \[
  \sum_{i \in P_r} g_i[v; P] = \sum_{j \in P_s} g_j[v; P];
  \]
- **quotient game property**: for all \( [v; P] \in \mathcal{G}_N^{cs} \),
  \[
  \sum_{i \in P_k} g_i[v; P] = g_k[v^P; P^m] \quad \text{for all } P_k \in P
  \]

(this property makes sense only for coalitional values defined for all \( N \); in this case, here and in the sequel we abuse the notation and use a unique symbol \( g \) on both \( \mathcal{G}_N^{cs} \) and \( \mathcal{G}_N^{c} \)).

**Remark 3.3.** The coalitional \( p \)-binomial total power property is a natural generalization of the \( p \)-binomial total power property (see Remark 2.3). If \( g \) is a coalitional value of a value \( f \), both defined for all \( N \) (as it is the case for \( \Omega^p \) and \( \psi^p \) for every \( p \in [0, 1] \)), the property is a consequence of the quotient game property, maybe much more compelling at first glance, and could be written as

\[
\sum_{i \in N} g_i[v; P] = \sum_{k \in M} f_k[v^P],
\]

thus establishing that the amount shared according to \( g \) in \([v; P]\) coincides with the amount shared according to \( f \) in the quotient game \( v^P \).

In Theorem 3.5 we give a first characterization of each symmetric coalitional \( p \)-binomial semivalue. We will need the next lemma.

**Lemma 3.4.** Let \( p \in [0, 1] \), \( \emptyset \neq S \subseteq N \), \( s = |S| \) and \( i \in N \). Then \( \psi_i^p[u_S] = p^{s-1} \) if \( i \in S \), and \( \psi_i^p[u_S] = 0 \) otherwise.

**Proof.** Let \( i \in S \). Then, if \( t = |T| \) when \( T \subseteq N \), and \( r = t - s + 1 \),

\[
\psi_i^p[u_S] = \sum_{T \subseteq N \setminus \{i\}} p^r (1 - p)^{n-r-1} [u_S(T \cup \{i\}) - u_S(T)]
\]

\[
= \sum_{t=s-1}^{n-1} \binom{n-s}{t-s+1} p^r (1 - p)^{n-r-1} = p^{s-1} \sum_{r=0}^{n-s} \binom{n-s}{r} p^r (1 - p)^{n-s-r} = p^{s-1}.
\]

If \( i \notin S \), the dummy player property of \( \psi^p \) yields \( \psi_i^p[u_S] = 0 \). \( \Box \)

**Theorem 3.5 (First axiomatic characterization).** Let \( p \in [0, 1] \). For each \( N \) there is a unique coalitional value on \( \mathcal{G}_N^{cs} \) that satisfies additivity, the dummy player property, balanced contributions within unions, the coalitional \( p \)-binomial total power property and symmetry in the quotient game. It is the symmetric coalitional \( p \)-binomial semivalue \( \Omega^p \).
Moreover, $\Omega_p^0$ satisfies the quotient game property, is a coalitional value of the $p$-binomial semivalue $\psi_p^0$ and yields
$\Omega_p^0[v; P^N] = \varphi[v]$ for all $v \in {\mathcal G}_N$.

**Proof.** (a) (Existence) It suffices to show that the coalitional value $\Omega_p^0$ satisfies the five properties enumerated in the statement.

1. Additivity. It merely follows from the expression of $\Omega_p^0[v; P]$.

2. Dummy player property. Let $i \in N$ be a dummy player in game $v$ and $P$ be any coalition structure. Assume $i \in P_k$.

Then $v(Q \cup T \cup \{i\}) - v(Q \cup T) = v(\{i\})$ for all $R$ and $T$. As, moreover,

$$\sum_{R \subseteq M \setminus \{k\}} p'(1 - p)^{m - r - 1} = 1 \quad \text{and} \quad \sum_{T \subseteq P_i \setminus \{i\}} \frac{1}{p_k^{p_k-1}} = 1,$$

we conclude that $\Omega_p^0[v; P] = v(\{i\})$.

