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Hochster's theta invariant and the Hodge–Riemann bilinear relations

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Abstract

Let R be an isolated hypersurface singularity, and let M and N be finitely generated R-modules. As R is a hypersurface, the torsion modules of M against N are eventually periodic of period two (i.e., $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_{i+2}^R(M,N)$ for $i \gg 0$). Since R has only an isolated singularity, these torsion modules are of finite length for $i \gg 0$. The theta invariant of the pair (M,N) is defined by Hochster to be $\mathrm{length}(\mathrm{Tor}_{2i}^R(M,N)) - \mathrm{length}(\mathrm{Tor}_{2i+1}^R(M,N))$ for $i \gg 0$. H. Dao has conjectured that the theta invariant is zero for all pairs (M,N) when R has even dimension and contains a field. This paper proves this conjecture under the additional assumption that R is graded with its irrelevant maximal ideal giving the isolated singularity. We also give a careful analysis of the theta pairing when the dimension of R is odd, and relate it to a classical pairing on the smooth variety $\mathrm{Proj}(R)$.

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1. Introduction

If R is a hypersurface — that is, a quotient of a regular ring T by a single element — and M and N are finitely generated R-modules, then the long exact sequence

$$\cdots \to \operatorname{Tor}_n^T(M,N) \to \operatorname{Tor}_n^R(M,N) \to \operatorname{Tor}_{n-2}^R(M,N) \to \operatorname{Tor}_{n-1}^T(M,N) \to \cdots \tag{1.1}$$

coming from [4, Chapter XV] shows that

$$\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_{i+2}^R(M,N) \quad \text{for } i \gg 0.$$

When these torsion modules are of finite length, M. Hochster [16, Theorem 1.2] defines

$$\theta^R(M,N) = \operatorname{length}\bigl(\operatorname{Tor}_{2i}^R(M,N)\bigr) - \operatorname{length}\bigl(\operatorname{Tor}_{2i+1}^R(M,N)\bigr) \quad \text{for } i \gg 0.$$

If R has at most a finite number of singularities, then $\theta^R(M,N)$ is defined for all pairs (M,N) of finitely generated R-modules. Typically we write just θ for θ^R.

Hochster introduced the θ pairing in his study of the Direct Summand Conjecture: if $A \subseteq B$ is a module finite ring extension of a regular ring A, then A is a direct summand of the A-module B. The Direct Summand Conjecture is known if A is equicharacteristic [15, Theorem 2] or has dimension at most three [14]. Hochster showed that the Direct Summand Conjecture holds provided $\theta(S/\mathfrak{p}, -)$ is the zero function for a particular prime $\mathfrak{p} \in \operatorname{Spec}(S)$, where S belongs to an explicit family of (mixed characteristic) local hypersurfaces.

Hochster [16, Theorem 1.4] showed that if R is an *admissible*[3] hypersurface and $\operatorname{length}(M \otimes_R N) < \infty$, then $\theta(M,N) = 0$ if and only if $\dim(M) + \dim(N) \leqslant \dim(R)$. H. Dao [7,8] studied the vanishing of θ for admissible local hypersurfaces R which have only isolated singularities. These papers motivated our work; in particular we address the following conjecture of Dao:

Conjecture 1.1. *(See [7, Conjecture 3.15].) Let R be an isolated hypersurface singularity. Assume that* $\dim(R)$ *is even and R contains a field. Then* $\theta(M,N) = 0$ *for all pairs of finitely generated R-modules M and N.*

In this paper, we prove this conjecture when R is a graded, finitely generated algebra over a field k that is non-singular away from its irrelevant maximal ideal; see Theorem 3.2 for our precise statement.

The θ pairing induces a symmetric bilinear form on the Grothendieck group of R. When $n = \dim(R)$ is odd and k is separably closed, we prove θ factors through the Chern character map taking values in étale cohomology; see Theorem 3.3. Moreover, when $\operatorname{char} k = 0$, we show in Theorem 3.4 that $(-1)^{\frac{n+1}{2}}\theta$ is positive semi-definite, and when $k = \mathbb{C}$, we identify its kernel using the Hodge–Riemann bilinear relations.

In Section 6 we extend our results on the θ pairing (under the same assumptions on the ground field) to graded hypersurfaces $S = k[y_1, \ldots, y_n]/(g)$ where $\deg y_i = e_i \geqslant 1$, and g is homogeneous with respect to this grading.

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[3] R is *admissible* if a completion $\hat{R}$ of R at a maximal ideal satisfies $\hat{R} \cong T/(f)$, and the *dimension inequality*, *vanishing*, and *positivity* of Serre [18, V.5.1] hold for T. Serre showed that these conditions on T hold when T is a regular local ring containing a field.

Another source of interest in the θ pairing comes from a result of Dao [7, Proposition 2.8], which provides a connection between the vanishing of θ and the *rigidity*[4] of Tor. Namely, when R is an admissible hypersurface and M, N are R-modules such that $\theta(M, N)$ is defined, then $\theta(M, N) = 0$ implies rigidity of the pair (M, N). Our results imply that, if R is a graded k-algebra as above, with $\operatorname{char} k = 0$, then an R-module M is rigid if $\theta(M, M) = 0$; see Corollary 3.15.

Finally, for readers familiar with the Herbrand difference, perhaps through [3, Section 10.3], we note that the Herbrand difference and θ are closely related. In detail, each can be interpreted as coming from a pairing on the graded rational Chow group, $CH^\bullet(R)_{\mathbb{Q}}$, of R. On the component $CH^j(R)_{\mathbb{Q}}$, they coincide for j odd and differ by a sign for j even. In particular, our results show that, over a field of characteristic zero, the Herbrand difference is a negative semi-definite form; see Example 3.14.

2. Background

Throughout the rest of this paper, we make the following assumptions:

$$\begin{array}{l} \bullet\ k \text{ is a field.} \\ \bullet\ R = k[x_0, \dots, x_n]/(f(x_0, \dots, x_n)) \text{ where } \deg x_i = 1 \text{ for all } i \text{ and } f \\ \quad \text{is a homogeneous polynomial of degree } d. \\ \bullet\ X = \operatorname{Proj}(R) \text{ is a smooth } k\text{-variety.} \\ \bullet\ \mathfrak{m} = (x_0, \dots, x_n) \text{ is the only non-regular prime of } R. \end{array} \tag{2.1}$$

For each $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, $R_{\mathfrak{p}}$ is regular. Therefore, for a finitely generated R-module M, the $R_{\mathfrak{p}}$-projective dimension of $M_{\mathfrak{p}}$ is finite. If N is also finitely generated, then for $i \gg 0$, the module $\operatorname{Tor}_i^R(M, N)$ is supported on $\{\mathfrak{m}\}$ and hence has finite length. Thus θ is defined for all pairs of finitely generated R-modules M and N.

The variety X is smooth if and only if the radical of the homogeneous ideal $(f, \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$ is $\mathfrak{m}$ [13, Theorem 5.3]. In particular, the fourth assumption in (2.1) follows from the third. Moreover, assumptions (2.1) remain valid upon passing to any field extension of k. Further, for finitely generated R-modules M and N and any field extension $k \subset k'$, we have

$$\theta^R(M, N) = \theta^{R \otimes_k k'}\big(M \otimes_k k', N \otimes_k k'\big). \tag{2.2}$$

In many of our results and constructions, we assume k is separably closed. Some of our results apply only when $k = \mathbb{C}$.

2.1. Geometry

Let $p : Y \to \operatorname{Spec}(R)$ be the blow-up of $\operatorname{Spec}(R)$ at the point $\mathfrak{m}$, so that $Y = \operatorname{Proj}(\bigoplus_{i \geqslant 0} \mathfrak{m}^i t^i)$. Note that $\mathfrak{m}^i = \bigoplus_{j \geqslant i} R_j$. The exceptional fiber of p is

$$\operatorname{Proj}\left(\bigoplus_{i \geqslant 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}\right) \cong \operatorname{Proj}(R) = X.$$

[4] A pair of modules (M, N) is *rigid* if for any integer $i \geqslant 0$, $\operatorname{Tor}_i^R(M, N) = 0$ implies $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geqslant i$. A module M is *rigid* if for all N the pair (M, N) is rigid.

Moreover, Y is the geometric line bundle over X associated to the rank one locally free coherent sheaf $\mathcal{O}_X(1)$, and the inclusion $i : X \hookrightarrow Y$ is the zero section of this line bundle [6, Lemma 2.2]. The projection $\pi : Y \twoheadrightarrow X$ comes from the inclusion of graded rings $R \hookrightarrow \bigoplus_{i \geqslant 0} \mathfrak{m}^i t^i$ given by identifying R with $\bigoplus_{i \geqslant 0} R_i t^i$.

Assume k is infinite, so that there is a k-rational point $Q \in \mathbb{P}^n \setminus X$. Then linear projection away from Q determines a regular map $\mathbb{P}^n \setminus \{Q\} \to \mathbb{P}^{n-1}$, and we write $\rho : X \to \mathbb{P}^{n-1}$ for its restriction to X. The map ρ is finite and dominant of degree d. The following diagrams summarize the situation:

$$\begin{tikzcd} X \arrow[r, hook, shift left, "i"] \arrow[d, "q"'] & Y \arrow[l, two heads, shift left, "\pi"] \arrow[d, "p"] \\ \mathrm{Spec}(k) \arrow[r, "j"'] & \mathrm{Spec}(R) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, hook] \arrow[d, "\rho"'] \arrow[dr, "q"] & \mathbb{P}^n \arrow[d] \\ \mathbb{P}^{n-1} \arrow[r, "s"'] & \mathrm{Spec}(k). \end{tikzcd} \tag{2.3}$$

The map ρ will be used in the proofs of the main results of this paper.

For a Noetherian scheme Z, let $G(Z)$ denote the Grothendieck group of coherent sheaves on Z. Thus $G(Z)$ is the abelian group generated by isomorphism classes of coherent sheaves modulo relations coming from short exact sequences. We write $K(Z)$ for the Grothendieck group of locally free coherent sheaves on Z. Recall that $K(Z)$ is a ring under tensor product. If Z is a smooth k-variety, the canonical map $K(Z) \to G(Z)$ is an isomorphism. We write $G(R)$ for $G(\mathrm{Spec}(R))$, so that $G(R)$ is the usual Grothendieck group of finitely generated R-modules.

