On eulerian and regular perfect path double covers of graphs

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Abstract

A perfect path double cover (PPDC) of a graph $G$ is a family $P$ of paths of $G$ such that every edge of $G$ belongs to exactly two paths of $P$ and each vertex of $G$ occurs exactly twice as an endpoint of a path in $P$. Li (J. Graph Theory 14 (1990) 645–650) has shown that every simple graph has a PPDC. A regular perfect path double cover (RPPDC) of a graph $G$ is a PPDC of $G$ in which all paths are of the same length. For a path double cover $P$ of a graph $G$, the associated graph $H_P(G)$ of $P$ is defined as a graph having the same vertex set as $G$, with two vertices $x$ and $y$ adjacent if and only if there is a path in $P$ with endpoints $x$ and $y$. An eulerian perfect path double cover (EPPDC) of a graph $G$ is a PPDC of $G$ whose associated graph is a cycle. If a PPDC is both eulerian and regular, it is called an ERPPDC. In this paper, we will discuss EPPDCs and RPPDCs for certain types of graphs. In particular, we will describe a construction for an ERPPDC of the line graph of a complete graph.

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1. Introduction

A path cover of a graph $G$ is a family $P$ of paths such that each edge of $G$ belongs to at least one path of $P$. A path cover $P$ in which every edge belongs to exactly one path in $P$
is called a path decomposition of \( G \). The definitions of a cycle decomposition and a cycle cover of a graph are analogous.

A path double cover of a graph \( G \) is a path cover \( P \) of \( G \) such that every edge of \( G \) belongs to exactly two paths in \( P \). In 1990, Bondy [1] posed several conjectures about path double covers of graphs. He conjectured that every simple graph admits a path double cover \( P \) such that each vertex occurs exactly twice as an end of a path in \( P \); a perfect path double cover (PPDC). This conjecture has been proved for \( k \geq 3 \) [1] and \( k = 4 \) [3], but is still open for larger values of \( k \).

Bondy also conjectured that every \( k \)-regular simple graph admits a path double cover \( P \) such that every path in \( P \) has length \( k \) and each vertex of the graph occurs exactly twice as an end of a path in \( P \); a regular perfect path double cover (RPPDC). This conjecture has been proved for \( k \leq 3 \) [1] and \( k = 4 \) [3], but is still open for larger values of \( k \).

For a path double cover \( P \) of a graph \( G \), the associated graph \( A_P(G) \) of \( P \) is defined as a graph having the same vertex set as \( G \), with two vertices \( x \) and \( y \) adjacent in \( A_P(G) \) if and only if there is a path in \( P \) with endpoints \( x \) and \( y \). Clearly, \( A_P(G) \) is a 2-regular (multi-)graph. A PPDC is called an eulerian perfect path double cover (EPPDC) if its associated graph is a cycle. Bondy [1] conjectured that every simple connected graph has an EPPDC. If a path double cover is both eulerian and regular, we call it an ERPPDC. Seyffarth [5] proved that complete graphs admit ERPPDC. Any terminology and notation not defined in this article follow that of [2].

2. An inductive result

**Proposition 1.** Suppose that \( G \) is a graph with an eulerian perfect path double cover. Then for \( 1 \leq d(x) \leq 3 \), \( G + x \) has an eulerian perfect double cover.

**Proof.** Assume \( G \) has an EPPDC \( P \). Orient the cycle \( A_P(G) \), and then add a vertex \( x \) to \( G \) with \( 1 \leq d(x) \leq 3 \).

If \( d(x) = 1 \), let \( x_1 \) be the vertex adjacent to \( x \) in \( G + x \), and let \( P_1 \in P \) have endpoint \( x_1 \). By replacing \( P_1 \) with \( P_1x \), and adding a new path \( xx_1 \), we obtain an EPPDC of \( G + x \).

Suppose \( d(x) = 2 \) and that \( x \) is adjacent to \( x_1 \) and \( x_2 \). Assume that, in \( A_P(G) \), the vertex preceding \( x_i \) is \( y_i, i = 1, 2 \). Then there is a path \( P_i \in P \) from \( y_i \) to \( x_i \). Replace \( P_1 \) by \( P_1x \), \( P_2 \) by \( P_2x \) and add a new path \( x_1x_2 \). The result is an EPPDC of \( G + x \), where the section in \( A_P(G) \) from \( y_1 \) to \( x_2 \) becomes \( y_1x_2 \).

Suppose \( d(x) = 3 \) and that \( x \) is adjacent to \( x_1, x_2 \) and \( x_3 \). Let \( y_i, 1 \leq i \leq 3 \), be the vertex preceding \( x_i \) in \( A_P(G) \), and denote the path from \( y_i \) to \( x_i \) by \( P_i \). We consider two cases.

**Case 1:** \( x_3 \) is not in \( P_2 \). Replace \( P_1 \) by \( P_1x \), \( P_2 \) by \( P_2x_3 \), \( P_3 \) by \( P_3x \), and add a new path \( x_1x_2 \). The result is an EPPDC of \( G + x \), where the section in \( A_P(G) \) from \( y_1 \) to \( x_3 \) becomes \( y_1x_3 \).

**Case 2:** \( x_3 \) is in \( P_2 \). Replace \( P_1 \) by \( P_1x \) and \( P_3 \) by \( P_3x \). If \( x_1 \) is not in the section \( x_3P_2x_2 \), then replace \( P_2 \) by two paths \( y_2P_2x_3x_2 \) and \( x_3P_2x_2x_1 \). The result is an EPPDC of \( G + x \), where the section in \( A_P(G) \) from \( y_1 \) to \( x_3 \) becomes \( y_1x_3 \).

If \( x_1 \) is in the section \( x_3P_2x_2 \), replace \( P_2 \) by two paths \( x_1P_2x_2x_3 \) and \( y_2P_2x_1x_2 \). The result is an EPPDC of \( G + x \), where the section in \( A_P(G) \) from \( y_1 \) to \( x_3 \) becomes \( y_1x_3 \).

\( \square \)
Corollary 2. Any connected graph with maximum degree at most three has an EPPDC.

Corollary 3. If $G$ is a $k$-tree, $k = 1, 2, 3$, then $G$ has an EPPDC.

3. Complete bipartite graphs

Proposition 4. For all $n \geq 1$, $K_{n,n}$ has an RPPDC. Moreover, if $n$ is odd, then $K_{n,n}$ has an ERPPDC.

