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The semicircle law for semiregular bipartite graphs

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Abstract

We give the (Ahumada type) Selberg trace formula for a semiregular bipartite graph G. Furthermore, we discuss the distribution on arguments of poles of zeta functions of semiregular bipartite graphs. As an application, we present two analogs of the semicircle law for the distribution of eigenvalues of specified regular subgraphs of semiregular bipartite graphs.

Keywords: Zeta function; Selberg trace formula; Semiregular graph; Semicircle law

1. Introduction

Graphs and digraphs treated here are finite. Let G be a connected graph and D the symmetric digraph corresponding to G. Set $D(G) = \{(u,v),(v,u) \mid uv \in E(G)\}$. We also refer D as a graph G. For $e = (u,v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v,u)$ be the *inverse* of $e = (u,v)$.

A *path P of length n* in D (or G) is a sequence $P = (e_1, \dots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})$ $(1 \leqslant i \leqslant n-1)$. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called $(o(P), t(P))$*-path*. We say that a path $P = (e_1, \dots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some i $(1 \leqslant i \leqslant n-1)$. A (v,w)-path is called a v*-cycle*

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(or *v-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \dots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \dots, e_1^{-1})$.

We introduce an equivalence relation between cycles. Such two cycles $C_1 = (e_1, \dots, e_m)$ and $C_2 = (f_1, \dots, f_m)$ are called *equivalent* if $f_j = e_{j+k}$ for all j. The inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G for a vertex v of G. Then the (Ihara) zeta function $\mathbf{Z}(G, u)$ of a graph G is defined to be the function of $u \in \mathbf{C}$ with u sufficiently small, given by

$$\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1}, \tag{1}$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G (cf. [6,7,9,13]).

Ihara [9] defined zeta functions of graphs, and showed that the reciprocals of zeta functions of regular graphs are explicit polynomials. Hashimoto [6] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph G. Stark and Terras [12] gave an elementary proof of this formula, and discussed three different zeta functions of any graph.

Let G be a connected graph with n vertices $v_1, \dots, v_n$, and $n \in \mathbf{N}$. The *adjacency matrix* $\mathbf{A} = \mathbf{A}(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Let $Spec(G)$ be the set of all eigenvalues of $\mathbf{A}(G)$. Let $\mathbf{D} = (d_{ij})$ be the diagonal matrix with $d_{ii} = deg_G\, v_i$, and $\mathbf{Q} = \mathbf{D} - \mathbf{I}$. The *degree* $deg_G\, v = deg\, v$ of a vertex v in G is defined by $deg_G\, v = |\{w \mid vw \in E(G)\}|$. A graph H is called *k-regular* if $deg_H\, v = k$ for each vertex $v \in V(H)$.

Theorem 1 (Ihara). *Let G be a connected $(q+1)$-regular graph with n vertices. Set $Spec(G) = \{\lambda_1, \dots, \lambda_n\}$. Then the reciprocal of the zeta function of G is*

$$\begin{aligned}\mathbf{Z}(G, u)^{-1} &= (1 - u^2)^{m-n}\, det(\mathbf{I}_n - u\mathbf{A}(G) + qu^2\mathbf{I}_n) \\ &= (1 - u^2)^{(q-1)n/2} \prod_{j=1}^{n} (1 - \lambda_j u + qu^2)\end{aligned}$$

where $m = |E(G)|$.

The Selberg trace formula for a connected graph G is closely related to the zeta function of G. Ahumada [1] gave the Selberg trace formula for a regular graph (cf. [15,16]). For a semiregular bipartite graph G, Hashizume [8] presented the Selberg trace formula.

Now, let G be a connected $(q+1)$-regular graph. Furthermore, let $h(\theta)$ be a complex function on $\mathbf{R}$ which satisfies the following properties:

1. $h(\theta+2\pi)=h(\theta)$,
2. $h(-\theta)=h(\theta)$,
3. $h(\theta)$ is analytically continuable to an analytic function over $Im\,\theta<\frac{1}{2}\log q+\varepsilon$ $(\varepsilon>0)$.

For this $h(\theta)$, we define its *Fourier transform* by

$$\hat{h}(k)=\frac{1}{2\pi}\int_0^{2\pi} h(\theta)e^{\sqrt{-1}\,k\theta}\,d\theta,$$

where $k\in\mathbf{Z}$.

Theorem 2 (Ahumada). *Let G be a connected $(q+1)$-regular graph with n vertices. Set $Spec(G)=\{\lambda_1,\dots,\lambda_n\}$. Let $\lambda_1,\dots,\lambda_l$ be the eigenvalues of G which $1-\lambda_j u+qu^2=0$ has imaginary roots. Furthermore, for each λ_i $(1\leqslant i\leqslant l)$, let $q^{-\frac{1}{2}}e^{\sqrt{-1}\,\theta_i}$ be a root of $1-\lambda_j u+qu^2=0$. Then the following trace formula holds*:

$$\sum_{i=1}^{l} h(\theta_i)=\frac{2n}{\pi}\int_0^{\pi}\frac{\sin^2\theta}{(q+1)^2-4q\cos^2\theta}h(\theta)\,d\theta+\sum_{[C]}\sum_{m=1}^{\infty}|C|q^{-m|C|/2}\hat{h}(m|C|),$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Let G be a connected $(q+1)$-regular graph with m vertices. Furthermore, let $Spec(G)=\{\lambda_1,\dots,\lambda_m\}$. By Theorem 1, the poles of $\mathbf{Z}(u)$ are ±1 and roots of $1-\lambda_j u+qu^2=0$ $(1\leqslant j\leqslant m)$. Therefore, $u=q^{-1/2}e^{\sqrt{-1}\,\theta}$ is a pole of $\mathbf{Z}(u)$ if and only if $\lambda=2\sqrt{q}\cos\theta$ is an eigenvalue of G.