Since θ is biadditive [16, p. 98] and θ is defined for all pairs of finitely generated R-modules, it follows that θ determines a pairing on $G(R)$ and hence on $G(R)_{\mathbb{Q}} := G(R) \otimes_{\mathbb{Z}} \mathbb{Q}$,

$$\theta : G(R)_{\mathbb{Q}} \otimes_{\mathbb{Q}} G(R)_{\mathbb{Q}} \to \mathbb{Q}.$$

Since $\pi : Y \to X$ is a line bundle and X and Y are smooth, pull-back along π determines an isomorphism π^*: $K(X) \xrightarrow{\cong} K(Y)$, with the inverse map given by i^*. The composition $K(X) \xrightarrow{i_*} K(Y) \xrightarrow{i^*} K(X)$ is multiplication by the element $\alpha := [\mathcal{O}_X] - [\mathcal{O}_X(1)]$ of $K(X)$. We may also describe this class as $\alpha = -[\mathcal{O}_H(1)]$, where $H \subset X$ is a general hyperplane section of X. Finally, the map $j_* : G(k) \to G(R)$ is torsion. Indeed, we may find a homogeneous prime $\mathfrak{p}$ of height $n-1$ and a homogeneous element $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then the short exact sequence $R/\mathfrak{p} \overset{x}{\hookrightarrow} R/\mathfrak{p} \twoheadrightarrow R/(x, \mathfrak{p})$ shows that $0 = [R/(x, \mathfrak{p})] \in G(R)$. As $(x, \mathfrak{p})$ is homogeneous and $R/(x, \mathfrak{p})$ has finite length, a prime filtration shows $[R/(x, \mathfrak{p})] = \mathrm{length}(R/(x, \mathfrak{p})) \cdot [R/\mathfrak{m}] \in G(R)$. Hence the generator $[k]$ for $G(k)$, which maps to $[R/\mathfrak{m}]$, is annihilated by $\mathrm{length}(R/(x, \mathfrak{p})) \in \mathbb{N}$.

Applying the localization sequence for G-theory to the left-hand square in (2.3) yields the diagram with exact rows

$$\begin{tikzcd} {} \arrow[r, dashed] & G_1(U) \arrow[r] \arrow[d, "="] & G_0(X) \arrow[r, "i_*"] \arrow[d] & G_0(Y) \arrow[r] \arrow[d] & G_0(U) \arrow[r] \arrow[d, "="] & 0 \\ {} \arrow[r, dashed] & G_1(U) \arrow[r] & G_0(\mathrm{Spec}(k)) \arrow[r] & G_0(\mathrm{Spec}(R)) \arrow[r] & G_0(U) \arrow[r] & 0, \end{tikzcd}$$

where $U = Y \setminus X = \operatorname{Spec}(R) \setminus \operatorname{Spec}(k)$. (In this diagram, G_0 is the group written as G everywhere else in this paper and G_1 denotes the first higher K-group of the abelian category of coherent sheaves on a scheme.) This leads to the right exact sequence

$$G(X) \to G(Y) \oplus G(k) \to G(R) \to 0$$

and since $G(k)_{\mathbb{Q}} \to G(R)_{\mathbb{Q}}$ is the zero map, we obtain the right exact sequence

$$G(X)_{\mathbb{Q}} \xrightarrow{i_*} G(Y)_{\mathbb{Q}} \xrightarrow{p_*} G(R)_{\mathbb{Q}} \to 0.$$

Since $K(X) \cong G(X)$, $i^* : K(Y) \to K(X)$ is an isomorphism (whose inverse is π^*) and $i^* \circ i_*$ is multiplication by $\alpha \in K(X)$, we obtain the isomorphism

$$p_* \pi^* : K(X)_{\mathbb{Q}} / \langle \alpha \rangle \xrightarrow{\cong} G(R)_{\mathbb{Q}}, \tag{2.4}$$

which allows us to regard θ as a pairing on $K(X)_{\mathbb{Q}}/\langle \alpha \rangle$. One may verify that the isomorphism (2.4) is given by "forgetting the grading"; i.e., for a finitely generated graded R-module M with associated coherent sheaf $\widetilde{M}$ on X, it sends $[\widetilde{M}] \in K(X)_{\mathbb{Q}}$ to $[M] \in G(R)_{\mathbb{Q}}$. In particular, the vector space $G(R)_{\mathbb{Q}}$ is spanned by classes of *graded* R-modules.

2.2. Cohomology

Our main technique will involve factoring the θ pairing through cohomology (either étale or singular, depending on k) via the Chern character. In this section we review the concepts concerning these topics that we will need. For the assertions concerning singular cohomology made below, we refer the reader to [11,12]. For those concerning étale cohomology, the ultimate reference is SGA [19–21,5,9], but a good survey of this material can be found in [10]. The features of étale and singular cohomology we need are those common to any "Weil cohomology theory"; see [17, §3] for a precise description of what this means.

If k is a separably closed field, then for any prime $\ell \neq \operatorname{char} k$, we can consider the étale cohomology of X with coefficients in the ℓ-adic rationals. Using μ_r to denote the étale sheaf of r-th roots of unity, the maps $\mu_{\ell^{m+1}}^{\otimes i} \twoheadrightarrow \mu_{\ell^m}^{\otimes i}$ given by taking ℓ-th powers form an inverse system of étale sheaves. When $i = 0$, take $\mu_{\ell^m}^{\otimes 0}$ to be $\mathbb{Z}/\ell^m$ and the map $\mathbb{Z}/\ell^{m+1} \twoheadrightarrow \mathbb{Z}/\ell^m$ to be the canonical one. By definition,

$$H^{2i}_{\text{ét}}\bigl(X, \mathbb{Q}_\ell(i)\bigr) = H^{2i}_{\text{ét}}\bigl(X, \mathbb{Z}_\ell(i)\bigr) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \quad \text{and} \quad H^{2i}_{\text{ét}}\bigl(X, \mathbb{Z}_\ell(i)\bigr) = \varprojlim_m H^{2i}_{\text{ét}}\bigl(X, \mu_{\ell^m}^{\otimes i}\bigr).$$

We write $H^{ev}_{\text{ét}}(X, \mathbb{Q}_\ell)$ for $\bigoplus_{i \geqslant 0} H^{2i}_{\text{ét}}(X, \mathbb{Q}_\ell(i))$, which is a commutative algebra under cup product $\cup$.

When $k = \mathbb{C}$, let $X(\mathbb{C})$ be the complex manifold associated to X. The singular cohomology $H^\bullet(X(\mathbb{C}), \mathbb{Q})$ of the manifold $X(\mathbb{C})$ is a graded-commutative $\mathbb{Q}$-algebra under cup product. We write $H^{ev}(X(\mathbb{C}), \mathbb{Q})$ for $\bigoplus_{i \geqslant 0} H^{2i}(X(\mathbb{C}), \mathbb{Q})$, the even degree subalgebra of $H^\bullet(X(\mathbb{C}), \mathbb{Q})$.

The Chow group of cycles modulo rational equivalence on X is $CH^\bullet(X)$. Since X is smooth, $CH^\bullet(X)$ is a ring under intersection of cycles. The étale and topological cycle class maps are ring homomorphisms

$$cy_{\text{ét}} : CH^\bullet(X)_{\mathbb{Q}_\ell} \to H^{ev}_{\text{ét}}(X, \mathbb{Q}_\ell) \quad \text{and} \quad cy_{\text{top}} : CH^\bullet(X)_{\mathbb{Q}} \to H^{ev}\bigl(X(\mathbb{C}), \mathbb{Q}\bigr)$$

which commute with both push-forward and pull-back maps for morphisms of smooth, projective varieties. (The map $cy_{\text{ét}}$ is defined when k is separably closed, and cy_{top} is defined when $k = \mathbb{C}$.)

Let $ch : K(X)_{\mathbb{Q}} \to CH^{\bullet}(X)_{\mathbb{Q}}$ be the Chern character taking values in the Chow ring [11, p. 282]. The étale and topological Chern characters

$$ch_{\text{ét}} : K(X)_{\mathbb{Q}} \to H^{ev}_{\text{ét}}(X, \mathbb{Q}_\ell) \quad \text{and} \quad ch_{\text{top}} : K(X)_{\mathbb{Q}} \to H^{ev}\big(X(\mathbb{C}), \mathbb{Q}\big)$$

are defined so that

$$\begin{tikzcd} K(X)_{\mathbb{Q}} \arrow[d, "ch"'] \arrow[dr, "ch_{\text{ét}}"] & \\ CH^{\bullet}(X)_{\mathbb{Q}} \arrow[r, "cy_{\text{ét}}"'] & H^{ev}_{\text{ét}}(X, \mathbb{Q}_\ell) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K(X)_{\mathbb{Q}} \arrow[d, "ch"'] \arrow[dr, "ch_{\text{top}}"] & \\ CH^{\bullet}(X)_{\mathbb{Q}} \arrow[r, "cy_{\text{top}}"'] & H^{ev}(X(\mathbb{C}), \mathbb{Q}) \end{tikzcd} \tag{2.5}$$

commute. These characters are ring homomorphisms from $K(X)_{\mathbb{Q}}$ taking values in graded rings. Let $\beta \in CH^1(X)_{\mathbb{Q}}$ denote the class of a generic hyperplane section of X. Let $\gamma = cy_{\text{ét}}(\beta) \in H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$; then γ is the étale cohomology class of the divisor given by a generic hyperplane section of X. Since $ch(\mathcal{O}_X) = 1$ and $ch(\mathcal{O}_X(1)) = e^{\beta} = 1 + \beta + \frac{\beta}{2!} + \cdots$, we have

$$ch(\alpha) = \beta \cdot u \quad \text{where } u = -1 - \frac{\beta}{2!} - \frac{\beta^2}{3!} - \cdots \in CH^{\bullet}(X).$$

Since u is a unit in the Chow ring of X, the ideals of $CH^{\bullet}(X)$ generated by $ch(\alpha)$ and β coincide. Likewise $ch_{\text{top}}(\alpha)$ and γ agree up to a unit factor in the cohomology ring of X.