3. Balanced contributions within unions. Let us take $[v; P] \in {\mathcal G}_N^{CS}$, with $P = \{P_1, P_2, \ldots, P_m\}$ and $M = \{1, 2, \ldots, m\}$. Let $P_k \in P$ and $i, j \in P_k$. Then we have

$$\Omega_p^0[v; P] \quad \Omega_p^0[v; P_{-j}]\quad \Omega_p^0[v; P], \quad \Omega_p^0[v; P_{-j}], \quad \Omega_p^0[v; P], \quad \Omega_p^0[v; P_{-j}], \quad \Omega_p^0[v; P],$$

where $Q = \bigcup_{r \in R} P_r$. We now consider

$$P_{-j} = \{P_1', P_2', \ldots, P_{m'+1}'\},$$

where $P_h' = P_h$ for $h \in \{1, \ldots, k - 1, k + 1, \ldots, m\}$, $P_j' = P_k \setminus \{j\}$, $P_{m+1}' = \{j\}$ and $M' = \{1, 2, \ldots, m + 1\}$, and get, in a similar way,

$$\Omega_p^0[v; P_{-j}], \quad \Omega_p^0[v; P_{-j}], \quad \Omega_p^0[v; P], \quad \Omega_p^0[v; P_{-j}], \quad \Omega_p^0[v; P],$$

Thus

$$\Omega_p^0[v; P] - \Omega_p^0[v; P_{-j}], \quad \Omega_p^0[v; P], \quad \Omega_p^0[v; P_{-j}], \quad \Omega_p^0[v; P],$$

where

$$A_1 = p'(1 - p)^{m - r - 1}(p_k - t - 1)!{(p_k - 1)!} \quad A_2 = p'(1 - p)^{m - r - 1}(p_k - t - 2)!{(p_k - 1)!}$$

and yields

$$v(Q \cup T \cup \{i\}) - v(Q \cup T) = v(\{i\})$$

for all $v \in {\mathcal G}_N$. 


and

\[
A_2 = p^r (1 - p)^{m-r-1} \frac{(p_k - t - 2)! (t + 1)!}{p_k!} - p^{r+1} (1 - p)^{m-r-1} \frac{(p_k - t - 2)! t!}{(p_k - 1)!}.
\]

It is easy to check that \(A_2 = -A_1\), so that

\[
\Omega^P_i[v; P] - \Omega^P_i[v; P_{-j}] = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i, j\}} A_1[v(Q \cup T \cup \{i\}) + v(Q \cup T \cup \{j\}) - v(Q \cup T) - v(Q \cup T \cup \{i\} \cup \{j\})].
\]

Since this expression depends on \(i\) in the same way as it depends on \(j\),

\[
\Omega^P_i[v; P] - \Omega^P_i[v; P_{-j}] = \Omega^P_i[v; P] - \Omega^P_j[v; P_{-i}].
\]

4. Coalitional \(p\)-binomial total power property. Let \([v; P] \in \mathcal{G}_N^S\). Fixing \(k \in M\), for every \(R \subseteq M \setminus \{k\}\) we consider the game \(v_R \in \mathcal{G}_{P_k}\) defined by

\[
v_R(T) = v(Q \cup T) - v(Q) \quad \text{for all } T \subseteq P_k.
\]

The Shapley value gives, for each \(i \in P_k\),

\[
\varphi_i[v_R] = \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{p_k^{n-1}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)].
\]

Using the efficiency of \(\varphi\), we get

\[
\sum_{i \in P_k} \varphi_i[v_R] = v_R(P_k) = v(Q \cup P_k) - v(Q) = v^P(R \cup \{k\}) - v^P(R).
\]

Hence

\[
\sum_{i \in P_k} \Omega^P_i[v; P] = \sum_{R \subseteq M \setminus \{k\}} p^r (1 - p)^{n-r-1} [v^P(R \cup \{k\}) - v^P(R)] = \psi^P_k[v^P]
\]

and, finally,

\[
\sum_{i \in M} \Omega^P_i[v; P] = \sum_{k \in M} \sum_{R \subseteq M \setminus \{k\}} p^r (1 - p)^{n-r-1} [v^P(R \cup \{k\}) - v^P(R)].
\]

5. Symmetry in the quotient game. It readily follows from the relationship

\[
\sum_{i \in P_k} \Omega^P_i[v; P] = \psi^P_k[v^P],
\]

stated in the previous point, and the anonymity of the \(p\)-binomial semivalue \(\psi^P\).

(b) (Uniqueness) Let \(g\) be a coalitional value that satisfies the above five properties. We will see that \(g\) is uniquely determined, so that \(g = \Omega^P\).