Sunada [14] gave an analogue of the semicircle law for the distribution on eigenvalues of regular graphs by using Theorem 2. Let

$$\phi(\lambda):=\begin{cases}\frac{q+1}{2\pi}\frac{\sqrt{4q-\lambda^2}}{(q+1)^2-\lambda^2} & \text{if } |\lambda|\leqslant 2\sqrt{q},\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 3 (Sunada). *Let $\{G_n\}_{n=1}^{\infty}$ be a family of $(q+1)$-regular graphs such that $\lim_{n\to\infty} g(G_n)=\infty$, where $g(G_n)$ is the girth of G_n. For $a,b\in\mathbf{R}$ $(a<b)$, let*

$$\phi_n([a,b])=|\{\lambda\in Spec(G_n)\mid a\leqslant\lambda\leqslant b\}|.$$

Then

$$\lim_{n\to\infty}\frac{1}{|V(G_n)|}\phi_n([a,b])=\int_a^b \phi(\lambda)\,d\lambda.$$

For a family $\{G_n\}_{n=1}^{\infty}$ of $(q+1)$-regular graphs such that $\lim_{n\to\infty}|V(G_n)|=\infty$ and $\lim_{n\to\infty} g_k(G_n)/|V(G_n)|=0$ for each $k\geqslant 3$, McKay [11] determined the limiting probability density $f(\lambda)$ for the eigenvalues of G_n as $n\to\infty$ and showed that $f(\lambda)=\phi(\lambda)$. Here $g_k(G_n)$ is the number of cycles with length k in G_n.

Furthermore, Sunada [14] presented the semicircle law for the distribution of eigenvalues of regular graphs when their girths and degrees are divergent. Let

$$\psi(x) := \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x|\leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4 (Sunada). *Let $\{G_n\}_{n=1}^{\infty}$ be a family of (q_n+1)-regular graphs such that*

$$\lim_{n\to\infty} g(G_n) = \lim_{n\to\infty} q_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\sqrt{q_n}}{\log g(G_n)} = 0.$$

For $a,b\in\mathbf{R}$ $(a<b)$, let

$$\psi_n([a,b]) = |\{\lambda\in Spec(G_n)\mid 2q_n^{1/2}a\leqslant\lambda\leqslant 2q_n^{1/2}b\}|.$$

Then

$$\lim_{n\to\infty}\frac{1}{|V(G_n)|}\psi_n([a,b]) = \int_a^b \psi(x)\,dx.$$

Using different methods, Godsil and Mohar [4] determined the expected distribution of the eigenvalues of a large random (q_1+1,q_2+1)-semiregular bipartite graph, and showed that the discrete part of the distribution is supported at 0 while the continuous part is supported on the set $|\sqrt{q_1}-\sqrt{q_2}|\leqslant\lambda\leqslant\sqrt{q_1}+\sqrt{q_2}$. This gives another proof of our Theorem 8. Furthermore, Li and Solé [10] showed that the continuous spectrum of the universal covering of (q_1+1,q_2+1)-semiregular bipartite graphs does not contain 0 if $q_1\neq q_2$.

In this paper, we give the (Ahumada type) Selberg trace formula for a semiregular bipartite graph G. Furthermore, we discuss the distribution of arguments of poles of zeta functions of semiregular bipartite graphs. As an application, we present two analogue of the semicircle law for the distribution of eigenvalues of specified regular subgraphs of semiregular bipartite graphs.

For a general theory of spectra of graphs and the Selberg trace formula, the reader is referred to [3] and [15], respectively.

2. The Selberg trace formulas for semiregular bipartite graphs

We present the Selberg trace formula for a semiregular bipartite graph G.

A graph G is called *bipartite*, denoted by $G=(V_1,V_2)$ if there exists a partition $V(G)=V_1\cup V_2$ of $V(G)$ such that $uv\in E(G)$ if and only if $u\in V_1$ and $v\in V_2$. A bipartite graph $G = (V_1, V_2)$ is called $(q_1 + 1, q_2 + 1)$-semiregular if $deg_G\, v = q_i + 1$ for each $v \in V_i$ $(i = 1, 2)$. Furthermore, $q_1 + 1$ and $q_2 + 1$ are called the *degrees* of G. For a $(q_1 + 1, q_2 + 1)$-semiregular bipartite graph $G = (V_1, V_2)$, let $G^{[i]}$ be the graph with vertex set V_i and edge set $\{P$: reduced path $|\,|P| = 2;\ o(P), t(P) \in V_i\}$ for $i = 1, 2$. Note that $G^{[1]}$ is a $(q_1 + 1)q_2$-regular graph, and $G^{[2]}$ is a $(q_2 + 1)q_1$-regular.

Let $G = (V_1, V_2)$ be a connected $(q_1 + 1, q_2 + 1)$-semiregular bipartite graph. Set $|V_1| = n$ and $|V_2| = m$ $(n \leqslant m)$. Let $\mathbf{A}^{[i]} = \mathbf{A}(G^{[i]})$ be the adjacency matrix of $G^{[i]}$ $(i = 1, 2)$.

Theorem 5 (Hashimoto). *Let $G = (V_1, V_2)$ be a connected $(q_1 + 1, q_2 + 1)$-semiregular bipartite graph with v vertices and ε edges, $|V_1| = n$ and $|V_2| = m$ $(n \leqslant m)$. Then*

$$\begin{aligned}\mathbf{Z}(G,u)^{-1} &= (1-u^2)^{\varepsilon - v}(1+q_2u^2)^{m-n} \\ &\quad \times \prod_{j=1}^{n} (1 - (\lambda_j^2 - q_1 - q_2)u^2 + q_1q_2u^4) \\ &= (1-u^2)^{\varepsilon - v}(1+q_2u^2)^{m-n}\, det(\mathbf{I}_n - (\mathbf{A}^{[1]} - (q_2 - 1)\mathbf{I}_n)u^2 + q_1q_2u^4\mathbf{I}_n) \\ &= (1-u^2)^{\varepsilon - v}(1+q_1u^2)^{n-m}\, det(\mathbf{I}_m - (\mathbf{A}^{[2]} - (q_1 - 1)\mathbf{I}_m)u^2 + q_1q_2u^4\mathbf{I}_m),\end{aligned}$$

where $Spec(G) = \{\pm\lambda_1, \ldots, \pm\lambda_n, 0, \ldots, 0\}$ and $\mathbf{A}^{[i]} = \mathbf{A}(G^{[i]})$ $(i = 1, 2)$.

In Theorem 5, let $Spec(G^{[1]}) = \{\mu_1, \ldots, \mu_n\}$. If $Spec(G) = \{\pm\lambda_1, \ldots, \pm\lambda_n, 0, \ldots, 0\}$, then we have

$$\mu_j = \lambda_j^2 - q_1 - 1 \quad (1 \leqslant j \leqslant n)$$

and so

$$\mathbf{Z}(G,u)^{-1} = (1-u^2)^{\varepsilon - v}(1+q_2u^2)^{m-n} \prod_{j=1}^{n} (1 - (\mu_j - q_2 + 1)u^2 + q_1q_2u^4)$$

(see [6]).