Since X is a smooth hypersurface in $\mathbb{P}^n$, Poincaré duality and the weak Lefschetz theorem show that the even degree étale cohomology groups of X are spanned by powers of γ, except possibly in degree $n-1$ when n is odd. That is, the following equations hold:

$$H^{2i}_{\text{ét}}\big(X, \mathbb{Q}_\ell(i)\big) = \mathbb{Q}_\ell \cdot \gamma^i, \quad \text{for all } i \text{ except when } n \text{ is odd and } 2i = n-1. \tag{2.6}$$

When $k = \mathbb{C}$, we also write γ for the element $cy_{\text{top}}(\beta) \in H^2(X(\mathbb{C}), \mathbb{Q})$. Equations analogous to (2.6) hold for singular cohomology when $k = \mathbb{C}$. Since $ch(\alpha)$ coincides with β up to a unit factor, there are induced maps on quotient rings from $K(X)_{\mathbb{Q}}/\langle\alpha\rangle$ to each of $CH^{\bullet}(X)_{\mathbb{Q}}/\langle\beta\rangle$, $H^{ev}_{\text{ét}}(X, \mathbb{Q}_\ell)/\langle\gamma\rangle$, and $H^{ev}(X(\mathbb{C}), \mathbb{Q})/\langle\gamma\rangle$. We write these induced maps also as $ch, ch_{\text{ét}}$, and ch_{top}, respectively.

Recall that for a (possibly singular) variety Y, the Grothendieck–Riemann–Roch isomorphism

$$\tau : G(Y)_{\mathbb{Q}} \xrightarrow{\cong} CH^{\bullet}(Y)_{\mathbb{Q}},$$

is functorial for push-forwards along proper maps [11, Corollary 18.3.2]. If Y happens to be smooth, so that $K(Y) \cong G(Y)$, we write τ also for the composition of isomorphisms

$$K(Y)_{\mathbb{Q}} \xrightarrow{\cong} G(Y)_{\mathbb{Q}} \xrightarrow{\tau} CH^{\bullet}(Y)_{\mathbb{Q}}.$$

It is useful for our purposes to compare ch with τ for the variety X.

Lemma 2.1. *For X as in* (2.1)*, the isomorphisms ch and τ from $K(X)_{\mathbb{Q}}$ to $CH(X)_{\mathbb{Q}}$ each map $\langle\alpha\rangle$ isomorphically onto $\langle\beta\rangle$. Moreover, they induce the same isomorphism $K(X)_{\mathbb{Q}}/\langle\alpha\rangle \xrightarrow{\cong} CH^{\bullet}(X)_{\mathbb{Q}}/\langle\beta\rangle$.*

Proof. Since X is smooth, $\tau(a) = ch(a) \cup \mathrm{Td}(\mathcal{T}_X)$ [11, p. 287] where $\mathrm{Td}(\mathcal{T}_X) \in CH^{\bullet}(X)$ is the Todd class of the tangent bundle $\mathcal{T}_X$ of X [11, Example 3.2.13]. Since X is a smooth hypersurface of degree d in $\mathbb{P}^{n-1}$, we have $[\mathcal{T}_X] = (n+1)[\mathcal{O}_X(1)] - [\mathcal{O}_X] - [\mathcal{O}_X(d)] \in K(X)$. (See [11, Examples 3.2.11, 3.2.12 & Appendix B.7.1].) Hence $\mathrm{Td}(\mathcal{T}_X) = \mathrm{Td}(\mathcal{O}_X(1))^{n+1}/\mathrm{Td}(\mathcal{O}_X(d))$. Since $\mathrm{Td}(\mathcal{O}_X(1))$ and $\mathrm{Td}(\mathcal{O}_X(d))$ are in $1 + \beta CH^{\bullet}(X)_{\mathbb{Q}}$, so is $\mathrm{Td}(\mathcal{T}_X)$. □

For a smooth projective variety Z over a separably closed field k with structure map $q: Z \to \mathrm{Spec}(k)$, we write

$$\int_Z : H^{ev}_{\text{ét}}(Z, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$$

for push-forward along q; it takes values in $H^{ev}_{\text{ét}}(\mathrm{Spec}(k), \mathbb{Q}_\ell) = H^0_{\text{ét}}(\mathrm{Spec}(k), \mathbb{Q}_\ell) = \mathbb{Q}_\ell$. For the analogous map on singular cohomology, we also write $\int_Z$.

Example 2.2. As γ is the class of a hyperplane on the $(n-1)$-dimensional variety X, the degree of X is equal to $\int_X \gamma^{n-1} = d$.

3. Statement of main results

We continue with the notation from Section 2. In this section, we state the main results of the paper, postponing several of the proofs until Section 5.

In light of the isomorphism (2.4), we write simply $\theta(x, y)$ for $\theta(p_*\pi^*x, p_*\pi^*y)$ when $x, y \in K(X)_{\mathbb{Q}}/\langle\alpha\rangle$. The following proposition shows that the pairing $\theta(x, y)$ on $G(R)_{\mathbb{Q}}$ factors through cohomology.

Proposition 3.1. *Let R and X be as in* (2.1) *with k a separably closed field. For any x and y in $K(X)_{\mathbb{Q}}/\langle\alpha\rangle$, we have*

$$\theta(x, y) = \int_{\mathbb{P}^{n-1}} \big(\rho_*(ch_{\text{ét}}x) \cup \rho_*(ch_{\text{ét}}y) - d \cdot \rho_*\big(ch_{\text{ét}}(x \cdot y)\big)\big).$$

If $k = \mathbb{C}$, the analogous formula involving ch_{top} also holds.

This proposition is useful in proving the following theorem, which establishes Dao's Conjecture 1.1 for those isolated hypersurface singularities satisfying (2.1).

Theorem 3.2. *Let R and X be as in* (2.1) *with k an arbitrary field. If n is even, then θ vanishes; i.e., for every pair of finitely generated modules M and N,*

$$\theta(M, N) = \mathrm{length}\big(\mathrm{Tor}^R_{2i}(M, N)\big) - \mathrm{length}\big(\mathrm{Tor}^R_{2i+1}(M, N)\big) = 0 \quad \textit{for all } i \gg 0.$$

We now give a precise description of θ when n is odd and k is a separably closed field. In this case, we define a symmetric pairing $\theta_{\text{ét}}$ on the $\mathbb{Q}_\ell$-vector space $H^{n-1}_{\text{ét}}(X, \mathbb{Q}_\ell(\frac{n-1}{2}))$ by setting

$$\theta_{\text{ét}}(a,b) = \left(\int_X a \cup \gamma^{\frac{n-1}{2}}\right)\left(\int_X b \cup \gamma^{\frac{n-1}{2}}\right) - d\left(\int_X a \cup b\right). \tag{3.1}$$

When $n = 1$, by γ^0 we mean $1 \in H^0_{\text{ét}}(X, \mathbb{Q}_\ell)$. When $k = \mathbb{C}$, the same expression on the right-hand side of (3.1) defines a symmetric bilinear pairing θ_{top} on the singular cohomology group $H^{n-1}(X(\mathbb{C}), \mathbb{Q})$. By Example 2.2, if either a or b is a $\mathbb{Q}_\ell$-multiple of $\gamma^{\frac{n-1}{2}}$, then $\theta_{\text{ét}}(a,b) = 0$; similarly, when $k = \mathbb{C}$, if a or b is a $\mathbb{Q}$-multiple of $\gamma^{\frac{n-1}{2}}$, then $\theta_{\text{top}}(a,b) = 0$. Thus $\theta_{\text{ét}}$ and θ_{top} induce pairings on

$$\frac{H^{n-1}_{\text{ét}}(X, \mathbb{Q}_\ell(\frac{n-1}{2}))}{\mathbb{Q}_\ell \cdot \gamma^{\frac{n-1}{2}}} \quad \text{and} \quad \frac{H^{n-1}(X(\mathbb{C}), \mathbb{Q})}{\mathbb{Q} \cdot \gamma^{\frac{n-1}{2}}}, \quad \text{respectively;}$$

we retain the $\theta_{\text{ét}}$ and θ_{top} notation for these pairings.

Theorem 3.3. *Let R and X be as in* (2.1) *with k a separably closed field. If n is odd, then there is a commutative diagram:*

$$\begin{tikzcd}
G(R)_{\mathbb{Q}}^{\otimes 2} \arrow[d, "\theta"'] & \left(\frac{K(X)_{\mathbb{Q}}}{\langle\alpha\rangle}\right)^{\otimes 2} \arrow[l, "(p_*\pi^*)^{\otimes 2}"', "\cong"] \arrow[d, "(ch_{\text{ét}}^{\frac{n-1}{2}})^{\otimes 2}"] \\
\mathbb{Q}_\ell & \left(\frac{H^{n-1}_{\text{ét}}(X, \mathbb{Q}_\ell(\frac{n-1}{2}))}{\mathbb{Q}_\ell \cdot \gamma^{\frac{n-1}{2}}}\right)^{\otimes 2} \arrow[l, "\theta_{\text{ét}}"]
\end{tikzcd}.$$

When $k = \mathbb{C}$, the analogous diagram involving $H^{n-1}(X(\mathbb{C}), \mathbb{Q})$, $ch_{\text{top}}^{\frac{n-1}{2}}$, and θ_{top} also commutes; i.e., $\theta \circ (p_*\pi^*)^{\otimes 2} = \theta_{\text{top}} \circ (ch_{\text{top}}^{\frac{n-1}{2}})^{\otimes 2}$.

When $k = \mathbb{C}$, we use the Hodge–Riemann bilinear relations [22, p. 165] to analyze the pairing θ_{top} in more detail, obtaining:

Theorem 3.4. *For R and X as in* (2.1) *with $k = \mathbb{C}$, if n is odd, then the restriction of the pairing $(-1)^{\frac{n+1}{2}}\theta_{\text{top}}$ to*

$$\operatorname{im}\left(ch_{\text{top}}^{\frac{n-1}{2}} : \frac{K(X)_{\mathbb{Q}}}{\langle\alpha\rangle} \longrightarrow \frac{H^{n-1}(X(\mathbb{C}), \mathbb{Q})}{\mathbb{Q} \cdot \gamma^{\frac{n-1}{2}}}\right)$$

is positive definite; i.e., for v in this image, $(-1)^{\frac{n+1}{2}}\theta_{\text{top}}(v,v) \geqslant 0$ with equality holding if and only if $v = 0$.

In particular, for R as in (2.1) *with k an arbitrary field of characteristic zero, the pairing $(-1)^{\frac{n+1}{2}}\theta$ on $G(R)_{\mathbb{Q}}$ is positive semi-definite.*

Remark 3.5. If the "Hodge standard conjecture" for ℓ-adic étale cohomology were known (see [17, §5]), then the evident analogue of this theorem involving an algebraically closed field of characteristic p could be proven. The proof would be nearly identical to the one given in Section 5 below.

With Theorem 3.4 serving as evidence, we propose the following conjecture about the θ pairing in general.

Conjecture 3.6. *Let S be an admissible isolated hypersurface singularity of dimension n. If n is odd, then $(-1)^{\frac{n+1}{2}}\theta$ is positive semi-definite on $G(S)_{\mathbb{Q}}$.*

The results above can be applied to give a relation between the theta pairing and Chow groups.