Using additivity and the fact that the unanimity games form a basis of \(\mathcal{G}_N\), it suffices to see that \(g\) is uniquely determined on each pair of the form \([\lambda u_T; P]\). So let \(\lambda \in \mathbb{R}, \emptyset \neq T \subseteq N\) and \(P \in P(N)\). Let \(R = \{k \in M : T \cap P_k \neq \emptyset\}\) and \(R_k = T \cap P_k\) for each \(k \in R\).

Using the dummy player property it follows that \(g_i[\lambda u_T; P] = 0\) if \(i \notin T\). Now we apply the coalitional \(p\)-binomial total power property:

\[
\sum_{i \in N} g_i[\lambda u_T; P] = \sum_{k \in M} \sum_{S \subseteq M \setminus \{k\}} p^s (1 - p)^{m-s-1} [(\lambda u_T)^P(S \cup \{k\}) - (\lambda u_T)^P(S)].
\]
It is easy to see that \((\lambda u_T)^P = \lambda u_T^P\). Then, by the definition of the \(p\)-binomial semivalue and its linearity, we have
\[
\sum_{i \in N} g_i[\lambda u_T; P] = \sum_{k \in M} \psi_k^P[\lambda u_T^P] = \lambda \sum_{k \in M} \psi_k^P[u_T^P].
\]
As, moreover, \(u_T^P = u_R\), using Lemma 3.4 yields
\[
\sum_{i \in N} g_i[\lambda u_T; P] = \lambda \sum_{k \in M} \psi_k^P[u_R] = \lambda \sum_{k \in R} p^{r-1} = \lambda r p^{r-1}.
\]
Let \(k \in R\). From the dummy player property and symmetry in the quotient game we get
\[
\sum_{i \in R_k} g_i[\lambda u_T; P] = \sum_{i \in P_k} g_i[\lambda u_T; P] = \lambda r p^{r-1}.
\]
It remains to see that \(g_i[\lambda u_T; P] = \lambda r p^{r-1}/r_k\) for all \(i \in R_k\). To this end, we use induction on \(r_k = |R_k|\).
If \(r_k = 1\) it is obvious because \(R_k = \{i\}\). So, let \(r_k > 1\). If \(i, j \in R_k\), from balanced contributions within unions it follows that
\[
g_i[\lambda u_T; P] - g_i[\lambda u_T; P_{-j}] = g_j[\lambda u_T; P] - g_j[\lambda u_T; P_{-i}] = g_j[\lambda u_T; P_{-j}]
\]
Now, the cardinality of the corresponding subsets \((R_{-i})_k\) and \((R_{-j})_k\), for both \(P_{-i}\) and \(P_{-j}\), is \(r_k - 1\), whereas \(|R_{-i}| = |R_{-j}| = r + 1\), so that, by the inductive hypothesis,
\[
g_i[\lambda u_T; P_{-j}] = \frac{\lambda r p^{r-1}}{r_k - 1} = g_j[\lambda u_T; P_{-i}]
\]
and hence
\[
g_i[\lambda u_T; P] = \frac{\lambda r p^{r-1}}{r_k} = g_j[\lambda u_T; P].
\]
This completes the uniqueness proof.
(c) First, if \(P = P^N\) then
\[
\Omega_i^p[v; P^N] = \sum_{T \subseteq N \setminus \{i\}} \frac{1}{n(r + 1)} [v(T \cup \{i\}) - v(T)] = \varphi_i[v]
\]
for all \(i \in N\) and all \(v \in \mathcal{G}_N\). Analogously, \(\Omega^p\) is a coalitional value of the \(p\)-binomial semivalue \(\psi^p\). Indeed, for \(P = P^n\),
\[
\Omega_i^p[v; P^n] = \sum_{R \subseteq N \setminus \{i\}} p^r(1 - p)^{m-r-1} [v(R \cup \{i\}) - v(R)] = \psi_i^p[v].
\]
Finally, the quotient game property; as we have seen when showing the symmetry in the quotient game in part (a) of this proof, and using the preceding property for \(\mathcal{G}^c_M\),
\[
\sum_{i \in P_k} \Omega_i^p[v; P] = \psi_k^p[v^P] = \Omega_k^p[v^P; P^m]. \quad \Box
\]

In [19], it was shown that the Owen value is the unique coalitional value of the Shapley value that satisfies the properties of quotient game and balanced contributions within unions. Analogously, in [4] it was proven that the symmetric coalitional Banzhaf value is the unique coalitional value of the Banzhaf value that satisfies these two properties. In Theorem 3.6 we generalize this result.