Let $G = (V_1, V_2)$ be a connected $(q_1 + 1, q_2 + 1)$-semiregular bipartite graph. Set $|V_1| = n$ and $|V_2| = m$ $(n \leqslant m)$. Let

$$\mathbf{Z}(v) = \mathbf{Z}(G, \sqrt{v}). \tag{2}$$

By Theorem 5, we have

$$\mathbf{Z}(v)^{-1} = (1-v)^{\varepsilon - v}(1+q_2v)^{m-n} \prod_{j=1}^{n} (1 - (\mu_j - q_2 + 1)v + q_1q_2v^2). \tag{3}$$

Theorem 6. *Let $G = (V_1, V_2)$ be a connected $(q_1 + 1, q_2 + 1)$-semiregular bipartite graph with v vertices and ε edges. Set $|V_1| = n$ and $|V_2| = m$ $(n \leqslant m)$. Furthermore, let $h(\theta)$ be an even complex function on $\mathbf{R}$ with period 2π which is analytically continuable to an analytic function over $Im\,\theta < \frac{1}{2}\log q_1q_2 + \varepsilon$ $(\varepsilon > 0)$. Set $Spec(G^{[1]}) = \{\mu_1, \ldots, \mu_n\}$. Let $\mu_1, \ldots, \mu_l$ be the eigenvalues of $G^{[1]}$ which $1 - (\mu_j - q_2 + 1)v + q_1q_2v^2 = 0$ has* *imaginary roots. Furthermore, for each* μ_i $(1 \leqslant i \leqslant l)$, *let* $(q_1 q_2)^{-1/2} e^{\sqrt{-1}\,\theta_i}$ *be a root of* $1 - (\mu_i - q_2 + 1)v + q_1 q_2 v^2 = 0$. *Then the following trace formula holds*:

$$\begin{aligned}\sum_{i=1}^{l} h(\theta_i) &= \frac{2n}{\pi} q_1 q_2 (q_1+1) \\ &\quad \times \int_0^{\pi} \frac{\sin^2\theta}{(q_1+q_2)(q_1q_2+1) + 2\sqrt{q_1q_2}(q_1-1)(q_2-1)\cos\theta - 4q_1q_2\cos^2\theta} \\ &\quad \times h(\theta)\,d\theta + \frac{1}{2}\sum_{[C]}\sum_{m=1}^{\infty} |C|(q_1q_2)^{-m|C|/4}\hat{h}(m|C|/2),\end{aligned}$$

where $[C]$ *runs over all equivalence classes of prime*, *reduced cycles of* G, *and*

$$\hat{h}(k) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\exp^{\sqrt{-1}\,k\theta}\,d\theta.$$

Proof. The argument is an analogue of Venkov and Nikitin's method [16]. By (3), we have

$$\begin{aligned}\mathbf{Z}(v/\sqrt{q_1q_2})^{-1} &= (1 - v/\sqrt{q_1q_2})^{\varepsilon - v}(1 + q_2 v/\sqrt{q_1q_2})^{m-n} \\ &\quad \times \prod_{j=1}^{n}(1 - (\mu_j - q_2 + 1)v/\sqrt{q_1q_2} + v^2).\end{aligned}$$

Thus,

$$\begin{aligned}-\log\mathbf{Z}(v/\sqrt{q_1q_2}) &= (\varepsilon - v)\log(1 - v/\sqrt{q_1q_2}) + (m-n)\log(1 + q_2v/\sqrt{q_1q_2}) \\ &\quad + \sum_{j=1}^{n}\log v/\sqrt{q_1q_2}(\sqrt{q_1q_2}(v+v^{-1}) - (\mu_j - q_2 + 1)).\end{aligned}$$

Therefore,

$$\begin{aligned}&-\sum_{j=1}^{n}\frac{d}{dv}\log(\sqrt{q_1q_2}(v+v^{-1}) - (\mu_j - q_2 + 1)) \\ &\quad = -\frac{\varepsilon - v}{\sqrt{q_1q_2} - v} + \frac{m-n}{\sqrt{q_1q_2} + q_2 v} + \frac{n}{v} + \frac{d}{dv}\log\mathbf{Z}(v/\sqrt{q_1q_2}).\end{aligned}$$

Since G is bipartite, we have $n(q_1+1) = m(q_2+1)$, i.e., $mq_2 - nq_1 = n - m$. Thus,

$$-\frac{\varepsilon - v}{\sqrt{q_1q_2} - v} + \frac{m-n}{\sqrt{q_1q_2} + q_2v} + \frac{n}{v} = \frac{nq_1q_2(1-v^2)}{(\sqrt{q_1q_2} - v)(\sqrt{q_1q_2} + q_2v)v}.$$

Next, since $Tr(\log(\mathbf{I} - \mathbf{B})) = \log det(\mathbf{I} - \mathbf{B})$, (1) and (2) imply that

$$\log\mathbf{Z}(v) = \log\mathbf{Z}_G(v^{1/2}) = -\sum_{[C]}\log(1 - v^{|C|/2}) = \sum_{[C]}\sum_{m=1}^{\infty}\frac{1}{m}u^{|C|m/2}.$$

Then,

$$\frac{d}{dv}\log \mathbf{Z}(v) = \frac{1}{2}v^{-1}\sum_{[C]}\sum_{m=1}^{\infty}|C|v^{|C|m/2},$$

where C runs over all prime, reduced cycles of G. Thus,

$$\frac{d}{dv}\log \mathbf{Z}(v/\sqrt{q_1q_2}) = \frac{1}{2}v^{-1}\sum_{[C]}\sum_{m=1}^{\infty}|C|(q_1q_2)^{-m|C|/4}v^{m|C|/2}.$$

Therefore,

$$\begin{aligned}&-\sum_{j=1}^{n}\frac{d}{dv}\log(\sqrt{q_1q_2}(v+v^{-1})-(\mu_j-q_2+1))\\ &= \frac{nq_1q_2(1-v^2)}{(\sqrt{q_1q_2}-v)(\sqrt{q_1q_2}+q_2v)v}+\frac{1}{2}v^{-1}\sum_{[C]}\sum_{m=1}^{\infty}|C|(q_1q_2)^{-m|C|/4}v^{m|C|/2}. \qquad (4)\end{aligned}$$