Corollary 3.7. *Let R and X be as in* (2.1) *with k an arbitrary field, and assume n is odd. Then the theta pairing is induced from a pairing, which we call $\theta_{CH(X)}$, on $(\frac{CH^\bullet(X)_{\mathbb{Q}}}{\langle\beta\rangle})^{\frac{n-1}{2}}$, the $\frac{n-1}{2}$ part of the graded ring $CH^\bullet(X)_{\mathbb{Q}}/\langle\beta\rangle$. That is, there is a commutative diagram:*

$$\begin{tikzcd} G(R)_{\mathbb{Q}}^{\otimes 2} \arrow[d, "\theta"'] & \left(\frac{K(X)_{\mathbb{Q}}}{\langle\alpha\rangle}\right)^{\otimes 2} \arrow[l, "(p_*\pi^*)^{\otimes 2}"', "\cong"] \arrow[d, "(ch^{\frac{n-1}{2}})^{\otimes 2}"] \\ \mathbb{Q} & \left(\left(\frac{CH^\bullet(X)_{\mathbb{Q}}}{\langle\beta\rangle}\right)^{\frac{n-1}{2}}\right)^{\otimes 2}. \arrow[l, "\theta_{CH(X)}"] \end{tikzcd} \tag{3.2}$$

Proof. We first argue under the assumption that k is separably closed. In this case, define $\theta_{CH(X)}$ to be $\theta_{\text{ét}} \circ (cy_{\text{ét}}^{\frac{n-1}{2}})^{\otimes 2}$. By Theorem 3.3 and (2.5), the diagram similar to (3.2), but with the $\mathbb{Q}$ in the lower-left corner replaced by $\mathbb{Q}_\ell$, commutes. As the image of $\theta \circ (p_*\pi^*)^{\otimes 2}$ is contained in $\mathbb{Q} \subseteq \mathbb{Q}_\ell$, so too is the image of $\theta_{CH(X)}$.

For an arbitrary field k, let k_{sep} be a separable closure of k, let $X_{\text{sep}} = X \otimes_k k_{\text{sep}}$, and let $\theta_{CH(X)}$ be

$$\left(\left(\frac{CH^\bullet(X)_{\mathbb{Q}}}{\langle\beta\rangle}\right)^{\frac{n-1}{2}}\right)^{\otimes 2} \longrightarrow \left(\left(\frac{CH^\bullet(X_{\text{sep}})_{\mathbb{Q}}}{\langle\beta\rangle}\right)^{\frac{n-1}{2}}\right)^{\otimes 2} \xrightarrow{\theta_{CH(X_{\text{sep}})}} \mathbb{Q}.$$

The commutativity of (3.2) follows from the fact that $p_*\pi^*$ and ch are natural with respect to pull-back along $X_{\text{sep}} \to X$. $\square$

Since $p_*\pi^*$ induces an isomorphism from $(CH^\bullet(X)_{\mathbb{Q}}/\langle\beta\rangle)^{\frac{n-1}{2}}$ to $CH^{\frac{n-1}{2}}(R)_{\mathbb{Q}}$, the previous corollary shows that the theta pairing on $G(R)_{\mathbb{Q}}$ factors through a pairing on $CH^{\frac{n-1}{2}}(R)_{\mathbb{Q}}$.

Corollary 3.8. *Let R be as in* (2.1) *with k an arbitrary field. If n is odd, then there exists a pairing $\theta_{CH(R)}$ on $CH^{\frac{n-1}{2}}(R)_{\mathbb{Q}}$, which corresponds to the pairing $\theta_{CH(X)}$ under the isomorphism $p_*\pi^*$, such that the triangle*

$$\begin{tikzcd} G(R)_{\mathbb{Q}}^{\otimes 2} \arrow[rr, two heads, "(\tau^{\frac{n-1}{2}})^{\otimes 2}"] \arrow[dr, "\theta"'] & & CH^{\frac{n-1}{2}}(R)_{\mathbb{Q}}^{\otimes 2} \arrow[dl, "\theta_{CH(R)}"] \\ & \mathbb{Q} & \end{tikzcd}$$

commutes. Here, $\tau^{\frac{n-1}{2}}$ is the degree $\frac{n-1}{2}$ component of the Grothendieck–Riemann–Roch isomorphism $\tau : G(R)_{\mathbb{Q}} \xrightarrow{\cong} CH^{\bullet}(R)_{\mathbb{Q}}$.

In particular, if $CH^{\frac{n-1}{2}}(R)$ is torsion, then $\theta = 0$.

Proof. We claim that the diagram

$$\begin{tikzcd} K(X)_{\mathbb{Q}}/\langle\alpha\rangle \arrow[r, "p_*\pi^*", "\cong"'] \arrow[d, "ch", "\cong"'] & G(R)_{\mathbb{Q}} \arrow[d, "\tau", "\cong"'] \\ CH^{\bullet}(X)_{\mathbb{Q}}/\langle\beta\rangle \arrow[r, "\cong", "p_*\pi^*"'] & CH^{\bullet}(R)_{\mathbb{Q}} \end{tikzcd} \tag{3.3}$$

commutes. Granting this, the result follows from Corollary 3.7, since the bottom arrow in this diagram is graded. In more detail, it follows from the equalities $\theta = \theta_{CH(X)} \circ ch^{\frac{n-1}{2}} \circ (p_*\pi^*)^{-1} = \theta_{CH(X)} \circ (p_*\pi^*)^{-1} \circ \tau^{\frac{n-1}{2}} = \theta_{CH(R)} \circ \tau^{\frac{n-1}{2}}$ (suppressing the $^{\otimes 2}$ notation).

To show that (3.3) commutes, we consider the diagram below, where the left-hand square commutes by [11, Theorem 18.2(3)] and the right-hand one commutes by [11, Theorem 18.2(1)].

$$\begin{tikzcd} K(X)_{\mathbb{Q}} \arrow[r, "\pi^*", "\cong"'] \arrow[d, "\tau", "\cong"'] & K(Y)_{\mathbb{Q}} \arrow[r, "p_*"] \arrow[d, "\tau", "\cong"'] & G(R)_{\mathbb{Q}} \arrow[d, "\tau", "\cong"'] \\ CH^{\bullet}(X)_{\mathbb{Q}} \arrow[r, "\mathrm{Td}(\mathcal{T}_\pi)\pi^*", "\cong"'] & CH^{\bullet}(Y)_{\mathbb{Q}} \arrow[r, "p_*"] & CH^{\bullet}(R)_{\mathbb{Q}} \end{tikzcd} \tag{3.4}$$

Here, $\mathcal{T}_\pi$ is the relative tangent bundle of π and $\mathrm{Td}(\mathcal{T}_\pi)$ is its Todd class [11, Example 3.2.4], which is a unit in the ring $CH^{\bullet}(Y)_{\mathbb{Q}}$. Since π is the geometric line bundle associated to the invertible sheaf $\mathcal{O}_X(1)$, it follows that $\mathcal{T}_\pi \cong \pi^*\mathcal{O}_X(-1)$ and hence that $\mathrm{Td}(\mathcal{T}_\pi) = \pi^* \mathrm{Td}(\mathcal{O}_X(-1))$. In particular, the map $\mathrm{Td}(\mathcal{T}_\pi)\pi^*$ in (3.4) is

$$\pi^*\left(\frac{-\beta}{1-\exp(\beta)}\right)\pi^* = \left(1 - \frac{1}{2}\left(\pi^*\beta\right) + \frac{1}{12}\left(\pi^*\beta\right)^2 - \cdots\right)\pi^*,$$

and so upon modding out by the ideals generated by β and $\mathrm{Td}(\mathcal{T}_\pi)\pi^*(\beta)$, the map $\mathrm{Td}(\mathcal{T}_\pi)\pi^*$ coincides with π^*. By Lemma 2.1, the left-most vertical map τ in (3.4) sends $\langle\alpha\rangle$ isomorphically onto $\langle\beta\rangle$ and the maps τ and ch coincide as maps on the quotients. The commutativity of (3.3) follows. $\square$

Remark 3.9. To provide clarity for the relations among the above results, we summarize them in the diagram below. For the sake of simplicity, the notation ${}^{\otimes 2}$ is suppressed. Note that all arrows to $\mathbb{Q}$ represent interpretations of θ. Recall that the arrows $ch_{\text{ét}}$ and ch_{top} require assumptions on k, namely that it is separably closed and equal to $\mathbb{C}$, respectively.

$$\begin{tikzcd}
\frac{K(X)_{\mathbb{Q}}}{\langle\alpha\rangle} \arrow[r, "\cong"] \arrow[drr, "\theta"] \arrow[dd, "ch^{\frac{n-1}{2}}"'] & G(R)_{\mathbb{Q}} \arrow[r, "\tau^{\frac{n-1}{2}}"] \arrow[dr, "\theta"] & CH^{\frac{n-1}{2}}(R)_{\mathbb{Q}} \arrow[d, "\theta_{CH(R)}"] \\
& \frac{H_{\text{ét}}^{\frac{n-1}{2}}(X,\mathbb{Q}_\ell)}{\mathbb{Q}_\ell\gamma^{\frac{n-1}{2}}} \arrow[r, "\theta_{\text{ét}}"] & \mathbb{Q} \\
\left(\frac{CH^\bullet(X)_{\mathbb{Q}}}{\langle\beta\rangle}\right)^{\frac{n-1}{2}} \arrow[ur, "cy_{\text{ét}}^{\frac{n-1}{2}}"] \arrow[urr, "\theta_{CH(X)}"] \arrow[r, "cy_{\text{top}}^{\frac{n-1}{2}}"'] & \frac{H^{\frac{n-1}{2}}(X(\mathbb{C}),\mathbb{Q})}{\mathbb{Q}\gamma^{\frac{n-1}{2}}} \arrow[ur, "\theta_{\text{top}}"'] &
\end{tikzcd} \tag{3.5}$$

We end this section with some further applications and examples. Recall that homological equivalence of cycles is defined so that the image of the cycle class map cy_{top} is isomorphic to the group of cycles modulo homological equivalence: $\operatorname{im} cy_{\text{top}} \cong CH^\bullet(X)/(\text{hom} \sim 0)$; see [11, Definition 19.1].

Corollary 3.10. *Let R and X be as in* (2.1) *with $k = \mathbb{C}$. The rational vector space $CH^{\frac{n-1}{2}}(X)_{\mathbb{Q}}/$* (hom ~ 0) *is spanned by the $\frac{n-1}{2}$-st multiple of the class of a hyperplane section if and only if $\theta = 0$.*

For $n = 3$, the divisor class group $CH^1(R)$ is torsion if and only if $\theta = 0$.