Proof. Assume that $K_{n,n}$ has bipartition $(X, Y)$, where $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. Suppose that $n$ is odd. For each $i$, $i = 0, 1, \ldots, n-1$, let

$$P_i = x_i y_{n-1-i} x_{i+1} y_{n-2-i} \ldots y_{(n-3)/2+i} x_{(n-1)/2+i} y_{(n-1)/2+i}.$$ 

Then $P = \{P_0, P_1, \ldots, P_{n-1}\}$ is a path decomposition of $K_{n,n}$. Exchanging the $x$’s and $y$’s results in a second path decomposition $P'$ of $K_{n,n}$. The union of the two path decompositions forms an RPPDC of $K_{n,n}$, and the edge set of $A_{P \cup P'}(K_{n,n})$ is

$$\{x_i y_{i+(n-1)/2}, y_{i+i+(n-1)/2} : 0 \leq i \leq n-1\},$$

(subscripts modulo $n$). Since $i + (n-1)/2 + (n-1)/2 \equiv i - 1 \pmod{n}$, $A_{P \cup P'}(K_{n,n})$ is a cycle. Therefore $P \cup P'$ is an ERPPDC.

If $n$ is even, it is impossible to construct an ERPPDC because a path of even length has both endpoints in the same part of $K_{n,n}$. Let

$$P_i = x_i y_{n-1+i} x_{i+1} y_{n-2+i} \ldots y_{n/2+i} x_{n/2+i}.$$ 

Then $P = \{P_0, P_1, \ldots, P_{n-1}\}$ is a path decomposition of $K_{n,n}$. By exchanging the $x$’s and $y$’s we obtain a second path decomposition of $K_{n,n}$. As in the case $n$ odd, it is easy to see that the union of these two path decompositions gives an RPPDC of $K_{n,n}$. □

Proposition 5. For all $m, n \geq 1$, $K_{m,n}$ has an EPPDC.

Proof. First suppose that $m \neq n$; without loss of generality, we may assume that $m < n$. Let $K_{m,n}$ have bipartition $(X, Y)$, where $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{m-1}\}$. For each $i, i = 0, 1, \ldots, n-2$, let

$$P_i = x_i y_0 x_{i+2} y_1 x_{i+3} y_2 \ldots x_{m+i} y_{m-1} x_{i+1}.$$ 

Then $P = \{P_0, P_1, \ldots, P_{n-1}\}$ is a path double cover of $K_{m,n}$, and it is easy to see that $A_P(K_{m,n})$ is a cycle. For $i = 0, 1, \ldots, m - 1$, break $P_i$ into two paths at $y_i$. The result is a collection of $m + n$ paths that form an EPPDC of $K_{m,n}$.

Now suppose that $m = n$. If $n$ is odd then by Proposition 4, $K_{n,n}$ has an ERPPDC. However, if $n$ is even, an alternate construction is required.

Suppose $n$ is even. For $i = 0, 1, \ldots, n-1$, define

$$\alpha_i = \{x_0 y_i, x_1 y_1+i, \ldots, x_{n-1} y_{i-1}\}.$$
Then \( x_i \) is a perfect matching in \( K_{n,n} \), and
\[
H_i = x_{2i-1} \cup x_{2i}, \quad 1 \leq i \leq \frac{n}{2}
\]
is a Hamilton cycle. It is well known that the collection \( \{H_1, H_2, \ldots, H_{n/2}\} \) is a decomposition of \( K_{n,n} \) into the \( n/2 \) Hamilton cycles. Define an \( n \)-path in \( K_{n,n} \) as follows:
\[
P = \begin{cases}
  x_0 y_{n-1} x_2 y_{n-3} \ldots x_{n/2-1} y_{n/2} x_n/2 y_{n/2-2} x_{n/2} x_{n/2+2} & \text{if } n \equiv 2 \pmod{4}, \\
  \ldots x_{n-3} y_1 x_n-1 & \\
  x_0 y_{n-1} x_2 y_{n-3} \ldots x_{n/2-2} y_{n/2+1} x_{n/2} y_{n/2} x_{n/2+2} y_{n/2-2} x_{n/2+2} y_{n/2-2} & \text{if } n \equiv 0 \pmod{4}.
\end{cases}
\]

Note that edges in \( P \) belong, respectively, to \( H_{n/2}, H_{n/2-1}, \ldots, H_1, \) and that between any two consecutive vertices in \( P \) there is a Hamilton path in \( K_{n,n}: H_{n/2} - \{x_0 y_{n-1}\} \) is a Hamilton path from \( x_0 \) to \( y_{n-1} \), \( H_{n/2-1} - \{x_2 y_{n-1}\} \) is a Hamilton path from \( y_{n-1} \) to \( x_2 \), etc. These \( n \) Hamilton paths plus \( P \) form a path double cover \( Q \) of \( K_{n,n} \) such that \( AQ(K_{n,n}) \) is a cycle. An EPPDC is obtained from \( Q \) as follows. Suppose a vertex \( x \) has degree zero in \( AQ(K_{n,n}) \). Then \( x \) must appear as a vertex in a path between \( y \) and \( z \), where \( y \) and \( z \) are adjacent vertices in \( AQ(K_{n,n}) \). Split the path from \( y \) to \( z \) into two paths, one from \( y \) to \( x \) and the other from \( x \) to \( z \). The result is a path double cover where the associated graph is a longer cycle. In this way, paths are split to lengthen the cycle that is the associated graph until the cycle contains all vertices of \( K_{n,n} \). Thus, \( K_{n,n} \) has an EPPDC. \( \square \)

4. **Line graphs**

**Proposition 6.** If \( G \) is a \( k \)-regular graph, \( k \geq 1 \), then \( L(G) \) has an RPPDC.

**Proof.** If \( k = 1 \) or \( 2 \), the conclusion is obvious. Thus we may assume \( k \geq 3 \). The edge set of \( L(G) \) can be partitioned into \( k \)-cliques, where each vertex of \( L(G) \) belongs to exactly two cliques and any two cliques have at most one vertex in common. Each of these cliques has an ERPPDC \([1,5]\). Give each clique an orientation as follows: write the vertices of the clique as a row \( e_0, e_1, \ldots, e_{k-1} \) if there is a \((k-1)\)-path in the ERPPDC of the clique from \( e_i \) to \( e_{i+1} \) (subscripts modulo \( k \)), and call \( e_{i+1} \) the *succeeding* vertex of \( e_i \). The result is an \( n \times k \) matrix \( M \), where \( n \) is the order of \( G \). Since every vertex of \( L(G) \) occurs in two rows of \( M \), and any two rows of \( M \) have at most one element in common, each vertex \( x \) has exactly two succeeding vertices \( y \) and \( z \), in two different rows of \( M \). Since two rows of \( M \) have at most one element in common, \( y \neq z \). Thus there is a \((k-1)\)-path between \( x \) and \( y \) in the ERPPDC of one clique, and a \((k-1)\)-path between \( x \) and \( z \) in the ERPPDC of another clique. By joining these two paths at \( x \) we obtain a \( 2(k-1) \)-path \( P_x \) between \( y \) and \( z \) (\( x \) is called the middle point of this path). This process is repeated for every \( x \in V(L(G)) \) to obtain a set \( P \) of \( 2(k-1) \)-paths in \( L(G) \) with \(|P| = n(n-1)/2\). The paths in \( P \) are distinct because \( x \) is the middle vertex of \( P_x \) and the paths in \( P \) have distinct middle vertices.