Now, for $\mu_j \in Spec(\mathbf{A}_{\rho}^{[1]})$ $(1\leqslant j\leqslant n)$, let z_j be a root of

$$(q_1q_2)^{1/2}(t+t^{-1}) = \mu_j - q_2 + 1.$$

If z_j is not contained in $\mathbf{R}$, then $|z_j| = 1$, i.e., $z_j = e^{\sqrt{-1}\,\theta_j}$. Let $z_1, \ldots, z_l$ be not contained in $\mathbf{R}$, and $z_{l+1}, \ldots, z_n$ be real numbers. Moreover, let $h(\theta)$ be an even complex function on $\mathbf{R}$ with period 2π which is analytically continuable to an analytic function over $Im\,\theta<\frac{1}{2}\log q_1q_2+\varepsilon$ $(\varepsilon>0)$. Furthermore, let K be a simple closed curve traced in the positive direction(counterclockwise) which contains $z_1, \ldots, z_l, \sqrt{q_1q_2}, -\sqrt{q_1/q_2}$, and does not contain z_j $(l+1\leqslant j\leqslant n), z_1^{-1}, \ldots, z_n^{-1}, 0$. We consider the following three contour integrals:

$$Q(h,j) = -\frac{1}{2\pi i}\oint_{-K} h(-\sqrt{-1}\log v)\frac{d}{dv}\log(\sqrt{q_1q_2}(v+v^{-1})-(\mu_j-q_2+1))\,dv,$$

$$I(h) = \frac{1}{2\pi i}\oint_{-K} h(-\sqrt{-1}\log v)\frac{1-v^2}{(\sqrt{q_1q_2}-v)(\sqrt{q_1q_2}+q_2v)v}\,dv,$$

$$H(h,m) = \frac{1}{2\pi i}\oint_{-K} h(-\sqrt{-1}\log v)v^{m|C|/2-1}\,dv.$$

Then, by (4), we have

$$\sum_{j=1}^{n} Q(h,j) = nq_1q_2I(h)+\frac{1}{2}\sum_{[C]}\sum_{m\geqslant 1}|C|(q_1q_2)^{-m|C|/4}H(h,m).$$

By the property of the residue theorem, we have

$$Q(h,j) = \frac{1}{2\pi i}\oint_{K} h(-\sqrt{-1}\log v)\frac{d}{dv}\log\sqrt{q_1q_2}/v(v-z_j)(v-1/z_j)\,dv = h(\theta_j)$$

if z_j is not contained in $\mathbf{R}$. Moreover, if $z_j \in \mathbf{R}$, then $Q(h,j) = 0$.

Furthermore, let K_1 be the circle of radius 1 traced in the positive direction. Then we have

$$H(h,m) = \frac{1}{2\pi i}\oint_{K_1} h(-\sqrt{-1}\log v)v^{m|C|/2-1}\,dv.$$

Set $v = e^{\sqrt{-1}\,\theta}$ $(0 \leqslant \theta \leqslant 2\pi)$. Then

$$H(h,m) = \frac{1}{2\pi i}\int_0^{2\pi} h(\theta)e^{\sqrt{-1}\,m|C|/2}d\theta = \hat{h}(m|C|/2).$$

Now, we have

$$I(h) = \frac{1}{2\pi i}\oint_{K_1} h(-\sqrt{-1}\log v)\frac{1-v^2}{(1-v/\sqrt{q_1q_2})(1+v/\sqrt{q_1/q_2})v}dv.$$

Set $v = e^{\sqrt{-1}\,\theta}(-\pi \leqslant \theta \leqslant \pi)$. Then we have

$$I(h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\frac{1-e^{2\sqrt{-1}\,\theta}}{(\sqrt{q_1q_2}-e^{\sqrt{-1}\,\theta})(\sqrt{q_1q_2}+q_2e^{\sqrt{-1}\,\theta})}d\theta.$$

But,

$$\frac{1-e^{2\sqrt{-1}\,\theta}}{(\sqrt{q_1q_2}-e^{\sqrt{-1}\,\theta})(\sqrt{q_1q_2}+q_2e^{\sqrt{-1}\,\theta})} = \frac{2q_2(1+q_1)\sin^2\theta - (2\sqrt{q_1q_2}(q_2-1)\sin\theta + q_2(q_1-1)\sin 2\theta)\sqrt{-1}}{q_2(q_1q_2-2\sqrt{q_1q_2}\cos\theta+1)(q_1+q_2+2\sqrt{q_1q_2}\cos\theta)}.$$

Let $g(\theta) = (q_1q_2 - 2\sqrt{q_1q_2}\cos\theta + 1)(q_1+q_2+2\sqrt{q_1q_2}\cos\theta)$. Since $g(\theta)$ is an even function, we have

$$\begin{aligned} I(h) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\frac{2(1+q_1)\sin^2\theta}{g(\theta)}d\theta \\ &- \frac{\sqrt{-1}}{2\pi}\int_{-\pi}^{\pi} h(\theta)\frac{2\sqrt{q_1q_2}(q_2-1)\sin\theta + q_2(q_1-1)\sin 2\theta}{q_2 g(\theta)}d\theta \\ =& \frac{2}{\pi}(q_1+1)\int_0^{\pi} h(\theta)\frac{\sin^2\theta}{g(\theta)}d\theta. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \sum_{j=1}^{l} h(\theta_j) =& n\frac{2}{\pi}q_1q_2(q_1+1)\int_0^{\pi} h(\theta)\frac{\sin^2\theta}{g(\theta)}d\theta \\ &+ \frac{1}{2}\sum_{[C]}\sum_{m\geqslant 1}^{\infty} |C|(q_1q_2)^{-m|C|/4}\hat{h}(m|C|/2). \end{aligned}$$

We show that the second term in the right side converges.

Since $G^{[1]}$ is $(q_1+1)q_2$-regular, Theorem 2 implies that

$$\sum_{[C_1]}\sum_{m\geqslant 1}^{\infty}|C_1|(q_1q_2+q_2-1)^{-m|C_1|/2}\hat{h}(m|C_1|)$$

converges, where $[C_1]$ runs over all equivalence classes of prime, reduced cycles in $G^{[1]}$. Each prime, reduced cycle with length $2m$ in G corresponds to a unique prime, reduced cycle with length m in $G^{[1]}$. Thus, the following series converges:

$$\sum_{[C]}\sum_{m\geqslant 1}^{\infty}\frac{1}{2}|C|(q_1q_2+q_2-1)^{-m|C|/4}\hat{h}(m|C|/2).$$

Since $(q_1q_2+q_2-1)^{-m|C|/4}\fallingdotseq(q_1q_2)^{-m|C|/4}$ for sufficiently large $m|C|$, the second term in the right side converges. $\square$

3. The distribution of poles of zeta functions of semiregular bipartite graphs

We shall state the notion of graph covering of a graph.