Proof. The first assertion follows from Theorem 3.4 and Corollary 3.7. Since X is a smooth hypersurface in projective space, homological and rational equivalence coincide on codimension one cycles: $CH^1(X)_{\mathbb{Q}}$ is isomorphic to $CH^1(X)_{\mathbb{Q}}/(\text{hom} \sim 0)$. The second assertion therefore follows from the isomorphism $CH^1(X)_{\mathbb{Q}}/\mathbb{Q} \cdot \beta \cong CH^1(R)_{\mathbb{Q}}$. $\square$

Example 3.11. If $n = 1$, then since X is smooth, it consists of d distinct points $P_1, \ldots, P_d$ in $\mathbb{P}^1$. We have $K(X)_{\mathbb{Q}} = CH^\bullet(X)_{\mathbb{Q}} = CH^0(X)_{\mathbb{Q}} = \mathbb{Q}^d$ and the map $ch_{\text{ét}} : CH^\bullet(X)_{\mathbb{Q}_\ell} \to H_{\text{ét}}^{ev}(X, \mathbb{Q}_\ell) = H_{\text{ét}}^0(X, \mathbb{Q}_\ell)$ is an isomorphism. A basis of $CH^0(X)_{\mathbb{Q}}$ is given by the classes of the P_i's. Theorem 3.3, or direct calculation, gives

$$\theta(P_i, P_j) = 1 - d \cdot \delta_{ij}.$$

Since $\beta^0 = 1 = P_1 + \cdots + P_d$, a basis for $CH^\bullet(X)_{\mathbb{Q}}/\langle\beta^0\rangle$ is given by $(P_1 - P_i)/\sqrt{2d-2}$ for $i > 1$. With this basis, θ is represented by the matrix $-I_{d-1}$, and so is negative definite.

Example 3.12. Let k be a separably closed field and $n=3$, so that X is a smooth surface of degree d in $\mathbb{P}^3$. In this case, it follows from Corollary 3.7 that the θ pairing on $K(X)_{\mathbb{Q}}$ is induced from a pairing $\theta_{CH(X)}$ on $\operatorname{Pic}(X)_{\mathbb{Q}} = CH^1(X)_{\mathbb{Q}}$ via the map

$$K(X)_{\mathbb{Q}} \xrightarrow{ch^1=c_1} CH^1(X)_{\mathbb{Q}},$$

where c_1 is the first Chern class. Observe that for a curve C on X, we have $\int_X cy_{\text{ét}}(C) \cup \gamma^{\frac{n-1}{2}} = \deg C$. Thus, from Theorem 3.3 and Corollary 3.7, we get

$$d \cdot (C \cap D) = \deg(C)\deg(D) - \theta_{CH(X)}(C, D), \tag{3.6}$$

where C and D are curves on X and $C \cap D$ denotes the number of points of their intersection, counted with multiplicity.

It is perhaps useful to think of (3.6) as a generalization of Bézout's Theorem. If $d=1$, then $X = \mathbb{P}^2_{\mathbb{C}}$ and $\theta \equiv 0$ (since R is regular), so that the resulting equation is the classical Bézout's Theorem. For $d>1$, θ gives the "error" term of this generalized version of Bézout's Theorem.

We also remark that the positive definiteness of

$$\theta_{CH(X)}(C, D) = \deg(C)\deg(D) - d \cdot (C \cap D)$$

on $CH^1(X)_{\mathbb{Q}}$ is a consequence of the Hodge Index Theorem; see [22, p. 165].

Example 3.13. In these examples, lower case letters refer to the images of the upper case variables in the quotients.

(1) ($n=3$): Let R be the ring $\mathbb{C}[X,Y,U,V]/(XU+YV)$, so that $X = \mathbb{P}^1 \times \mathbb{P}^1$, embedded in $\mathbb{P}^3$ via the Segre embedding. Since $CH^1(R) \cong \mathbb{Z}$, θ does not vanish, and moreover it is positive definite since $\frac{n+1}{2}$ is even. Set $M = R/(x,y)$. Matrices for the minimal resolution of M eventually alternate between $\left[\begin{smallmatrix} x & y \\ v & -u \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u & y \\ v & -x \end{smallmatrix}\right]$. It is now easy to calculate that $\theta(M,M)=1$.

(2) ($n=5$): Let R be $\mathbb{C}[X,Y,Z,U,V,W]/(XU+YV+ZW)$. Since $\frac{n+1}{2}$ is odd, θ is negative definite. Set $M = R/(x,y,z)$. Then $\theta(M,M) = -1$. Here the matrices for the minimal resolution of M eventually alternate between

$$\begin{bmatrix} x & y & z & 0 \\ v & -u & 0 & z \\ -w & 0 & u & y \\ 0 & -w & v & -x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u & y & -z & 0 \\ v & -x & 0 & -z \\ w & 0 & x & y \\ 0 & w & v & -u \end{bmatrix}.$$

Example 3.14. In unpublished notes [3, §10.4], Buchweitz studies the Herbrand difference pairing $h(-,-)$ on $G(R)$ for R and X as in (2.1) with $n=3$ and $f(x_0,x_1,x_2,x_3)$ of degree three. As with the pairing θ, h can be interpreted as a pairing on $K(X)$ induced from a pairing on $\operatorname{Pic}(X) = CH^1(X)$. In this example, $\operatorname{Pic}(X)$ is free of rank six, with basis given by the classes of six lines on X. Let M and M' be cyclic modules defining two lines L and L' on X. With ω as the canonical divisor of X, by direct calculation, Buchweitz obtains

$$h\big(M, M'\big) = -\frac{1}{3}(3L + \omega) \cap \big(3L' + \omega\big),$$

and remarks that "there should exist a more conceptual proof for this". We show how our Theorem 3.3 leads to Buchweitz's formula.

Since X is a degree three hypersurface in $\mathbb{P}^3$, we have $\omega = -\beta$ and hence $L \cap \omega = L' \cap \omega = -1$ and $\omega \cap \omega = 3$. It follows that Buchweitz's formula for h is equivalent to

$$h\big(M, M'\big) = 1 - 3\big(L \cap L'\big).$$

As mentioned in the introduction, θ and h coincide on $CH^1(X)$, and hence they coincide as pairings on $G(R)$ in this example. Buchweitz's formula for h thus follows from the formula for θ given in Example 3.12 (with $C = L$ and $D = L'$, so that $\deg(C) = \deg(D) = 1$ and $d = 3$).

The following corollary shows that, at least when $\operatorname{char}(k) = 0$, to check rigidity of a module one needs only to check θ of the module against itself.

Corollary 3.15. *Let R be as in* (2.1) *with k of characteristic* 0 *and let n be odd. If M is a finitely-generated R-module with $\theta(M, M) = 0$, then M is rigid.*

Proof. If ϕ is a positive semi-definite form on a $\mathbb{Q}$-vector space V and $v \in V$, then $\phi(v, v) = 0$ implies $\phi(v, -) \equiv 0$. Since $(-1)^{\frac{n+1}{2}}\theta$ is positive semi-definite by Theorem 3.4, if $\theta(M, M) = 0$, then $\theta(M, N) = 0$ for all finitely generated R-modules N. The result now follows from [7, Proposition 2.8]. $\square$

Remark 3.16. Theorem 3.2 and [7, Proposition 2.8] imply that when R is as in (2.1), with k arbitrary and n even, then every finitely generated R-module is rigid.

4. Hilbert series, Hilbert polynomials and related invariants

This section establishes some technical results that will be used in Section 5 to prove the main results of this paper. Let R be as in (2.1). For the first part of this section, we assume only that the field k is infinite. Recall that the map $\rho : X \to \mathbb{P}^{n-1}$ is defined as projection away from a k-rational point of $\mathbb{P}^n \setminus X$.

Throughout this section we identify $K(\mathbb{P}^{n-1})_{\mathbb{Q}}$ with $\mathbb{Q}[t]/(1-t)^n$ under the ring isomorphism sending t to $[\mathcal{O}_{\mathbb{P}^{n-1}}(-1)]$ (see [13, Exercise III.5.4]). For example, given $x \in K(X)_{\mathbb{Q}}$, we interpret $\rho_*(x)$ as being a truncated polynomial in t, i.e., an element of $\mathbb{Q}[t]/(1-t)^n$, and we use the fact that the ring homomorphism ρ^* satisfies $\rho^*(t) = [\mathcal{O}_X(-1)] \in K(X)_{\mathbb{Q}}$.

For a finitely generated graded R-module M, its *Hilbert series*

$$\mathrm{H}_M(t) = \sum_{l \in \mathbb{Z}} \dim_k(M_l) t^l$$

is a rational function with a pole of order equal to $m := \dim(M)$ at $t = 1$. In fact,

$$\mathrm{H}_M(t) = \frac{e_M(t)}{(1-t)^m},$$

where $e_M(t)$ is a Laurent polynomial [1, (1.1)]. The *Hilbert polynomial* of M is the polynomial $\mathrm{P}_M(l)$ of degree $m-1$ such that

$$\mathrm{H}_M(t) = \text{some Laurent polynomial in } t + \sum_{l \geqslant 0} \mathrm{P}_M(l) t^l.$$

For $j \geqslant 1$, let $q_j(l)$ be the degree $j-1$ polynomial with $\mathbb{Q}$ coefficients given by

$$q_j(l) = \binom{l+j-1}{l} = \frac{(l+j-1)\cdots(l+1)}{(j-1)!}.$$

For $j \leqslant 0$, let $q_j = 0$. So $q_j(0) = 1$ for all $j \geqslant 1$, and $q_j(0) = 0$ for all $j \leqslant 0$. Thus

$$\frac{1}{(1-t)^j} = \sum_{l \geqslant 0} q_j(l) t^l. \tag{4.1}$$

We now assume M is non-negatively graded (i.e., $M_l = 0$ for all $l < 0$). Then $e_M(t) = \sum_{i \geqslant 0} a_i (1-t)^i$ is a polynomial. Hence for these rational numbers a_i,

$$\mathrm{H}_M(t) = \frac{a_0}{(1-t)^m} + \frac{a_1}{(1-t)^{m-1}} + \cdots + \frac{a_{m-1}}{(1-t)^1} + \text{a polynomial}.$$

Using (4.1), we get

$$\mathrm{P}_M(l) = a_0 q_m(l) + \cdots + a_{m-1} q_1(l). \tag{4.2}$$

Recall that the *first difference* of a polynomial $q(l)$ is the polynomial $q^{(1)}(l) = q(l) - q(l-1)$, and recursively one defines $q^{(i)} = (q^{(i-1)})^{(1)}$. For all j, by induction on i, one may prove $q_j^{(i)}(l) = q_{j-i}(l)$, and so,

$$q_j^{(i)}(0) = q_{j-i}(0) = \begin{cases} 1 & \text{if } j > i \text{ and} \\ 0 & \text{if } j \leqslant i. \end{cases}$$

Thus, still assuming M is non-negatively graded, (4.2) gives

$$\mathrm{P}_M^{(i)}(0) = a_0 + \cdots + a_{m-i-1}. \tag{4.3}$$

The lemma below shows that the coefficients of $\rho_*([\widetilde{M}])$ in the basis $1, 1-t, \ldots, (1-t)^{n-1}$ of $\mathbb{Q}[t]/(1-t)^n \cong K(\mathbb{P}^{n-1})_{\mathbb{Q}}$ coincide, up to order, with the coefficients of $\mathrm{P}_M(l)$ in the basis $q_1(l), \ldots, q_n(l)$ of $\mathbb{Q}$-polynomials of degree at most $n-1$; see (4.4).