Since each \((k-1)\)-path in the ERPPDC of every clique is used once and every edge is covered twice by these paths, every edge is covered twice by paths in \( P \). Every vertex \( y \) occurs in two rows of \( M \): it is the succeeding vertex of \( x_1 \) in one row and the succeeding vertex of \( x_2 \) in another row. Since two rows have at most one element in common, \( x_1 \neq x_2 \),
and \( y \) are endpoints of \( P_{x_1} \) and \( P_{x_2} \). Thus every vertex of \( L(G) \) occurs twice as endpoints of paths in \( P \), and hence \( P \) is an RPPDC of \( L(G) \). □

**Lemma 7.** Let \( G \) be a graph with \( m \) edges. Suppose \( 2G \) has an Euler circuit \( e_1, e_2, \ldots, e_{2m} \) such that \( S_1 = \{e_1, e_3, \ldots, e_{2m-1}\} \) and \( S_2 = \{e_2, e_4, \ldots, e_{2m}\} \) are both the set \( E(G) \) of all edges of \( G \). Furthermore, suppose that for each \( v \in V(G) \) there is an ordering, \( C(v) \), of the edges incident to \( v \) such that every pair of consecutive edges in \( C(v) \) occurs exactly once as a pair of consecutive edges in the Euler circuit. Then \( L(G) \) has an EPPDC.

**Proof.** Each vertex \( v \) of \( G \) corresponds to a clique \( K(v) \) in \( L(G) \), and the ordering \( C(v) \) corresponds to a Hamilton cycle in \( K(v) \). By [5] there is an ERPPDC, \( P \), of \( K(v) \) so that \( A_P(K(v)) \) is the cycle \( C(v) \).

The Euler circuit in \( 2G \) corresponds to the list \( e_1, e_2, \ldots, e_{2m} \) of vertices of \( L(G) \), and each of \( S_1 \) and \( S_2 \) is the set of all vertices of \( L(G) \). For each pair \( e_i, e_{i+2} \) in \( S_1 \), \( e_i \) and \( e_{i+1} \) have some vertex \( u \) in common, and \( e_{i+1} \) and \( e_{i+2} \) have some vertex \( v \) in common. Thus there is a path in the ERPPDC of \( K(u) \) from \( e_i \) to \( e_{i+1} \) and there is a path in the ERPPDC of \( K(v) \) from \( e_{i+1} \) to \( e_{i+2} \). These two paths have no vertices in common except \( e_{i+1} \), and thus concatenating them results in a path from \( e_i \) to \( e_{i+2} \). This procedure is repeated for all consecutive pairs of vertices of \( L(G) \) in \( S_1 \) (include \( e_{2m-1} \) and \( e_1 \)). Every path in the ERPPDC of \( K(v) \) is used once for every \( v \), resulting in \( m \) paths that form an EPPDC for \( L(G) \). □

**Proposition 8.** For all \( m \geq 2 \), \( L(K_m) \) has an ERPPDC.

**Proof.** We consider \( m \) even and \( m \) odd separately.

*Case 1:* Suppose that \( m \) is even; i.e., \( m = 2n \) for some \( n \geq 1 \). Label the vertices of \( K_{2n} \) by \( v_∞, v_0, v_1, \ldots, v_{2n-2} \). It is well known that \( K_{2n} \) has a 1-factorization with \( n - 1 \) factors \( F_0, F_1, \ldots, F_{2n-2} \) where

\[
F_i = \{v_∞v_i, v_{i+1}v_{i-1}, v_{i+2}v_{i-2}, \ldots, v_{i+n-1}v_{i-n+1}\}.
\]

For vertex \( v_∞ \), set

\[
C(v_∞) = (f_0, f_2, \ldots, f_{2n-2}, f_1, f_3, \ldots, f_{2n-3}),
\]

where \( f_i \) is the edge incident to \( v_∞ \) belonging to the 1-factor \( F_i \). For any vertex \( v_j \) with \( j \neq ∞ \), we set

\[
C(v_j) = (f_0, f_1, \ldots, f_{2n-2}),
\]

where \( f_j \) is the edge incident to \( v_j \) belonging to the 1-factor \( F_j \).

Let \( H_i = F_i \cup F_{i+1} \) for \( i = 0, 1, \ldots, 2n - 2 \). Then \( H_i \) is a Hamilton cycle in \( K_m \). These cycles are used to construct an Euler circuit of \( 2K_m \).

In \( H_i = F_i \cup F_{i+1} \), we start at the edge \( v_∞v_{i+1} \) and then list all other edges along \( H_i \). We use the same symbol \( H_i \) to denote this list. Then \( H_0, H_1, \ldots, H_{2n-2} \) is an Euler circuit of \( 2K_{2n} \). It is routine to verify that the conditions in Lemma 7 are satisfied by the construction described, and thus \( L(K_m) \) has an EPPDC (in fact, \( L(K_m) \) has an ERPPDC).
Case 2: Now suppose that $m$ is odd. In this case, we show that it is possible to choose an ordering $C(v)$ for each $v \in V(K_m)$, and construct an Euler circuit, so that the conditions in Lemma 7 are satisfied. To simplify the notation, we use a matrix to describe the construction, where each row of the matrix corresponds to an ordering of $C(v), v \in V(K_m)$.

Denote the vertices of $L(K_m)$ by the integers $1, 2, \ldots, m(m - 1)/2$, and construct an $m \times (m - 1)$ matrix $M$ (with row and column indices starting at zero) as follows. The zeroth row of $M$ is $1, 2, \ldots, m - 1$. The zeroth column of $M$ beginning at row one is $1, 2, \ldots, m - 1$. Row one of $M$, beginning at column one is $m, m + 1, \ldots, 2m - 3$, and column one, beginning at the row two, is $m, m + 1, \ldots, 2m - 3$. Similarly, the $(2, 2)$-entry and $(3, 2)$-entry are equal to $2m - 2$, and so on (Example 9 illustrates the matrix when $m = 7$).

Every row of $M$ corresponds to a clique, and two distinct rows have exactly one element in common. As in the proof of Proposition 6, for every vertex $x$ we have a $2(m - 1)$-path $P_x$, and these paths form a RPPDC $P$ of $L(K_m)$.