Let G be a connected graph, and let $N(v)=\{w\in V(G)\mid vw\in E(G)\}$ for any vertex v in G. A graph H is called a *covering* of G with projection $\pi: H\to G$ if there is a surjection $\pi: V(H)\to V(G)$ such that $\pi|_{N(v')}: N(v')\to N(v)$ is a bijection for all vertices $v\in V(G)$ and $v'\in\pi^{-1}(v)$. When a finite group Π acts on a graph (digraph) G, the *quotient graph* (*digraph*) G/Π is a simple graph (digraph) whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi: H\to G$ is said to be a *regular covering* of G if there is a subgroup B of the automorphism group $Aut\,H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and A a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha: D(G)\to A$ is called an *ordinary voltage assignment* if $\alpha(v,u)=\alpha(u,v)^{-1}$ for each $(u,v)\in D(G)$. The pair (G,α) is called an *ordinary voltage graph*. The *derived graph* G^{α} of the ordinary voltage graph (G,α) is defined as follows:

$$V(G^{\alpha})=V(G)\times A \text{ and } ((u,h),(v,k))\in D(G^{\alpha}) \text{ if and only if } (u,v)\in D(G) \text{ and } k=h\alpha(u,v),$$

where $V(G)$ is the vertex set of G (see [5]). Also, G^{α} is called an A-covering of G. The *natural projection* $\pi: G^{\alpha}\to G$ is defined by $\pi(v,h)=v$ for all $(v,h)\in V(G)\times A$. Then the covering $\pi: G^{\alpha}\to G$ is a regular covering of G.

For a graph G, the *girth* $g(G)$ of G is the minimum of the lengths of prime, reduced cycles in G. Let $ord(g)$ be the *order* of an element g in a group.

Lemma 1. *Let G be a connected graph with v vertices and ε edges. Then there exists some finite group Γ and an ordinary voltage assignment $\alpha : D(G) \to \Gamma$ such that*

$$g(G^{\alpha}) > g(G).$$

Proof. Let T be a spanning tree of G. We give each of the $r = \varepsilon - v + 1$ edges in $G \backslash E(T)$ a direction and label $e_1, \ldots, e_r$. Let $e_{r+j} = e_j^{-1}$ $(1 \leqslant j \leqslant r)$.

Any prime, reduced cycle on G is uniquely determined by the ordered sequence of e_k's it passes through. Specially, if e_i and e_j are two consecutive e_k's in this sequence, then the part of the cycle between e_i and e_j is the unique reduced path on T joining $t(e_i)$ and $o(e_j)$. Note that the free group of rank r generated by the e_k's is identified with the fundamental group π_1 (G, v) $(v \in V(G))$.

Let $\Gamma = \langle \gamma_1 \rangle \times \cdots \times \langle \gamma_r \rangle$ be the direct product of r cyclic groups $\langle \gamma_1 \rangle, \ldots, \langle \gamma_r \rangle$, where $ord(\gamma_i) > 1, i = 1, \ldots, r$. Furthermore, let $\alpha : D(G) \to \Gamma$ be the ordinary voltage assignment such that

$$\alpha(e_i) = \gamma_i \ (1 \leqslant i \leqslant r); \quad \alpha(e) = 1, \quad e \in D(T).$$

Let $\tilde{C}$ be any prime, reduced cycle in G^{α}. Then there exists some prime, reduced cycle C in G such that

$$\pi_{\alpha}(\tilde{C}) = C^f, \quad f \geqslant 0.$$

Let $e_{i_1}, \ldots, e_{i_s}$ be the sequence of e_k's corresponding to C. Then we have

$$\alpha(C) = \alpha(e_{i_1}) \cdots \alpha(e_{i_s})$$

and

$$t = ord(\alpha(C)) > 1.$$

By Gross and Tucker [5, Theorem 2.1.3], the preimage $\pi_{\alpha}^{-1}(C)$ of C in G^{α} is the union of $|\Gamma|/t$ disjoint cycles with length $t|C|$. Since $|C| \geqslant g(G)$, we have $t = f$, and so

$$|\tilde{C}| = t|C| \geqslant tg(G) > g(G).$$

Therefore, it follows that

$$g(G^{\alpha}) > g(G). \qquad \square$$

In Lemma 1, G^{α} is $(q_1 + 1, q_2 + 1)$-semiregular bipartite if G is $(q_1 + 1, q_2 + 1)$-semiregular bipartite.

Corollary 1. *There exists a family $\{G_n\}_{n=1}^{\infty}$ of $(q_1 + 1, q_2 + 1)$-semiregular bipartite graphs such that*

$$\lim_{n \to \infty} g(G_n) = \infty.$$

Now, we consider a family $\{G_n\}_{n=1}^{\infty}$ of (q_1+1, q_2+1)-semiregular bipartite graphs such that $\lim_{n\to\infty} g(G_n) = \infty$. For $0 \leqslant \alpha < \beta \leqslant \pi$, let

$$\varphi_n([\alpha, \beta]) = |\{v = (q_1 q_2)^{-1/2} e^{\sqrt{-1}\,\theta} \mid v : a \ pole \ of \ \mathbf{Z}(v), \alpha \leqslant \theta \leqslant \beta\}|.$$

We give a property of the distribution on arguments of poles of zeta functions of semiregular bipartite graphs.

Theorem 7. *Let* $\{G_n\}_{n=1}^{\infty}$ *be a family of* (q_1+1, q_2+1)*-semiregular bipartite graphs such that* $\lim_{n\to\infty} g(G_n) = \infty$. *Then*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{|V(G_n^{[1]})|} \varphi_n([\alpha, \beta]) \\
&\quad = \frac{2}{\pi} q_1 q_2 (q_1+1) \\
&\quad \times \int_{\alpha}^{\beta} \frac{\sin^2\theta}{(q_1+q_2)(q_1q_2+1) + 2\sqrt{q_1q_2}(q_1-1)(q_2-1)\cos\theta - 4q_1q_2\cos^2\theta}\, d\theta.
\end{aligned}$$

Proof. Let $G = G_a = (V_1, V_2)$, $|V_1| = n$ and $|V_2| = m$ $(n \leqslant m)$. Furthermore, let

$$h(\theta) := \begin{cases} 1 & \text{if } \alpha \leqslant \theta \leqslant \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $h(0) = h(\pi) = 0$.