**Theorem 3.6 (Second axiomatic characterization).** Let \(p \in [0, 1]\). The symmetric coalitional \(p\)-binomial semivalue \(\Omega^p\) is the unique coalitional value of the \(p\)-binomial semivalue \(\psi^p\) defined for all \(N\) that satisfies balanced contributions within unions and the quotient game property.
Appendix) fall out of this class. The only distinction between the Owen value and the symmetric coalitional binomial

Theorem 3.5 with so that

\[ ck \]

satisfying the above two properties. Let \( g^1 \) and \( g^2 \) satisfy the quotient game property, for all \( k \in M \) we have

\[
\sum_{i \in P_k} g^h_i[v; P] = g^h_k[v^P; P^m] \quad \text{for } h = 1, 2,
\]

and, from both being coalitional values of \( \psi^p \) (also on \( M \), of course),

\[
\sum_{i \in P_k} g^1_i[v; P] = \psi^p_k[v^P] = \sum_{i \in P_k} g^2_i[v; P] \quad \text{for all } k \in M,
\]

so that \( g^1 \) and \( g^2 \) coincide (say, additively) on each union \( P_k \). If \( P_k = \{i\} \) then \( g^1_i[v; P] = g^2_i[v; P] \). If \( p_k > 1 \), let \( i, j \in P_k \) be distinct. Using the property of balanced contributions within unions,

\[
g^h_i[v; P] - g^h_j[v; P] = g^h_i[v; P_{-j}] - g^h_j[v; P_{-i}] \quad \text{for } h = 1, 2.
\]

By the maximality of \( m \), it follows that

\[
g^1_i[v; P_{-j}] - g^1_j[v; P_{-i}] = g^2_i[v; P_{-j}] - g^2_j[v; P_{-i}]
\]

and hence

\[
g^1_i[v; P] - g^1_j[v; P] = g^2_i[v; P] - g^2_j[v; P],
\]

that is,

\[
g^1_i[v; P] - g^2_i[v; P] = c_k \quad \text{(a constant) \ for all } i \in P_k.
\]

However

\[
0 = \sum_{i \in P_k} g^1_i[v; P] - \sum_{i \in P_k} g^2_i[v; P] = p_k c_k,
\]

so that \( c_k = 0 \) and therefore \( g^1 \) and \( g^2 \) coincide on each player of \( P_k \); thus, \( g^1 = g^2 \) on \( N \), a contradiction. \( \square \)

Remark 3.7 (A third axiomatic characterization). A further axiomatic characterization of each symmetric coalitional \( p \)-binomial semivalue \( \Omega^p \) was carried out in [9] by replacing the property of balanced contributions within unions in Theorem 3.5 with

- symmetry within unions: if \( i, j \in P_k \) are symmetric in \( v \) then \( g_i[v; P] = g_j[v; P] \).

Remark 3.8. Let us call symmetric coalitional values to those coalitional values that satisfy additivity, the dummy player property, balanced contributions within unions, symmetry within unions, symmetry in the quotient game and the quotient game property. Among them we find the Owen value, all symmetric coalitional binomial semivalues and, in particular, the symmetric coalitical Banzhaf value. Instead, the Banzhaf--Owen value and its counterpart (see the Appendix) fall out of this class. The only distinction between the Owen value and the symmetric coalitional binomial semivalues is that the former satisfies efficiency whereas each one of the latter satisfies a coalitional \( p \)-binomial total power property. This parallels the distinction between the Shapley value and the binomial semivalues (efficiency vs. total power property) which generalizes Feltkamp’s work [11], and extends the comparison between the classical four coalitional values (derived from combining the Shapley and Banzhaf values) that may be found in [3].

Remark 3.9 (Restriction to simple games). Simple games form an especially interesting class of cooperative games. Not only as a test bed for many cooperative concepts, but also for the variety of their interpretations, some of which are
far from game theory. In particular, they have been often applied to describe and analyze collective decision-making mechanisms—weighted majority games have played a crucial role here—and the notion of voting power has been closely attached to them.