To show this RPPDC is eulerian, we define a permutation $f$ on $\mathbb{Z}_m \times \mathbb{Z}_{m-1}$ as follows:

$$f(i, j) = \begin{cases} 
(i + 2, i) & \text{if } i = j \text{ and } 0 \leq i \leq m - 3, \\
(0, n - 2) & \text{if } i = j = m - 2, \\
(i + 1, j - 1) & \text{if } i \neq j \text{ and } i < m - 1, \\
(1, n - 2) & \text{if } i = m - 1 \text{ and } j = 0, \\
(0, n - 2) & \text{if } i = m - 1 \text{ and } 2 \leq j \leq m - 2.
\end{cases}$$

**Example 9.** An illustration for the construction described in Case 2 of the proof of Proposition 8 in the case $m = 7$.

$$M = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 7 & 8 & 9 & 10 & 11 \\
2 & 7 & 12 & 13 & 14 & 15 \\
3 & 8 & 12 & 16 & 17 & 18 \\
4 & 9 & 13 & 16 & 19 & 20 \\
5 & 10 & 14 & 17 & 19 & 21 \\
6 & 11 & 15 & 18 & 20 & 21
\end{bmatrix}$$

$$\begin{array}{c|cccccccc}
\mathbb{Z}_7 \times \mathbb{Z}_6 & 0, 0 & 0, 1 & 0, 2 & 0, 3 & 0, 4 & 0, 5 & 1, 0 & 1, 1 & 1, 2 & 1, 3 & 1, 4 \\
\hline
0, 0 & (2, 0) & (1, 0) & (1, 1) & (1, 2) & (1, 3) & (1, 4) \\
0, 1 & (2, 5) & (3, 1) & (2, 1) & (2, 2) & (2, 3) & (2, 4) \\
0, 2 & (3, 5) & (3, 0) & (4, 2) & (3, 2) & (3, 3) & (3, 4) \\
0, 3 & (4, 5) & (4, 0) & (4, 1) & (5, 3) & (4, 3) & (4, 4) \\
0, 4 & (5, 5) & (5, 0) & (5, 1) & (5, 2) & (6, 4) & (5, 4) \\
0, 5 & (6, 5) & (6, 0) & (6, 1) & (6, 2) & (6, 3) & (6, 4) \\
0, 6 & (1, 5) & (0, 5) & (0, 0) & (0, 1) & (0, 2) & (0, 3)
\end{array}$$

**the permutation $f$ of $\mathbb{Z}_7 \times \mathbb{Z}_6$**
One can verify that \( f \) is a permutation of \( \mathbb{Z}_m \times \mathbb{Z}_{m-1} \). We claim that for all \((i, j) \in \mathbb{Z}_m \times \mathbb{Z}_{m-1}\), \( P_M(i, j) \) and \( P_M(f(i, j)) \) are two paths in \( P \) that are disjoint except for having a common endpoint. It is a case by case argument. For example, if \( M(i, i) = x \), then \( M(i+1, i) = x \), and hence \( P_x \) has endpoints \( M(i, i+1) \) and \( M(i+1, i+1) \). If \( M(i+2, i) = y \), then \( M(i, i+1) = y \), and \( P_y \) has endpoints \( M(i, i+2) \) and \( M(i+2, i+1) \). By the definition of \( M \), \( M(i+1, i+1) = M(i+2, i+1) \), and thus \( P_x \) and \( P_y \) have a common endpoint.

Let \( S = \{(f^t(0, 0)) : t = 0, 1, 2, \ldots, k-1\} \), where \( k = m(m-1)/2 \), and let \( T = \{M(i, j) : (i, j) \in S\} \). Since \( P_M(i, j) \) and \( P_f(M(i, j)) \) have one common endpoint, \( P \) is eulerian provided that \( T \) is a set of \( m(m-1)/2 \) distinct elements. This is a consequence of the following observations:

(i) \((0, 0) \in S\);
(ii) if \((0, j) \in S\), then \((0, j+2) \in S\);
(iii) \((1, j+1) \in S\);
(iv) by the definition of \( f \), \((i+1, j-1) \in S\) whenever \((i, j) \in S\) for \( i \neq j \) and \((i+2, i) \in S\) whenever \((i, i) \in S\) and
(v) induction on \( i \) can be used to show that \((i, j) \in S\) whenever \( i \) and \( j \) have the same parity.

Using the definition of the matrix \( M \), it follows that the elements in \( T \) are distinct. It now follows from Lemma 7 that \( P \) is an EPPDC; the fact that \( P \) is an ERPPDC follows easily from the fact that the paths of \( P \) are all of length \( 2(m-1) \). \( \square \)

Another family of regular line graphs is the family of \( L(K_{m,n}) \) for all positive integers \( m \) and \( n \).

**Proposition 10.** For all \( m, n \geq 1 \), \( L(K_{m,n}) \) has an RPPDC. Furthermore, if \( \gcd(n, m) = 1 \) or \( \gcd(n, n - m + 2) = 1 \), then \( L(K_{m,n}) \) has an ERPPDC.

**Proof.** Let \( K_{m,n} \) have bipartition \((X, Y)\) with \(|X| = m\), \(|Y| = n\), \( m \leq n \). Consider the \( m \) cliques of \( L(K_{m,n}) \) corresponding to the vertices of \( X \). Any two of these \( m \) cliques are vertex disjoint, and so can be represented as the rows of an \( m \times n \) matrix \( M_1 \), where the \( m \times n \) entries are distinct elements. Without loss of generality, \( M_1(i, j) = i \times n + (j + 1), 0 \leq i \leq m - 1, 0 \leq j \leq n - 1 \). Let \( M_2 = M_1^t \). Then each row of \( M_2 \) represents a clique of \( L(K_{m,n}) \) corresponding to a vertex of \( Y \). For each clique, choose an ERPPDC of the clique so that if \( e_1, e_2, \ldots, e_k \) (where \( k = m \) or \( n \)) is a row of \( M_1 \) or \( M_2 \) then there is a \((k-1)\)-path from \( e_i \) to \( e_{i+1} \) in the ERPPDC of the clique.

For any element \( x \), if \( x = M_1(i, j) = M_2(j, i) \), then there is an \((n-1)\)-path between \( x \) and \( M_1(i, j+1) \) in the ERPPDC of one clique and an \((m-1)\)-path between \( x \) and \( M_2(j, i+1) \) in the ERPPDC of the other clique (where the first coordinate of the indices is taken modulo \( m \), and the second coordinate is taken modulo \( n \)). The union of these two paths forms an \((m+n-2)\)-path \( P_x \). By forming such a path for each element of the matrix \( M_1 \), we obtain a set \( P \), of \((m+n-2)\)-paths, with \(|P| = mn \). It is clear that each path in the ERPPDC of the cliques is used once, and it is easy to verify that every vertex is the endpoint of two paths in \( P \). Therefore, \( P \) is an RPPDC of \( L(K_{m,n}) \).
What remains is to show that $\mathbf{P}$ is eulerian when $\gcd(n, m) = 1$ or $\gcd(n, n - m + 2) = 1$. We begin with the case $\gcd(n, m) = 1$; the case when $\gcd(n, n - m + 2) = 1$ requires a small change in the construction of $\mathbf{P}$, but is otherwise analogous.