Theorem 6 implies that

$$\begin{aligned}
\varphi_a([\alpha, \beta]) = & \frac{2n}{\pi} q_1 q_2 (q_1+1) \int_{\alpha}^{\beta} \frac{\sin^2\theta}{g(\theta)}\, d\theta \\
& + \frac{1}{2} \sum_{[C]} \sum_{m \geqslant 1}^{\infty} (q_1 q_2)^{-m|C|/4} \hat{h}(m|C|/2),
\end{aligned} \tag{5}$$

where $g(\theta) = (q_1+q_2)(q_1q_2+1) + 2\sqrt{q_1q_2}(q_1-1)(q_2-1)\cos\theta - 4q_1q_2\cos^2\theta$, and $[C]$ runs over all equivalence classes of prime, reduced cycles of G. Then we have

$$\hat{h}(m|C|/2) = \frac{1}{2\pi} \int_{\alpha}^{\beta} e^{\sqrt{-1}\, m|C|\theta/2}\, d\theta = \frac{1}{\pi m|C|\sqrt{-1}} \left(e^{\sqrt{-1}\, m|C|\beta/2} - e^{\sqrt{-1}\, m|C|\alpha/2} \right).$$

Thus,

$$\begin{aligned}
K_a &:= \frac{1}{2} \sum_{[C]} \sum_{m \geqslant 1}^{\infty} (q_1 q_2)^{-m|C|/4} \hat{h}(m|C|/2) \\
&= \frac{1}{2\pi\sqrt{-1}} \sum_{[C]} \sum_{m \geqslant 1}^{\infty} \frac{1}{m} (q_1 q_2)^{-m|C|/4} \left(e^{\sqrt{-1}\, m|C|\beta/2} - e^{\sqrt{-1}\, m|C|\alpha/2} \right).
\end{aligned}$$

Therefore, we have

$$|K_a| \leqslant \frac{1}{2\pi} \sum_{[C]} \sum_{m \geqslant 1}^{\infty} \frac{1}{m} \{(q_1 q_2)^{-|C|/4}\}^m \cdot 2$$
$$= \frac{1}{\pi} \sum_{[C]} (-\log(1 - (q_1 q_2)^{-|C|/4}))$$
$$= \frac{1}{\pi} \log \prod_{[C]} (1 - (q_1 q_2)^{-|C|/4})^{-1}.$$

Since $|C| \geqslant g(G) = g$, we have

$$(q_1 q_2)^{|C|/4} \geqslant (q_1 q_2)^{g/4},$$

i.e.,

$$\log \prod_{[C]} (1 - (q_1 q_2)^{-|C|/4})^{-1} \leqslant \log \prod_{[C]} (1 - (q_1 q_2)^{-g/4})^{-1}.$$

The fact that $\lim_{n\to\infty} g(G_n) = \infty$ implies that

$$\forall \varepsilon, \ \exists N \in \mathbf{N}: \ a > N \Rightarrow \left(\frac{1}{q_1 q_2}\right)^{g(G_a)/4} < \varepsilon.$$

Thus, we have

$$0 < (1 - (q_1 q_2)^{-g(G_a)/4})^{-1} < (1 - \varepsilon)^{-1}$$

for any $a > N$. Since ε is any,

$$\lim_{a\to\infty} \log \prod_{[C]} (1 - (q_1 q_2)^{-g(G_a)/4})^{-1} \leqslant \log \prod_{[C]} \lim_{\varepsilon \to +0} (1 - \varepsilon)^{-1} = 0.$$

Therefore, it follows that

$$\lim_{a\to\infty} K_a = 0.$$

Hence, the result follows. □

4. Analogue of the semicircle law for the distribution of eigenvalues of specified regular subgraphs of semiregular bipartite graphs

In this section, we consider a sequence $G_1, G_2, \ldots, G_n \ldots$ of semiregular bipartite graphs, and then study the distribution of eigenvalues of regular subgraphs $G_n^{[1]}$ in G_n (cf. [4,10]).

Let $G=(V_1,V_2)$ be a connected (q_1+1,q_2+1)-semiregular bipartite graph. Set $|V_1|=n$ and $|V_2|=m$ $(n\leqslant m)$. Furthermore, let $Spec(G^{[1]})=\{\mu_1,\dots,\mu_n\}$. By (3), we have

$$\mathbf{Z}(v)^{-1}=(1-v)^{\varepsilon-v}(1+q_2v)^{m-n}\prod_{j=1}^{n}(1-(\mu_j-q_2+1)v+q_1q_2v^2).$$

Thus, the poles of $\mathbf{Z}(v)$ are $1,-1/q_2$ and roots of $1-(\mu_j-q_2+1)v+q_1q_2v^2=0$ $(1\leqslant j\leqslant n)$. Therefore, $v=(q_1q_2)^{-1/2}e^{\sqrt{-1}\,\theta}$ is a pole of $\mathbf{Z}(v)$ if and only if $\mu_j=2\sqrt{q_1q_2}\cos\theta+q_2-1$ is an eigenvalue of $G^{[1]}$.

Now, we consider a family $\{G_n\}_{n=1}^{\infty}$ of (q_1+1,q_2+1)-semiregular bipartite graphs such that $\lim_{n\to\infty}g(G_n)=\infty$. For $a,b\in\mathbf{R}$ $(a<b)$, let

$$\phi_n([a,b])=|\{\mu\in Spec(G_n^{[1]})\mid a\leqslant\mu\leqslant b\}|.$$

We present an analogue of the semicircle law for the distribution of eigenvalues of specified regular subgraphs of semiregular bipartite graphs. This is the same as the formula (5.2) in Godsil and Mohar [4], and we give a different proof of it.