While Theorem 3.6 translates without any change and gives an axiomatic characterization of each symmetric coalitional binomial semivalue as a coalitional power index (that is, restricted to simple games with a coalition structure), which we will not state explicitly, in the cases of Theorem 3.5 and Remark 3.7 we get “parallel” axiomatizations on the class of simple games by just replacing additivity with the

- transfer property: $g[v \lor v'; P] = g[v; P] + g[v'; P] - g[v \land v'; P]$ for all $v, v'$ and $P$.

The analogue of Theorem 3.5 for simple games is given below without proof (which is very similar to that of Theorem 3.5). In the case of the analogue of the characterization mentioned in Remark 3.7, we refer the reader again to [9] for a detailed statement.

**Theorem 3.10 (Axiomatic characterization on simple games).** Let $p \in [0, 1]$. For any $N$ there is a unique coalitional power index on $\mathcal{SG}^N$, the domain of all simple games with a coalition structure on $N$, that satisfies the transfer property, the dummy player property, balanced contributions within unions, the coalitional $p$-binomial total power property and symmetry in the quotient game. It is (the restriction to $\mathcal{SG}^N$ of) the symmetric coalitional $p$-binomial semivalue $\Omega^p$.

Moreover, $\Omega^p$ satisfies the quotient game property and yields, for every simple game $v$ on $N$, $\Omega^p[v; P^n] = \psi^p[v]$ and $\Omega^p[v; P^N] = \phi[v]$.

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**Appendix A.**

Games with a coalition structure were first considered by Aumann and Drèze [6], who extended the Shapley value to this new framework in such a manner that the game really splits into subgames played by the unions isolatedly from each other, and every player receives the payoff allocated to him by the Shapley value in the subgame he is playing within his union. A second approach was used by Owen [14] when introducing and axiomatically characterizing the coalitional value (Owen value). In this case, the unions play a quotient game among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value.

By applying a similar procedure to the Banzhaf value, Owen [15] got a second coalitional value, the modified Banzhaf value for games with a coalition structure or Banzhaf–Owen value. In this case the payoffs at both levels, that of the unions in the quotient game and that of the players within each union, are given by the Banzhaf value.

Alonso and Fiestras [4] realized that the Banzhaf–Owen value fails to satisfy two interesting properties of the Owen value: symmetry in the quotient game and the quotient game property. Then they suggested to modify the two-step allocation scheme and use the Banzhaf value for sharing in the quotient game and the Shapley value within unions. This gave rise to the symmetric coalitional Banzhaf value, the first mixed (or heterogeneous) coalitional value.

More or less simultaneously, but independently, Amer et al. [5] introduced a second mixed (or heterogeneous) coalitional value, which may now be viewed as a sort of “counterpart” of the symmetric coalitional Banzhaf value since it applies the Shapley value in the quotient game and the Banzhaf value within unions.

Later on, Albizuri and Zarzuelo [1] defined and studied coalitional semivalues, thus giving a general procedure for extending semivalues to the coalitional context. Each coalitional semivalue can be obtained by combining two arbitrary (maybe coincident) semivalues in a two-step procedure: the first one applies for unions in the quotient game and the other for players within each union. Therefore, the Owen value, the Banzhaf–Owen value, the symmetric coalitional Banzhaf value and its counterpart (which completes the four possibilities to combine the Shapley and Banzhaf values) fall within the notion of coalitional semivalue, although we should call homogeneous to the former two in order to distinguish them from the heterogeneous (mixed) latter two.
The idea of $p$-binomial semivalue was first given by Puente [17]. Carreras and Puente [9] extended this concept to games with a coalition structure and obtained at the same time a wide generalization of the Alonso–Fiestras value (essentially: $p \in [0, 1]$ instead of $p = \frac{1}{2}$), the family of symmetric coalitional $p$-binomial semivalues. Each one of them can also be considered as a particular case of coalitional semivalue since the payoff to unions is given by the $p$-binomial semivalue in the quotient game and is shared within each union according to the Shapley value.

It is worth to mention that, in the context of games with a coalition structure, the multilinear extension technique has been applied to computing the Owen value [16], the Banzhaf–Owen value [8], the symmetric coalitional Banzhaf value and its counterpart [2] and all symmetric coalitional binomial semivalues [9].

References