Define a permutation $f$ on $\mathbb{Z}_m \times \mathbb{Z}_n$ as follows:

$$f(i, j) = (i + 1, j - 1) \quad \text{for all } i, j.$$ 

We claim that $\mathbf{P}_{M_1(i, j)}$ and $\mathbf{P}_{M_1 f(i, j)}$ have a common endpoint. To see this, let $x = M_1(i, j) = M_2(j, i)$, and let $y = M_1(f(i, j)) = M_2(i + 1, j - 1) = M_2(j + 1, i + 1)$. Then $\mathbf{P}_x$ has endpoints $M_1(i, j + 1)$ and $M_2(j, i + 1)$, and $\mathbf{P}_y$ has endpoints $M_1(i + 1, j)$ and $M_2(j - 1, i + 2)$. Since $M_1(i + 1, j) = M_2(j, i + 1)$, $\mathbf{P}_x$ and $\mathbf{P}_y$ have a common endpoint. It now follows that $\mathbf{P}$ is eulerian provided

$$\{f^k(0, 0) : k = 0, 1, \ldots, mn - 1\}$$

is a set of $mn$ distinct elements. This fact follows from the observation that, for $k = 0, 1, \ldots, n - 1$, $f^{km}(0, 0) = (0, n - km)$, and for $n - km$ to run over all elements of $\mathbb{Z}_n$, it suffices that $m$ generate $\mathbb{Z}_n$; i.e., $\gcd(n, m) = 1$. Therefore, $\mathbf{P}$ is an ERPPDC of $K_{m, n}$.

Now suppose that $\gcd(n, n - m + 2) = 1$. We re-define the collection $\mathbf{P}$ as before, except in the case where $i = m - 1$; i.e., if $x = M_1(i, j) = M_2(j, i)$ where $i < m - 1$, then $\mathbf{P}_x$ is the union of an $(n - 1)$-path between $x$ and $M_1(i, j + 1)$ in the ERPPDC of one clique, and an $(m - 1)$-path between $x$ and $M_2(j, i + 1)$ in the ERPPDC of the other clique. However, in the case $i = m - 1$, if $x = M_1(m - 1, j) = M_2(j, m - 1)$, then $\mathbf{P}_x$ is the union of an $(n - 1)$-path between $M_1(m - 1, j - 1)$ and $x$ in the ERPPDC of one clique and an $(m - 1)$-path between $x$ and $M_2(j, 0)$ in the ERPPDC of the other clique. The verification that $\mathbf{P}$ is an RPPDC of $L(K_{m, n})$ is routine.

Define a permutation $g$ on $\mathbb{Z}_m \times \mathbb{Z}_n$ as follows:

$$g(i, j) = \begin{cases} (i + 1, j - 1) & \text{if } i \neq m - 1, \\ (i + 1, j + 1) & \text{if } i = m - 1. \end{cases}$$

Again, we claim that $\mathbf{P}_{M_1 g(i, j)}$ and $\mathbf{P}_{M_2 g(i, j)}$ have a common endpoint. The argument is identical to the one presented in the case $\gcd(n, m) = 1$, except when $i = m - 1$. Suppose that $x = M_1(m - 1, j) = M_2(j, m - 1)$, and $y = M_1(g(m - 1, j)) = M_2(0, j + 1) = M_2(j + 1, 0)$. Then $\mathbf{P}_x$ has endpoints $M_1(m - 1, j - 1)$ and $M_2(j, 0)$, and $\mathbf{P}_y$ has endpoints $M_1(0, j)$ and $M_2(j + 1, 1)$. Since $M_1(0, j) = M_2(j, 0)$, $\mathbf{P}_x$ and $\mathbf{P}_y$ have a common endpoint. It now follows that $\mathbf{P}$ is eulerian provided

$$\{g(0, 0) : k = 0, 1, \ldots, mn - 1\}$$

is a set of $mn$ distinct elements. Notice that $g^{km}(0, 0) = (0, n - km + 2k)$ for $k = 0, 1, \ldots, n - 1$, and for $n - km + 2k$ to run over all elements of $\mathbb{Z}_n$, it suffices that $2 - m$ generates $\mathbb{Z}_n$; i.e., $\gcd(n, n - m + 2) = 1$. Therefore, $\mathbf{P}$ is an ERPPDC of $K_{m, n}$. □

**Corollary 11.** For every positive odd integer $n$, $L(K_{n,n})$ has an ERPPDC.
5. The Cartesian product

Definition 12. Let $G$ be a graph of order $m$ with $V(G) = \{g_i : 0 \leq i \leq m - 1\}$, and let $H$ be a graph of order $n$ with $V(H) = \{h_j : 0 \leq j \leq n - 1\}$. The Cartesian product $G \Box H$ is defined to be the graph with vertex set $\{(g_i, h_j) : 0 \leq i \leq m - 1 \text{ and } 0 \leq j \leq n - 1\}$ and $(g_i, h_j)(g_s, h_t) \in E(G \Box H)$ if either $g_i = g_s$ and $h_jh_t \in E(H)$ or $h_j = h_t$ and $g_i g_s \in E(G)$.

Proposition 13.

(i) If $G$ and $H$ have RPPDCs, then $G \Box H$ has an RPPDC.
(ii) If $G$ and $H$ have EPPDCs and $(|G|, |H|) = 1$, then $G \Box H$ has an EPPDC.

Proof. Let $G$ be a graph with $V(G) = \{g_i : 0 \leq i \leq m - 1\}$, and $H$ a graph with $V(H) = \{h_j : 0 \leq j \leq n - 1\}$. Suppose that $P$ is a PPDC of $G$, and $Q$ is a PPDC of $H$. There exists a bijection $f_1 : V(G) \rightarrow P$ so that $f_1(g_i)$ has $g_i$ as one of its endpoints. Similarly, there exists a bijection $f_2 : V(H) \rightarrow Q$ so that $f_2(h_j)$ has $h_j$ as one of its endpoints.