Theorem 8. *Let* $\{G_n\}_{n=1}^{\infty}$ *be a family of* (q_1+1,q_2+1)*-semiregular bipartite graphs such that* $\lim_{n\to\infty}g(G_n)=\infty$. *Then*

$$\lim_{n\to\infty}\frac{1}{|V(G_n^{[1]})|}\phi_n([a,b])=\int_a^b\phi(\mu)\,d\mu,$$

where

$$\phi(\mu):=\begin{cases}\frac{q_1+1}{2\pi}\frac{\sqrt{4q_1q_2-(\mu-q_2+1)^2}}{(q_1q_2+1-(\mu-q_2+1))(q_1+q_2+(\mu-q_2+1))} & \text{if } |\mu-q_2+1|\leqslant 2\sqrt{q_1q_2},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let $0\leqslant\alpha<\beta\leqslant\pi$. Then Theorem 7 implies that

$$\lim_{n\to\infty}\frac{1}{|V(G_n^{[1]})|}\varphi_n([\alpha,\beta])=\frac{2}{\pi}q_1q_2(q_1+1)\times\int_\alpha^\beta\frac{\sin^2\theta}{g(\theta)}\,d\theta, \tag{6}$$

where $g(\theta)=(q_1+q_2)(q_1q_2+1)+2\sqrt{q_1q_2}(q_1-1)(q_2-1)\cos\theta-4q_1q_2\cos^2\theta$. Let $G=G_a=(V_1,V_2)$, $|V_1|=n$ and $|V_2|=m$ $(n\leqslant m)$. Furthermore, let $Spec(G^{[1]})=\{\mu_1,\dots,\mu_n\}$. For the root $v=(q_1q_2)^{-1/2}e^{\sqrt{-1}\,\theta_j}$ of $1-(\mu_j-q_2+1)v+q_1q_2v^2=0$ $(1\leqslant j\leqslant n)$,

$$\alpha\leqslant\theta_j\leqslant\beta\iff 2\sqrt{q_1q_2}\cos\beta\leqslant\mu_j-q_2+1\leqslant 2\sqrt{q_1q_2}\cos\alpha.$$

In (6), let

$$\mu=2\sqrt{q_1q_2}\cos\theta+q_2-1.$$

Then we have

$$\int_{\alpha}^{\beta} \frac{\sin^2\theta}{g(\theta)}\,d\theta = \frac{1}{\sqrt{q_1q_2}} \int_{\mu_\beta}^{\mu_\alpha} \frac{\sin\theta}{(q_1q_2+q_2-\mu)(q_1+1+\mu)}\,d\mu,$$

where $\mu_\alpha = 2\sqrt{q_1q_2}\cos\alpha + q_2 - 1$ and $\mu_\beta = 2\sqrt{q_1q_2}\cos\beta + q_2 - 1$. But,

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sqrt{4q_1q_2-(\mu-q_2+1)^2}}{2\sqrt{q_1q_2}}.$$

Thus,

$$\int_{\alpha}^{\beta} \frac{\sin^2\theta}{g(\theta)}\,d\theta = \frac{1}{4q_1q_2} \int_{\mu_\beta}^{\mu_\alpha} \frac{\sqrt{4q_1q_2-(\mu-q_2+1)^2}}{(q_1q_2+q_2-\mu)(q_1+1+\mu)}\,d\mu.$$

Therefore, it follows that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{|V(G_n^{[1]})|}\phi_n([\mu_\beta,\mu_\alpha]) &= \lim_{n\to\infty} \frac{1}{\left|V(G_n^{[1]})\right|}\varphi_n([\alpha,\beta]) \\ &= \frac{q_1+1}{2\pi}\int_{\mu_\beta}^{\mu_\alpha} \frac{\sqrt{4q_1q_2-(\mu-q_2+1)^2}}{(q_1q_2+q_2-\mu)(q_1+1+\mu)}\,d\mu. \qquad \square\end{aligned}$$

Next, we consider a sequence of semiregular bipartite graphs for which both their girths and degrees are divergent.

By Lemma 1, we obtain the following result.

Corollary 2. *There exist a family* $\{G_n\}_{n=1}^{\infty}$ *of* $(q_{n,1}+1, q_{n,2}+1)$*-semiregular bipartite graphs such that*

$$\lim_{n\to\infty} g(G_n) = \lim_{n\to\infty} q_{n,1} = \infty,$$

where $q_{n,1} \geqslant q_{n,2}$ *for each* $n \in \mathbf{N}$.

Let $\{G_n\}_{n=1}^{\infty}$ be a family of $(q_{n,1}+1, q_{n,2}+1)$-semiregular bipartite graphs such that

$$\lim_{n\to\infty} g(G_n) = \lim_{n\to\infty} q_{n,1} = \infty,$$

where $q_{n,1} \geqslant q_{n,2}$. For $-1 \leqslant a < b \leqslant 1$, let

$$\psi_n([a,b]) = |\{\mu = 2\sqrt{q_1q_2}\cos\theta + q_2 - 1 \in Spec(G_n^{[1]}) \mid a \leqslant \cos\theta \leqslant b\}|.$$

Furthermore, let

$$\psi(t) := \begin{cases} \frac{2}{\pi}\frac{\sqrt{1-t^2}}{1+\delta+2\sqrt{\delta}t} & \text{if } |t| \leqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\delta = \lim_{n\to\infty} \frac{q_{n,2}}{q_{n,1}}.$$

We present another analogue of the semicircle law for the distribution of eigenvalues of specified regular subgraphs of semiregular bipartite graphs.

Theorem 9. *Let* $\{G_n\}_{n=1}^{\infty}$ *be a family of* $(q_{n,1}+1, q_{n,2}+1)$*-semiregular bipartite graphs such that*

$$\lim_{n\to\infty} g(G_n) = \lim_{n\to\infty} q_{n,1} = \infty, \quad q_{n,1} \geqslant q_{n,2}$$

and

$$\lim_{n\to\infty} \frac{\sqrt[4]{q_{n,1}q_{n,2}}}{\log g(G_n)} = 0.$$

Suppose that $\delta = \lim_{n\to\infty} \frac{q_{n,2}}{q_{n,1}}$. *Then*

$$\lim_{n\to\infty} \frac{1}{|V(G_n^{[1]})|}\psi_n([a,b]) = \int_a^b \psi(t)\,dt.$$

Proof. Let $G = G_k = (V_1, V_2)$, $|V_1| = n$, $|V_2| = m$ $(n \leqslant m)$, $q_1 = q_{k,1}$ and $q_2 = q_{k,2}$. For $0 \leqslant \alpha < \beta \leqslant \pi$, let

$$h(\theta) := \begin{cases} 1 & \text{if } \alpha \leqslant \theta \leqslant \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $h(0) = h(\pi) = 0$. Then (5) implies that

$$\varphi_k([\alpha,\beta]) = \frac{2n}{\pi} q_1 q_2 (q_1+1) \int_\alpha^\beta \frac{\sin^2\theta}{g(\theta)}\,d\theta + \frac{1}{2}\sum_{[C]}\sum_{m\geqslant 1}^{\infty} (q_1 q_2)^{-m|C|/4}\hat{h}(m|C|/2),$$

where $g(\theta) = (q_1+q_2)(q_1q_2+1) + 2\sqrt{q_1q_2}(q_1-1)(q_2-1)\cos\theta - 4q_1q_2\cos^2\theta$.