Lemma 4.1. *Let R be as in* (2.1) *with k infinite, and let M be a finitely generated graded R-module. Then*

$$\rho_*\big([\widetilde{M}]\big) = (1-t)^n\,\mathrm{H}_M(t) \quad \textit{in } K\big(\mathbb{P}^{n-1}\big)_{\mathbb{Q}} \cong \mathbb{Q}[t]/(1-t)^n.$$

In particular,

$$\rho_*(1) = \rho_*\big([\mathcal{O}_X]\big) = e_R(t) = 1 + t + \cdots + t^{d-1} \quad \textit{in } \mathbb{Q}[t]/(1-t)^n.$$

Proof. We leave the second remark to the reader.

Using $\rho^*(t) = [\mathcal{O}_X(-1)]$ and the projection formula, we get

$$\rho_*\big(\big[\widetilde{M}(-i)\big]\big) = \rho_*\big([\widetilde{M}]\cdot\rho^*\big(t^i\big)\big) = \rho_*\big([\widetilde{M}]\big)t^i \quad \text{in } \mathbb{Q}[t]/(1-t)^n.$$

Likewise,

$$\mathrm{H}_{M(-i)}(t)(1-t)^n = t^i\,\mathrm{H}_M(t)(1-t)^n \quad \text{in } \mathbb{Q}[t]/(1-t)^n.$$

It follows that the lemma holds for M provided it holds for $M(-i)$ for any i. So taking i sufficiently large, we may assume M is non-negatively graded. Therefore, we have (4.3). Further

$$\begin{aligned}\mathrm{H}_M(t)(1-t)^n &= e_M(t)(1-t)^{n-m}\\ &= a_0(1-t)^{n-m} + a_1(1-t)^{n-m+1} + \cdots + a_{m-1}(1-t)^{n-1}\\ &\quad + \text{higher order terms in } (1-t).\end{aligned}$$

There are rational numbers b_i such that

$$\rho_*\big([\widetilde{M}]\big) = b_0 + b_1(1-t) + \cdots + b_{n-1}(1-t)^{n-1}.$$

To prove the lemma, it suffices to show

$$b_0 = \cdots = b_{n-m-1} = 0 \quad \text{and} \quad a_i = b_{n-m+i} \quad \text{for each } i = 0,\ldots,m-1. \tag{4.4}$$

Recall $q : X \to \mathrm{Spec}(k)$ and $s : \mathbb{P}^{n-1} \to \mathrm{Spec}(k)$ are the structure maps; see (2.3). The class of a coherent sheaf $\mathcal{F}$ on X maps to $\sum_i(-1)^i \dim_k H^i(X,\mathcal{F})$ under q_*, and similarly for s_*. The sheaf cohomology of the line bundles $\mathcal{O}_{\mathbb{P}^{n-1}}(i)$ [13, Exercise III.5.4] shows that s_* sends each $(1-t)^i \in \mathbb{Q}[t]/(1-t)^n \cong K(\mathbb{P}^{n-1})_{\mathbb{Q}}$ to 1, for $i = 0,\ldots,n-1$, and so

$$s_*\big(\rho_*\big([\widetilde{M}]\big)(1-t)^i\big) = b_0 + \cdots + b_{n-i-1}.$$

On the other hand, for a finitely generated graded R-module T, we have $q_*([\widetilde{T}]) = \mathrm{P}_T(0)$ and hence $q_*([\widetilde{T}(i)]) = \mathrm{P}_T(i)$ for all i [13, Exercise III.5.2]. It follows that

$$q_*\big([\widetilde{T}]\big(1 - \big[\mathcal{O}(-1)\big]\big)\big) = \mathrm{P}_T(0) - \mathrm{P}_T(-1) = \mathrm{P}_T^{(1)}(0).$$

From this we deduce that

$$s_*\big(\rho_*\big([\widetilde{M}]\big)(1-t)^i\big) = q_*\big([\widetilde{M}]\big(1-\big[\mathcal{O}(-1)\big]\big)^i\big) = \mathrm{P}_M^{(i)}(0).$$

Using (4.3), we conclude that

$$a_0 + \cdots + a_{m-i-1} = b_0 + \cdots + b_{n-i-1}$$

for all $i \geqslant 0$. Eqs. (4.4) follow. □

The preceding lemma relates $\rho_*([\widetilde{M}])$ to an invariant that is closely related to the Hilbert polynomial of M. The next one relates ρ_* to θ.

Lemma 4.2. *Let X be as in* (2.1) *with k infinite. Then for any pair of elements $x, y \in K(X)_{\mathbb{Q}}$, there is an equality*

$$\frac{(1-t)^{n-1}}{d^2}\theta(x,y) = \left(\frac{\rho_*(x)}{\rho_*(1)}\right)\left(\frac{\rho_*(y)}{\rho_*(1)}\right) - \left(\frac{\rho_*(x\cdot y)}{\rho_*(1)}\right)$$

in $K(\mathbb{P}^{n-1})_{\mathbb{Q}} \cong \mathbb{Q}[t]/(1-t)^n$.

Consequently,

$$\theta(x,y) = s_*\left[\left(\frac{d\cdot\rho_*(x)}{\rho_*(1)}\right)\left(\frac{d\cdot\rho_*(y)}{\rho_*(1)}\right) - d\left(\frac{d\cdot\rho_*(x\cdot y)}{\rho_*(1)}\right)\right]$$

where $s : \mathbb{P}^{n-1} \to \mathrm{Spec}(k)$ is the structure map.

Proof. The second equation follows from the first since $s_*((1-t)^{n-1}) = 1$. In this proof, we suppress the variable from the notation $\mathrm{H}_M(t)$ and simply write H_M.

As θ is bilinear and ρ_* is linear, we may assume $x = [\widetilde{M}]$ and $y = [\widetilde{N}]$ for finitely generated graded R-modules M and N. Let $\mathrm{H}_i = \mathrm{H}_{\mathrm{Tor}_i^R(M,N)}$, the Hilbert series of $\mathrm{Tor}_i^R(M,N)$. By [1, Lemma 7], we have

$$\sum_{i\geqslant 0}(-1)^i\,\mathrm{H}_i = \frac{\mathrm{H}_M\,\mathrm{H}_N}{\mathrm{H}_R}. \tag{4.5}$$

For a sufficiently large even integer E, the length of $\mathrm{Tor}_i^R(M,N)$ is finite and there is an isomorphism of graded R-modules

$$\mathrm{Tor}_i^R(M,N)(-d) \cong \mathrm{Tor}_{i+2}^R(M,N), \quad \text{for each } i \geqslant E.$$

As these torsion modules are finite length, and hence are non-zero in only finitely many graded degrees, it follows that H_E and H_{E+1} are polynomials. Moreover,

$$\mathrm{H}_{E+2j} = t^{dj}\,\mathrm{H}_E \quad \text{and} \quad \mathrm{H}_{E+1+2j} = t^{dj}\,\mathrm{H}_{E+1}, \quad \text{for all } j \geqslant 0.$$

Consequently

$$\sum_{i\geqslant E}(-1)^i\,\mathrm{H}_i = (\mathrm{H}_E - \mathrm{H}_{E+1})\big(1+t^d+t^{2d}+\cdots\big) = \frac{\mathrm{H}_E-\mathrm{H}_{E+1}}{1-t^d} = \frac{\mathrm{H}_E-\mathrm{H}_{E+1}}{e_R(t)(1-t)}.$$

Combining this with (4.5) gives

$$\sum_{i=0}^{E-1}(-1)^i\,\mathrm{H}_i + \frac{\mathrm{H}_E-\mathrm{H}_{E+1}}{e_R(t)(1-t)} = \frac{\mathrm{H}_M\,\mathrm{H}_N}{\mathrm{H}_R} = \frac{(1-t)^n\,\mathrm{H}_M\,\mathrm{H}_N}{e_R(t)}.$$

Multiplying both sides by $(1-t)^n/e_R(t)$ and rearranging the terms gives

$$\frac{(1-t)^n\,\mathrm{H}_M}{e_R(t)}\frac{(1-t)^n\,\mathrm{H}_N}{e_R(t)} - \sum_{i=0}^{E-1}(-1)^i\frac{(1-t)^n}{e_R(t)}\,\mathrm{H}_i = \frac{(\mathrm{H}_E-\mathrm{H}_{E+1})}{(e_R(t))^2}(1-t)^{n-1}. \tag{4.6}$$

Both sides of this equation are power series in powers of $1-t$ and we may thus take their images in $\mathbb{Q}[t]/(1-t)^n$. We claim doing so results in the equation in the statement of this lemma. For a finitely generated graded R-module T, Lemma 4.1 shows that

$$\rho_*\big([\widetilde{T}]\big)/\rho_*(1) = (1-t)^n\,\mathrm{H}_T\,/e_R(t) \quad \text{in } \mathbb{Q}[t]/(1-t)^n.$$

Since the coherent $\mathcal{O}_X$-sheaf associated to a graded module of finite length is zero, we have $\widetilde{\mathrm{Tor}_i^R(M,N)} = 0$ for all $i \geqslant E$. Hence, in the ring $K(X)_{\mathbb{Q}}$ we have

$$x\cdot y = \sum_{i=0}^{E-1}(-1)^i\big[\widetilde{\mathrm{Tor}_i^R(M,N)}\big].$$