Let $(g_a, h_i) \in V(G \Box H)$, and suppose that $f_1(g_a) = g_a u_1 u_2 \ldots u_k g_b \in P$ and $f_2(h_i) = h_i v_1 v_2 \ldots v_l h_j \in Q$. Then

$$P_{\alpha i} : (g_a, h_i), (u_1, h_i), (u_2, h_i), \ldots, (u_k, h_i), (g_b, h_i), (g_b, v_1), (g_b, v_2), \ldots, (g_b, v_l), (g_b, h_j)$$

is a path in $G \Box H$ from $(g_a, h_i)$ to $(g_b, h_j)$. It follows easily that the collection of paths $T = \{P_{\alpha i} : 0 \leq \alpha \leq m, 0 \leq i \leq n\}$ is a PPDC of $G \Box H$.

If $P$ is an RPPDC with all paths of length $\alpha$, and $Q$ is an RPPDC with all paths of length $\beta$, then every path in $T$ has length $\alpha + \beta$, implying that $T$ is an RPPDC of $G \Box H$.

If $P$ is an EPPDC (whose associated graph is, without loss of generality, the cycle $g_0 g_1 g_2 \ldots g_{m-1} g_0$) and $Q$ is an EPPDC (whose associated graph is, without loss of generality, the cycle $h_0 h_1 h_2 \ldots h_{n-1} h_0$), then $P_{\alpha i}$ has endpoints $(g_a, h_i)$ and $(g_{a+1}, h_{i+1})$. Thus, if $\gcd(m, n) = 1$, the associated graph for $T$ is a cycle, and thus $T$ is an EPPDC of $G \Box H$.

Proposition 14. If $G$ has an EPPDC, then the Cartesian product $G \Box K_2$ has an EPPDC.

Proof. Let $V(G) = \{g_0, g_1, \ldots, g_{n-1}\}$. If $n$ is odd, the result follows from Proposition 13. Thus we assume $n$ is even. Let $h_0$ and $h_1$ be the two vertices of $K_2$. Every vertex of $G \Box K_2$ has form $(g_i, h_j), 0 \leq i \leq n - 1, 0 \leq j \leq 1. Let P be an EPPDC of G; without loss of generality, we may assume that Ap(G) is the cycle $g_0 g_1 \ldots g_{n-1} g_0$.

The path in $P$ from $g_i$ to $g_{i+1}$ (subscripts modulo $n$) corresponds to two paths in $G \Box K_2$: one path $P_i$ from $(g_i, h_0)$ to $(g_{i+1}, h_0)$, and a second path $Q_i$ from $(g_i, h_1)$ to $(g_{i+1}, h_1)$. For each $i, 0 \leq i < n - 1$, let

$$P_i' = P_i \cup \{(g_{i+1}, h_0)(g_{i+1}, h_1)\},$$

$$Q_i' = Q_i \cup \{(g_{i+1}, h_1)(g_{i+1}, h_0)\}.$$
Finally, let

\[ P'_{n-1} = P_{n-1} \cup \{(g_0, h_0)(g_0, h_1)\} \cup Q_{n-1}, \]
\[ Q'_{n-1} = \{(g_0, h_0)(g_0, h_1)\}. \]

Certainly,

\[ P = (\bigcup_{i=0}^{n-1} P'_i) \cup (\bigcup_{j=0}^{n-1} Q'_j) \]

is a PPDC of \( G \Box K_2 \). Since \( A_P(G \Box K_2) \) is the cycle

\[ (g_0, h_0)(g_1, h_1)(g_2, h_0)(g_3, h_1) \ldots (g_{n-2}, h_0)(g_{n-1}, h_1) \]
\[ (g_{n-1}, h_0)(g_{n-2}, h_1) \ldots (g_1, h_0)(g_0, h_1)(g_0, h_0), \]

\( P \) is an EPPDC of \( G \Box K_2 \).

**Corollary 15.** For all \( n \geq 0 \), the n-cube, \( Q_n \), has an EPPDC.

### 6. The lexicographic product

**Definition 16.** Let \( G = (V, E) \) and \( G' = (V', E') \) be two graphs. The lexicographic product \( G[G'] \) has the vertex set \( V \times V' \), with vertices \((x, x')\) and \((y, y')\) joined by an edge if and only if \( xy \in E \), or \( x = y \) and \( x'y' \in E' \).

Informally, the lexicographic product \( G[G'] \) can be described as the graph obtained by taking \(|V(G)|\) copies of \( G' \) (corresponding to the vertices of \( G \)), and adding the edges of a \( K_{|V'|,|V'|} \) between two copies of \( G' \) if and only if the corresponding vertices of \( G \) are adjacent.

**Lemma 17.** Suppose \( K_{n,n} \) has bipartition \( (X, Y) \) with \( X = \{x_0, x_1, \ldots, x_{n-1}\} \) and \( Y = \{y_0, y_1, \ldots, y_{n-1}\} \), and let \( M = \{x_0 y_0, x_1 y_1, \ldots, x_{n-1} y_{n-1}\} \) be a matching in \( K_{n,n} \). Then for each \( i, 0 \leq i \leq n-1 \), there is a path \( P_i \) from \( x_i \) and \( y_i \), such that

1. \(|P_i| = 2n - 1, 0 \leq i \leq n - 1\) and
2. \( P = \{P_i : 0 \leq i \leq n - 1\} \) covers each edge of \( M \) once and covers each edge in \( E(K_{n,n}) - M \) exactly twice.

**Proof.** For each \( i, 0 \leq i \leq n - 1 \), define the path \( P_i \) as

\[ P_i = x_i y_{n+i-1} x_{i+1} y_{n+i-2} \ldots x_{n+i-2} y_{i+1} x_{n+i-1} y_i \]

(subscripts modulo \( n \)). It is routine to verify that these paths satisfy the conditions (1) and (2). □

Let \( C_k \) denote a cycle on \( k \) vertices, and \( S_n \) an independent set of size \( n \).
Proposition 18. The lexicographic product \( C_k[S_n] \) has a RPPDC. Moreover, if \( k \) and \( n \) are odd, \( C_k[S_n] \) has an ERPPDC.

Proof. Let \( V_0, V_1, \ldots, V_{k-1} \) denote the \( k \) copies of \( S_n \), where \( V_i = \{x_i, 0, x_i, 1, \ldots, x_i, n-1\} \), \( 0 \leq i \leq k-1 \). For each \( i, 1 \leq i \leq k-1 \), let

\[
M_i = \{x_{i-1,0}, x_{i-1,1}, x_{i-1,2}, \ldots, x_{i-1,n-1}x_{i,n-1}\}
\]

and let

\[
M_0 = \{x_{k-1,0}x_0, 0, x_{k-1,1}x_0, 2, \ldots, x_{k-1,n-1}x_0, 0\}.
\]

Then \( M_i \) is a matching in the \( K_{n,n} \) between \( V_{i-1} \) and \( V_i \), and \( M_0 \) is a matching in the \( K_{n,n} \) between \( V_{k-1} \) and \( V_0 \).