Let $Spec(G^{[1]}) = \{\mu_1, \ldots, \mu_n\}$. For the root $v = (q_1q_2)^{-1/2}e^{\sqrt{-1}\,\theta_j}$ of $1 - (\mu_j - q_2 + 1)v + q_1q_2v^2 = 0$ $(1 \leqslant j \leqslant n)$,

$$\alpha \leqslant \theta_j \leqslant \beta \iff 2\sqrt{q_1q_2}\cos\beta \leqslant \mu_j - q_2 + 1 \leqslant 2\sqrt{q_1q_2}\cos\alpha.$$

Thus, we have

$$\varphi_k([\alpha,\beta]) = \psi_k([\cos\beta, \cos\alpha]).$$

Therefore,

$$\lim_{k\to\infty}\frac{1}{|V(G_k^{[1]})|}\psi_k([\cos\beta,\cos\alpha]) = \lim_{k\to\infty}\frac{2}{\pi}q_1q_2(q_1+1)\int_\alpha^\beta \frac{\sin^2\theta}{g(\theta)}\,d\theta + \lim_{k\to\infty}\frac{1}{2}\sum_{[C]}\sum_{m\geqslant 1}^{\infty}(q_1q_2)^{-m|C|/4}\hat{h}(m|C|/2),$$

Let

$$K_k := \frac{1}{2}\sum_{[C]}\sum_{m\geqslant 1}^{\infty}(q_1q_2)^{-m|C|/4}\hat{h}(m|C|/2).$$

Then the proof of Theorem 7 implies that

$$|K_k| \leqslant \frac{1}{\pi}\log\prod_{[C]}(1-(q_1q_2)^{-g(G_k)/4})^{-1}.$$

Since $\lim_{n\to\infty}\frac{\sqrt[4]{q_{n,1}q_{n,2}}}{\log g(G_n)} = 0$, we have

$$\lim_{k\to\infty}|K_k| \leqslant \lim_{k\to\infty}\frac{1}{\pi}\log\prod_{[C]}(1-(q_1q_2)^{-g(G_k)/4})^{-1} = 0$$

and so

$$\lim_{k\to\infty}K_k = 0.$$

Next, let

$$t = \cos\theta.$$

Then we have

$$\frac{2}{\pi}q_1q_2(q_1+1)\int_\alpha^\beta\frac{\sin^2\theta}{g(\theta)}\,d\theta = \frac{2}{\pi}q_1q_2(q_1+1)\int_{\cos\beta}^{\cos\alpha}\frac{\sqrt{1-t^2}}{(q_1q_2+1-2\sqrt{q_1q_2}\,t)(q_1+q_2+2\sqrt{q_1q_2}\,t)}\,dt.$$

Thus,

$$\lim_{k\to\infty}\frac{2}{\pi}q_1q_2(q_1+1)\int_\alpha^\beta\frac{\sin^2\theta}{g(\theta)}\,d\theta = \lim_{k\to\infty}\frac{2}{\pi}\int_{\cos\beta}^{\cos\alpha}\frac{\sqrt{1-t^2}}{\left(1+\frac{1}{q_1q_2}-\frac{2}{\sqrt{q_1q_2}}\,t\right)\left(1+\frac{q_2/q_1-1/q_1}{1+1/q_1}+\frac{2\sqrt{q_2/q_1}}{1+1/q_1}\,t\right)}\,dt.$$

$$= \frac{2}{\pi}\int_{\cos\beta}^{\cos\alpha}\frac{\sqrt{1-t^2}}{1+\delta+2\sqrt{\delta}\,t}\,dt.$$

Therefore, it follows that

$$\lim_{k\to\infty} \frac{1}{|V(G_k^{[1]})|}\psi_k([\cos\beta,\cos\alpha]) = \int_{\cos\beta}^{\cos\alpha} \frac{2}{\pi}\frac{\sqrt{1-t^2}}{1+\delta+2\sqrt{\delta}t}\,dt. \qquad \square$$

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References

[1] G. Ahumada, Fonctions periodiques et formule des traces de Selberg sur les arbres, C. R. Acad. Sci. Parris Ser. I 305 (16) (1987) 709–712.
[2] H. Bass, The Ihara–Selberg zeta function of a tree lattice, Internat. J. Math. 3 (1992) 717–797.
[3] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1979.
[4] C.D. Godsil, B. Mohar, Walk generating functions and spectral measures of infinite graphs, Linear Algebra Appl. 107 (1988) 191–206.
[5] J.L. Gross, T.W. Tucker, Topological Graph Theory, Wiley–Interscience, New York, 1987.
[6] K. Hashimoto, in: Zeta Functions of Finite Graphs and Representations of *p*-Adic Groups, Advanced Studious in Pure Mathematics, Vol. 15, Academic Press, New York, 1989, pp. 211–280.
[7] K. Hashimoto, Artin-type *L*-functions and the density theorem for prime cycles on finite graphs, Internat. J. Math. 3 (1992) 809–826.
[8] M. Hashizume, Selberg trace formula for semiregular graphs, preprint.
[9] Y. Ihara, On discrete subgroups of the two by two projective linear group over *p*-adic fields, J. Math. Soc. Japan 18 (1966) 219–235.
[10] W.-C.W. Li, P. Solé, Spectra of regular graphs and hypergraphs and orthogonal polynomials, European J. Combin. 17 (1996) 461–477.
[11] B.D. McKay, The expected eigenvalue distribution of a large regular graph, Linear Algebra Appl. 40 (1981) 203–216.
[12] H.M. Stark, A.A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1996) 124–165.
[13] T. Sunada, in: *L*-Functions in Geometry and Some Applications, Lecture Notes in Mathematics, Vol. 1201, Springer, New York, 1986, pp. 266–284.
[14] T. Sunada, in: The Discrete and the Continuous, Sugaku Seminar (in Japanese), Vol. 40, Nippon Hyouronsha, Tokyo, 2001, pp. 48–51.
[15] A. Terras, Fourier Analysis on Finite Groups and Applications, Cambridge University Press, Cambridge, 1999.
[16] A.B. Venkov, A.M. Nikitin, The Selberg trace formula, Ramanujan graphs, and some problems of mathematical physics, St. Petersburg Math. J. 5 (3) (1994) 419–484.