The image of the left-hand side of (4.6) in the ring $\mathbb{Q}[t]/(1-t)^n$ is therefore

$$\frac{\rho_*(x)}{\rho_*(1)}\frac{\rho_*(y)}{\rho_*(1)} - \sum_{i=0}^{E-1}(-1)^i\frac{\rho_*([\widetilde{\mathrm{Tor}_i^R(M,N)}])}{\rho_*(1)} = \frac{\rho_*(x)}{\rho_*(1)}\frac{\rho_*(y)}{\rho_*(1)} - \frac{\rho_*(x\cdot y)}{\rho_*(1)}.$$

To simplify the right-hand side of (4.6), observe that $f(t) = (\mathrm{H}_E - \mathrm{H}_{E+1})/(e_R(t))^2$ is a rational function without a pole at $t=1$. Modulo $(1-t)^n$, we have

$$f(t)(1-t)^{n-1} = f(1)(1-t)^{n-1} + \frac{f(t)-f(1)}{1-t}(1-t)^n \equiv f(1)(1-t)^{n-1}.$$

Since $\mathrm{H}_E(1) = \mathrm{length}(\mathrm{Tor}_E^R(M,N))$, $\mathrm{H}_{E+1}(1) = \mathrm{length}(\mathrm{Tor}_{E+1}^R(M,N))$, and $e_R(1) = d$, it follows that

$$\frac{\rho_*(x)}{\rho_*(1)}\frac{\rho_*(y)}{\rho_*(1)} - \frac{\rho_*(x\cdot y)}{\rho_*(1)} = f(1)(1-t)^{n-1} = \frac{\theta(x,y)}{d^2}(1-t)^{n-1}. \qquad \square$$

Lemma 4.3. *Let X be as in* (2.1) *with k an infinite field. The diagram*

$$\begin{array}{ccc} K(X)_{\mathbb{Q}} & \xrightarrow{\frac{d}{\rho_*(1)}\cdot\rho_*} & K(\mathbb{P}^{n-1})_{\mathbb{Q}} \\ \Big\downarrow{\scriptstyle ch} & & \Big\downarrow{\scriptstyle ch} \\ CH^\bullet(X)_{\mathbb{Q}} & \xrightarrow[\rho_*]{} & CH^\bullet(\mathbb{P}^{n-1})_{\mathbb{Q}} \end{array} \tag{4.7}$$

commutes.

If k is separably closed, then $\rho_* \circ ch_{\text{ét}} = \frac{d}{\rho_*(1)} \cdot ch_{\text{ét}} \circ \rho_*$.

If $k = \mathbb{C}$, then $\rho_* \circ ch_{\text{top}} = \frac{d}{\rho_*(1)} \cdot ch_{\text{top}} \circ \rho_*$.

Proof. The maps $cy_{\text{ét}}$ and cy_{top} combine with (4.7) to give the last results since they commute with push-forwards.

By the Grothendieck–Riemann–Roch Theorem [11, 15.2], for any $x \in K(X)_{\mathbb{Q}}$,

$$\rho_*\big(ch x \cup \mathrm{Td}(\mathcal{T}_X)\big) = ch\,\rho_*(x) \cup \mathrm{Td}(\mathcal{T}_{\mathbb{P}^{n-1}})$$

where $\mathcal{T}_Y$ is the tangent bundle of a smooth variety Y. Now

$$ch\left(\frac{d\cdot\rho_*(x)}{\rho_*(1)}\right) = \frac{d\cdot ch\,\rho_*(x)}{ch\,\rho_*(1)} = \frac{d\cdot ch\,\rho_*(x)\cup \mathrm{Td}(\mathcal{T}_{\mathbb{P}^{n-1}})}{ch\,\rho_*(1)\cup \mathrm{Td}(\mathcal{T}_{\mathbb{P}^{n-1}})} = \frac{d\cdot \rho_*(ch x\cup \mathrm{Td}(\mathcal{T}_X))}{\rho_*(ch 1\cup \mathrm{Td}(\mathcal{T}_X))}.$$

But $\mathrm{Td}(\mathcal{T}_X) \in 1 + \beta CH^\bullet(X)_{\mathbb{Q}}$ as per Lemma 2.1, and hence $\mathrm{Td}(\mathcal{T}_X) = \rho^*(\xi)$ for some unit $\xi \in CH^\bullet(\mathbb{P}^{n-1})_{\mathbb{Q}}$. The projection formula gives

$$\frac{d\cdot \rho_*(ch x\cup \mathrm{Td}(\mathcal{T}_X))}{\rho_*(ch 1\cup \mathrm{Td}(\mathcal{T}_X))} = \frac{d\cdot \rho_*(ch x\cup \rho^*(\xi))}{\rho_*(ch 1\cup \rho^*(\xi))} = \frac{d\cdot \rho_*(ch x)\xi}{\rho_*(ch 1)\xi}.$$

Since $\rho_*(ch 1) = d \in CH^0(\mathbb{P}^{n-1})_{\mathbb{Q}}$, the diagram (4.7) commutes. □

5. Proofs of the main theorems

This section contains the proofs of the main results of this paper, which are stated in Section 3. We use the results developed in Section 4.

Proof of Proposition 3.1. Multiplying the first equation in Lemma 4.2 by d^2, applying $ch_{\text{ét}}$, and then simplifying using Lemma 4.3 yields

$$\rho_*(ch_{\text{ét}} x) \cup \rho_*(ch_{\text{ét}} y) - d\cdot\rho_*\big(ch_{\text{ét}}(x\cdot y)\big) = \theta(x,y) ch_{\text{ét}}(1-t)^{n-1}$$

for all $x, y \in K(X)_{\mathbb{Q}}$. As t corresponds to $[\mathcal{O}_{\mathbb{P}^{n-1}}(-1)]$ under the isomorphism $K(\mathbb{P}^{n-1})_{\mathbb{Q}} \cong \mathbb{Q}[t]/(1-t)^n$, the étale Chern character of t is $ch_{\text{ét}} t = \exp(-\varsigma) = 1 - \varsigma + \frac{\varsigma^2}{2} - \frac{\varsigma^3}{3!} + \cdots$

[11, Example 3.2.3], where $\varsigma \in H^2_{ét}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell(1))$ is the class of a hyperplane. Since $\varsigma^n = 0$ in $H^{ev}_{ét}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell)$, it follows that $ch_{ét}(1-t)^{n-1}$ is ς^{n-1}. Using this gives

$$\int\limits_{\mathbb{P}^{n-1}} \big(\rho_*(ch_{ét}x) \cup \rho_*(ch_{ét}y) - d \cdot \rho_*\big(ch_{ét}(x \cdot y)\big)\big) = \theta(x,y) \int\limits_{\mathbb{P}^{n-1}} \varsigma^{n-1}.$$

The first assertion of Proposition 3.1 follows from the fact that $\int_{\mathbb{P}^{n-1}} \varsigma^{n-1} = 1$.

For a proof of the second assertion, use ch_{top} in place of $ch_{ét}$. $\square$

Proof of Theorem 3.1. We may assume k is separably closed by applying (2.2) with $k' = k_{sep}$. Define a pairing ϕ on $H^{ev}_{ét}(X, \mathbb{Q}_\ell)$ by the formula

$$\phi(a,b) = \int\limits_{\mathbb{P}^{n-1}} \big(\rho_*(a) \cup \rho_*(b) - d \cdot \rho_*(a \cup b)\big). \tag{5.1}$$

Since $ch_{ét}$ is a ring homomorphism, Proposition 3.1 shows that for all x, y in $K(X)_{\mathbb{Q}}$, $\theta(x,y) = \phi(ch_{ét}x, ch_{ét}y)$. Thus Theorem 3.2 follows from the assertion that $\phi = 0$ when n is even.

Using the projection formula, we get that $\phi(a,b) = 0$ if either a or b lies in the image of the ring map $\rho^* : H^{ev}_{ét}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell) \to H^{ev}_{ét}(X, \mathbb{Q}_\ell)$. Recall that $\gamma = cy_{ét}(\beta) \in H^2_{ét}(X, \mathbb{Q}_\ell(1))$ is the étale cohomology class of the divisor given by a generic hyperplane section of X. Since γ lies in the image of ρ^*, we have $\phi(a,b) = 0$ if either a or b is a multiple of a power of γ. The theorem therefore follows from (2.6). $\square$

Proof of Theorem 3.3. We use the pairing ϕ introduced in (5.1). Using (2.6) and the fact that $\phi(a,b) = 0$ if either a or b is a multiple of a power of γ, we get that the pairing ϕ factors through the canonical surjection

$$H^{ev}_{ét}(X, \mathbb{Q}_\ell) \twoheadrightarrow \frac{H^{n-1}_{ét}(X, \mathbb{Q}_\ell(\frac{n-1}{2}))}{\mathbb{Q}_\ell \cdot \gamma^{\frac{n-1}{2}}}$$

when n is odd. We claim

$$\int\limits_{\mathbb{P}^{n-1}} \big(\rho_*(a) \cup \rho_*(b)\big) = \int\limits_X \big(a \cup \gamma^{\frac{n-1}{2}}\big) \int\limits_X \big(b \cup \gamma^{\frac{n-1}{2}}\big), \tag{5.2}$$

for all $a, b \in H^{n-1}_{ét}(X, \mathbb{Q}_\ell(\frac{n-1}{2}))$. To see this, first recall that $H^{ev}_{ét}(\mathbb{P}^{n-1}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell[\varsigma]/\langle \varsigma^n \rangle$ and that

$$\int\limits_{\mathbb{P}^{n-1}} \varsigma^i = \begin{cases} 0, & \text{for } i = 0, \dots, n-2 \text{ and} \\ 1, & \text{for } i = n-1. \end{cases}$$

If $\rho_*(a) = r\varsigma^{\frac{n-1}{2}}$ and $\rho_*(b) = r'\varsigma^{\frac{n-1}{2}}$ for $r, r' \in \mathbb{Q}_\ell$, then $\int_{\mathbb{P}^{n-1}} (\rho_*(a) \cup \rho_*(b)) = rr'$. On the other hand, since ρ factors as $X \hookrightarrow \mathbb{P}^n \setminus \{Q\} \twoheadrightarrow \mathbb{P}^{n-1}$ (with the second map being linear projection away from Q), we have $\rho^*(\varsigma) = \gamma$. The equation $\int_X = \int_{\mathbb{P}^{n-1}} \circ \rho_*$ and the projection formula give

$$\int_X \bigl(a \cup \gamma^{\frac{n-1}{2}}\bigr) = \int_{\mathbb{P}^{n-1}} \rho_*\bigl(a \cup \gamma^{\frac{n-1}{2}}\bigr) = \int_{\mathbb{P}^{n-1}} \rho_*(a) \cup \varsigma^{\frac{n-1}{2}} = r.$$

Similarly, $\int_X (b \cup \gamma^{\frac{n-1}{2}}) = r'$, and (5.2) follows.