Applying Lemma 17 to each copy of \( K_{n,n} \) gives, for each \( i \) and \( j \), \( 1 \leq i \leq k-1, 0 \leq j \leq n-1 \), a path \( P_{ij} \) of length \( 2n-1 \) from \( x_{i-1,j} \) to \( x_{i,j} \). Furthermore, for each \( j \), \( 0 \leq j \leq n-1 \), there is a path \( P_{0j} \) of length \( 2n-1 \) from \( x_{k-1,j} \) to \( x_{0,j+1} \). These paths cover every edge in the matchings \( M_i \) once, \( 0 \leq i \leq k-1 \), and cover every other edge in \( C_k[S_n] \) twice. For \( 0 \leq i \leq k-2 \), let \( P'_{ij} = P_{ij} \cup \{x_{i,j}, x_{i+1,j}\} \), and let \( P'_{k-1,j} = P_{k-1,j} \cup \{x_{k-2,j}, x_{0,j+1}\} \). Then the collection \( P' = \{P'_{ij}\} \) is an RPPDC of \( C_k[S_n] \). If \( k \) and \( n \) are odd, then \( A_P(C_k[S_n]) \) is the cycle

\[
\begin{align*}
x_0,0 & x_2,0 x_4,0 \ldots x_{k-1,0} x_1,0 x_3,0 \ldots x_{k-1,2} x_1,2 x_3,2 \ldots x_{k-1,3} x_1,3 x_3,1 \ldots x_{k-2,1} \\
x_0,2 & x_2,2 x_4,2 \ldots x_{k-1,2} x_1,3 x_3,3 \ldots x_{k-1,3} \\
\vdots & \vdots \vdots \vdots \vdots \vdots \\
x_0,n-1 & x_2,n-1 \ldots x_{k-1,n-1} \\
x_1,0 & x_3,0 \ldots x_{k-2,0} x_0,1 x_2,1 \ldots x_{k-1,1} \\
x_1,2 & x_3,2 \ldots x_{k-2,2} x_0,3 x_2,3 \ldots x_{k-1,3} \\
\vdots & \vdots \vdots \vdots \vdots \\
x_0,n-2 & x_2,n-2 \ldots x_{k-1,n-2} x_1,n-1 x_3,n-1 \ldots x_{k-2,n-1} x_0,0
\end{align*}
\]

and therefore the RPPDC is eulerian. \( \square \)

Lemma 19. Let \( n \) be even and suppose \( K_{n,n} \) has bipartition \( (X, Y) \) with \( X = \{x_0, \ldots, x_{n-1}\} \) and \( Y = \{y_0, \ldots, y_{n-1}\} \). There exists a path cover \( P = \{P_j, 1 \leq j \leq n\} \) of \( K_{n,n} \) such that

1. every edge of \( S = \{x_0y_1, x_1y_2, \ldots, x_{n-2}y_{n-1}, x_{n-1}y_0, y_{n-1}\} \) is in exactly one path of \( P \), and every edge of \( K_{n,n} - S \) is in exactly two paths of \( P \);
2. the associated graph of \( P \) has edge set \( \bar{S} \).

Proof. Suppose that \( K_{n,n} \) has bipartition \( (X, Y) \), \( X = \{a_0, a_1, \ldots, a_{n-1}\}, Y = \{b_0, b_1, \ldots, b_{n-1}\} \). Let \( x_i = \{a_0b_1, a_1b_1, \ldots, a_{n-1}b_{i-1}\}, i = 0, 1, \ldots, n-1 \). It is well known that \( x_i \) is a perfect matching and \( x_i \cup x_{i+1} \) is a Hamilton cycle, \( i = 0, 1, \ldots, n-1 \). There are two cases to consider.

Case 1. \( n \equiv 2 \) (mod 4). For \( i = 1, 2, \ldots, (n-2)/2 \), let

\[
Q_{2i-1} = x_{2i-1} \cup x_{2i} - \{a_{2i}b_{4i}\},
Q_{2i} = x_{2i} \cup x_{2i+1} - \{a_{2i+1}b_{4i+1}\}.
\]
Then $Q_{2i-1}$ is a $(2n-1)$-path from $a_{2i}$ to $b_{4i}$, and $Q_{2i}$ is a $(2n-1)$-path from $a_{2i+1}$ to $b_{4i+1}$. Note that \{a_{2i}b_{4i}, a_{2i+1}b_{4i+1} : 1 \leq i \leq (n-2)/2\} is an independent set of edges. Also, let

$$Q_{n-1} = x_{n-1} \cup x_0 - \{a_0 b_{n-1}\},$$
$$Q_0 = x_0 \cup x_1 - \{a_1 b_1\}.$$

Then $Q_{2n-1}$ is a $(2n-1)$-path from $a_0$ to $b_{n-1}$, and $Q_0$ is a $(2n-1)$-path from $a_1$ to $b_1$. By appropriately re-labelling the vertices of $X$ with $x_0, x_1, \ldots, x_{n-1}$, and the vertices of $Y$ with $y_0, y_1, \ldots, y_{n-1}$, the paths \{Q_i : 0 \leq i \leq n-1\} satisfy the lemma and the result follows.

**Case 2.** $n \equiv 0 \pmod{4}$. Without loss of generality, $n = 4k$ for some integer $k \geq 1$. For $i = 1, 2, \ldots, k - 2$, let

$$Q_{2i-1} = x_{2i-1} \cup x_{2i} - \{a_{2i} b_{4i}\},$$
$$Q_{2i} = x_{2i} \cup x_{2i+1} - \{a_{2i+1} b_{4i+1}\}.$$

For $i = 0, 1, \ldots, k - 2$, let

$$Q_{2(k+i)-1} = x_{2(k+i)-1} \cup x_{2(k+i)} - \{a_{2(k+i)-1} b_{4(k+i)-1}\},$$
$$Q_{2(k+i)} = x_{2(k+i)} \cup x_{2(k+i)+1} - \{a_{2(k+i)+1} b_{4(k+i)+2}\}.$$

Finally, let

$$Q_{2k-3} = x_{2k-3} \cup x_{2k-2} - \{a_{2k-2} b_{4k-4}\},$$
$$Q_{2k-2} = x_{2k-2} \cup x_{2k-1} - \{a_{2k-1} b_{4k-2}\},$$
$$Q_{4k-3} = x_{4k-3} \cup x_{4k-2} - \{a_{4k-3} b_{4k-5}\},$$
$$Q_{4k-2} = x_{4k-2} \cup x_{4k-1} - \{a_{4k-1} b_{4k-3}\},$$
$$Q_{4k-1} = x_{4k-1} \cup x_0 - \{a_0 b_{4k-1}\},$$
$$Q_0 = x_0 \cup x_1 - \{a_1 b_1\}.$$

The remainder of the proof is analogous to Case 1. □

**Proposition 20.** If $G$ is a graph with an EPPDC and $P_k$ is a path with $k$ vertices, then $P_k[G]$ has an EPPDC.

**Proof.** Let $V(G) = \{g_0, g_1, \ldots, g_{n-1}\}$ and let $P_k = h_0, h_1, \ldots, h_{k-1}$. Let $Q$ be an EPPDC of $G$; without loss of generality we may assume that $A_Q(G)$ is the cycle $g_0 g_1 \ldots g_{n-1}$. To reduce notation, we write $g_i^j$ for $(g_i, h_j)$. Note that for each $j$, $0 \leq j \leq k-1$, the graph $G^j$ induced by $\{g_i^j : 0 \leq i \leq n-1\}$ is isomorphic to $G$, and there is a copy of $K_{n,n}$ between $G^j$ and $G^{j+1}$ (superscripts modulo $k$, subscripts modulo $n$).

Let $Q = \{Q_i : 0 \leq i \leq n-1\}$, where $Q_i$ is the path from $g_i$ to $g_{i+1}$. Then for each $i$, $Q_i$ corresponds to a path $Q_i^j$, $0 \leq j \leq k-1$, from $g_i^j$ to $g_i^{j+1}$ in $G^j$.

**Case 1.** $k$ odd. For each $i$, $0 \leq i \leq n-1$, let

$$Q_i = Q_i^0 \cup \{g_i^{k+1} g_i^{k+1}\} \cup Q_i^1 \cup \{g_i^1 g_i^2\} \cup Q_i^2 \cup \{g_i^{2} g_i^{3}\} \cup Q_i^3 \cup \cdots \cup \{g_i^{k-2} g_i^{k-1}\} \cup Q_i^{k-1}.$$
Then $Q_i$ is a path in $P_k[G]$ from $g_i^0$ to $g_i^{k-1}$. The collection of paths \{\(Q_i : 0 \leq i \leq n - 1\)\} covers every edge in $G^j$ twice, $0 \leq j \leq k - 1$, and covers, once, every edge of a perfect matching $M$ in the copy of $K_{n,n}$ between $G^j$ and $G^{j+1}$. Let $j$ be an integer, $0 \leq j \leq k - 2$, and consider the copy of $K_{n,n}$ between $G^j$ and $G^{j+1}$. By Lemma 17, for each $i$, $0 \leq i \leq n - 1$, there is a path from $g_i^j$ to $g_i^{j+1}$, and these $n$ paths cover every edge in $M$ once, and cover every other edge between $G^j$ and $G^{j+1}$ twice.

These $kn$ paths form a PPDC of $P_k[G]$. The fact that this PPDC is eulerian follows because the associated graph is the cycle

\[
\begin{align*}
&g_0 \ g_1 \ g_2 \ g_3 \ \cdots \ g_1 \ g_2 \ g_3 \ g_4 \\
&\vdots \\
&g_{n-1} \ g_n \ g_0 \ g_1 \ g_2 \ g_3 \ \cdots \\
&g_n \ g_0 \ g_1 \ g_2 \ g_3 \ g_4.
\end{align*}
\]

**Case 2.** $k$ even and $n$ odd. The proof is analogous to Case 1, with $Q_i$ defined as

\[
Q_i = Q_i^0 \cup \{g_i^{0} g_i^{1}\} \cup Q_i^1 \cup \{g_i^{1} g_i^{2}\} \cup \ldots \cup Q_i^{k-2} \cup \{g_i^{k-2} g_i^{k-1}\} \cup Q_i^{k-1}.
\]

The resulting PPDC is eulerian because the associated graph is the cycle

\[
\begin{align*}
&g_0 \ g_1 \ g_2 \ g_3 \ \cdots \ g_1 \ g_2 \ g_3 \ g_4 \\
&\vdots \\
&g_{n-1} \ g_n \ g_0 \ g_1 \ g_2 \ g_3 \ \cdots \\
&g_n \ g_0 \ g_1 \ g_2 \ g_3 \ g_4.
\end{align*}
\]

**Case 3.** $k$ even and $n$ even. Again we proceed as in Case 1, with a slight variation. For each integer $i$, $0 \leq i \leq n - 2$, let

\[
Q_i = Q_i^0 \cup \{g_i^{0} g_i^{1}\} \cup Q_i^1 \cup \{g_i^{1} g_i^{2}\} \cup Q_i^2 \cup \{g_i^{2} g_i^{3}\} \cup Q_i^3 \\
\vdots \\
\cup \ldots \cup Q_i^{k-2} \cup \{g_i^{k-2} g_i^{k-1}\} \cup Q_i^{k-1},
\]

and let

\[
Q_{n-1} = Q_{n-1}^0 \cup \{g_n^{0} g_{n-1}^{1}\} \cup Q_{n-1}^1 \cup \{g_{n-1}^{1} g_{n-2}^{2}\} \cup Q_{n-1}^2 \cup \{g_{n-2}^{2} g_{n-3}^{3}\} \cup Q_{n-1}^3 \\
\vdots \\
\cup \ldots \cup Q_{n-1}^{k-2} \cup \{g_{n-1}^{k-2} g_{n-2}^{k-1}\} \cup Q_{n-1}^{k-1}.
\]

Then the collection of paths \{\(Q_i : 0 \leq i \leq n - 1\)\} covers every edge in $G^j$ twice, $0 \leq j \leq k - 1$. It also covers, once, every edge of a perfect matching $M$ in the copy of $K_{n,n}$ between $G^j$ and $G^{j+1}$, for $1 \leq j \leq k - 1$. Finally, it covers, once, the set of edges

\[
\{g_i^0 g_i^{1} : 1 \leq i \leq n - 1\} \cup \{g_n^0 g_{n-1}^{1}\}.
\]

Applying Lemma 17 to each $K_{n,n}$ between $G^j$ and $G^{j+1}$, and Lemma 19 to the copy of $K_{n,n}$ between $G^0$ and $G^1$, produces the remaining paths in the PPDC of $P_k[G]$. This PPDC is eulerian because the associated graph is the cycle

\[
\begin{align*}
&g_0 \ g_1 \ g_2 \ g_3 \ \cdots \ g_1 \ g_2 \ g_3 \ g_4 \\
&\vdots \\
&g_{n-1} \ g_n \ g_0 \ g_1 \ g_2 \ g_3 \ \cdots \\
&g_n \ g_0 \ g_1 \ g_2 \ g_3 \ g_4.
\end{align*}
\]
References