Since (5.2) holds, formula (5.1) becomes

$$\phi(a,b) = \int_X \bigl(a \cup \gamma^{\frac{n-1}{2}}\bigr) \int_X \bigl(b \cup \gamma^{\frac{n-1}{2}}\bigr) - d \cdot \int_X (a \cup b).$$

The first assertion of the theorem now follows from Proposition 3.1.

The proof of the second assertion is analogous. □

Proof of Theorem 3.4. Recall that for integers p, q with $p + q = n - 1$, the complex vector space $H^{p,q}(X(\mathbb{C}))$ is the (p,q)-part of the Hodge decomposition of $H^{n-1}(X(\mathbb{C}), \mathbb{Q})$. (See [12, §0.6] or [11, 19.3.6].) Define

$$W = H^{n-1}\bigl(X(\mathbb{C}), \mathbb{Q}\bigr) \cap H^{\frac{n-1}{2},\frac{n-1}{2}}\bigl(X(\mathbb{C})\bigr),$$

a $\mathbb{Q}$-vector subspace of $H^{n-1}(X(\mathbb{C}), \mathbb{Q})$. Since the image of $cy_{\mathrm{top}}^{\frac{n-1}{2}}$ is contained in W [11, 19.3.6], it follows that the image of $ch_{\mathrm{top}}^{\frac{n-1}{2}} : K(X)_{\mathbb{Q}} \to H^{n-1}(X(\mathbb{C}), \mathbb{Q})$ is also contained in W. We argue that the restriction of $(-1)^{\frac{n+1}{2}} \theta_{\mathrm{top}}$ to $W/\mathbb{Q} \cdot \gamma^{\frac{n-1}{2}}$ is positive definite.

Define an injection $e : W/\mathbb{Q} \cdot \gamma^{\frac{n-1}{2}} \hookrightarrow H^{n-1}(X(\mathbb{C}), \mathbb{Q})$ by setting, for $a \in W$,

$$e(a) = a - \frac{\int_X a \cup \gamma^{\frac{n-1}{2}}}{d} \gamma^{\frac{n-1}{2}} \in H^{n-1}\bigl(X(\mathbb{C}), \mathbb{Q}\bigr).$$

The image of e is contained in $H^{\frac{n-1}{2},\frac{n-1}{2}}(X(\mathbb{C}))$ since both W and $\gamma^{\frac{n-1}{2}}$ are. It is also contained in the primitive part of $H^{n-1}(X(\mathbb{C}), \mathbb{Q})$, i.e., in the subspace of elements v with $\gamma \cup v = 0$. Indeed, (2.6) implies that repeated cupping with γ, followed by the map $\int_X$, forms a sequence of isomorphisms:

$$H^{n+1}\bigl(X(\mathbb{C}), \mathbb{Q}\bigr) \xrightarrow[\cong]{\gamma\cup -} H^{n+3}\bigl(X(\mathbb{C}), \mathbb{Q}\bigr) \xrightarrow[\cong]{\gamma\cup -} \cdots \xrightarrow[\cong]{\gamma\cup -} H^{2n-2}\bigl(X(\mathbb{C}), \mathbb{Q}\bigr) \xrightarrow[\cong]{\int_X} \mathbb{Q}.$$

So the vanishing of $\gamma \cup e(a)$ follows from the vanishing of $\int_X \gamma^{\frac{n-1}{2}} \cup e(a)$, which is clear.

Let Q be the bilinear pairing on $H^{n-1}(X(\mathbb{C}), \mathbb{Q})$ given by $Q(x,y) = \int_X x \cup y$. Straightforward computation verifies that for $a, b \in W$,

$$\theta_{\mathrm{top}}(a,b) = -d \cdot Q\bigl(e(a), e(b)\bigr).$$

The Hodge–Riemann bilinear relations [22, p. 165] give that $(-1)^{\frac{(n-1)(n-2)}{2}} Q$ is positive definite on the primitive part of W and hence on the image of e.

For the last assertion of the theorem, let R be as in (2.1) with k a field of characteristic zero, and let M be a finitely generated R-module. There is a finitely generated field extension of $\mathbb{Q}$ that contains the coefficients of $f(x_0, \ldots, x_n)$ and the entries of a presentation matrix for M. Using (2.2) twice, we may assume $k \subseteq \mathbb{C}$ and then $k = \mathbb{C}$. The result follows from the first assertion of the theorem. $\square$

6. Generalization of the main theorems

The following theorem allows us, among other things, to extend our result to hypersurfaces that are homogeneous with respect to a non-standard grading.

Theorem 6.1. *Let R be as in* (2.1)*. Suppose $S \subset R$ is a subring such that the map $S \hookrightarrow R$ is finite, flat, and a local complete intersection (see* [11, B.7.6]*). Note that $n = \dim(R) = \dim(S)$. Assume S is also a hypersurface with isolated singularity.*

(1) *If n is even and k is an arbitrary field, then θ^S vanishes on $G(S)_{\mathbb{Q}}$.*
(2) *If n is odd and k is an arbitrary field, then θ^S on $G(S)_{\mathbb{Q}}$ is induced from $\theta^S_{CH(S)}$ on $CH^{\frac{n-1}{2}}(S)_{\mathbb{Q}}$ via the surjective map $\tau^{\frac{n-1}{2}} : G(S)_{\mathbb{Q}} \twoheadrightarrow CH^{\frac{n-1}{2}}(S)_{\mathbb{Q}}$.*
(3) *If n is odd and $\mathrm{char}(k) = 0$, then $(-1)^{\frac{n+1}{2}}\theta^S$ is positive semi-definite on $G(S)_{\mathbb{Q}}$.*
(4) *If $n = 3$ and $k = \mathbb{C}$, then $\theta^S_{CH(S)}$ is positive definite on $CH^1(S)_{\mathbb{Q}}$. In particular, $\theta^S = 0$ if and only if $CH^1(S)$ is torsion.*

Proof. Let $h : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ be the induced map. Since $h_* \circ h^*$ is multiplication by the degree of h, the map h^* is injective. By [11, 18.2] the diagram

$$\begin{tikzcd} G(S)_{\mathbb{Q}} \arrow[r, "\tau", "\cong"'] \arrow[d, "h^*"'] & CH^\bullet(S)_{\mathbb{Q}} \arrow[d, "\mathrm{Td}(T_h)h^*"] \\ G(R) \arrow[r, "\cong", "\tau"'] & CH^\bullet(R)_{\mathbb{Q}} \end{tikzcd}$$

commutes, where T_h is the virtual tangent bundle of h. Let $\mathfrak{n}$ denote the isolated singularity of S and let $c = \mathrm{length}_R(R/\mathfrak{n}R)$. Then the following triangle commutes.

$$\begin{tikzcd} G(S)_{\mathbb{Q}}^{\otimes 2} \arrow[rr] \arrow[dr, "\theta^S"'] & & G(R)_{\mathbb{Q}}^{\otimes 2} \arrow[dl, "\frac{1}{c}\theta^R"] \\ & \mathbb{Q} & \end{tikzcd}$$

The first assertion follows from this commutative triangle and Theorem 3.2.

The second assertion follows from the above two commutative diagrams and Corollary 3.8 by taking

$$\theta^S_{CH} = \frac{1}{c} \cdot \theta^R_{CH} \circ \left(\mathrm{Td}(T_h)h^*\right)^{\otimes 2}.$$

The last two assertions follow from Theorem 3.4 and Corollary 3.10, using the fact that $\mathrm{Td}(T_h)h^*$ is injective since $\mathrm{Td}(T_h)$ is a unit. $\square$

We now describe a general situation to which Theorem 6.1 applies.

Example 6.2. Let $S = k[y_0, \dots, y_n]/(g)$ be graded with $\deg(y_i) = e_i$ for some integers $e_i \geqslant 1$, and let $g(y_0, \dots, y_n)$ be homogeneous of degree $d \geqslant 1$. Set $R = k[x_0, \dots, x_n]/(f)$, where $\deg(x_i) = 1$ and $f = g(x_0^{e_0}, \dots, x_n^{e_n})$. Then S is a subring of R, with $y_i = x_i^{e_i}$.

Since $R \cong S[x_0, \dots, x_n]/\langle x_i^{e_i} - y_i \mid i = 0, \dots, n\rangle$, it follows that the map $S \hookrightarrow R$ is finite, flat, and a local complete intersection. When both S and R have isolated singularities, Theorem 6.1 applies; in particular, it applies to rings of the form

$$S = k[y_0, \dots, y_n]/\bigl(y_0^{m_0} + \dots + y_n^{m_n}\bigr), \quad \text{with } m_i \geqslant 1 \text{ for all } i. \tag{6.1}$$

Example 6.3 *(A specific application of Example 6.2).* J. Bingener and U. Storch [2, §12] show that for S given by (6.1) with $n = 3$ and k algebraically closed of characteristic zero, the group $CH^1(S)$ is finitely generated free abelian. They find bounds on its rank. For instance, if S is defined by the 4-tuple $(m_0, m_1, m_2, m_3) = (2, 3, 3, 6)$, then $CH^1(S) \cong \mathbb{Z}^4$, and therefore $\theta^S \neq 0$. In this case, the module $M = S/(y_0 - y_3^3, y_1 + y_2)$ determines a non-zero class in $CH^1(S)_{\mathbb{Q}}$ and hence $\theta^S(M, M) \neq 0$. On the other hand, if $m_3 = 7$ instead, then $CH^1(S) = 0$, and hence θ^S vanishes.

When m_0, m_1, m_2 are distinct primes and $m_3 = m_0m_1m_2$, Bingener and Storch find an upper bound on the rank of $CH^1(S)$; see [2, (12.9)(2)]. For most primes, the exact rank of $CH^1(S)$ is unknown, and $CH^1(S)$ may even be trivial. For

$$S = k[y_0, \dots, y_3]/\bigl(y_0^2 + y_1^3 + y_2^5 + y_3^{30}\bigr),$$

they show that $CH^1(S) \cong \mathbb{Z}^8$, and therefore $\theta^S \neq 0$. It appears difficult to find generators of $CH^1(S)$. Indeed, we were unable to find a single explicit module M for which $\theta^S(M, M) \neq 0$, even though our results show such modules must exist.

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