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A combinatorial Yamabe flow in three dimensions

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Abstract

A combinatorial version of Yamabe flow is presented based on Euclidean triangulations coming from sphere packings. The evolution of curvature is then derived and shown to satisfy a heat equation. The Laplacian in the heat equation is shown to be a geometric analogue of the Laplacian of Riemannian geometry, although the maximum principle need not hold. It is then shown that if the flow is nonsingular, the flow converges to a constant curvature metric.

Keywords: Curvature flow; Yamabe flow; Sphere packing; Laplacian; Discrete Riemannian geometry

1. Introduction

In his proof of Andreev's theorem in [29], Thurston introduced a conformal geometric structure on two-dimensional simplicial complexes which is an analogue of a Riemannian metric. He then used a version of curvature to prove the existence of circle packings (see also Marden and Rodin [23] for more exposition). Techniques very similar to elliptic partial differential equation techniques were used by Colin de Verdière [7] to study conformal structures and circle packings. Cooper and Rivin in [8] then defined a version of scalar curvature on three-dimensional simplicial complexes and used it to look at rigidity of sphere packings along the lines of Colin de Verdière.

Inspired by this work, Chow and Luo [5] defined several combinatorial Ricci flows on two-dimensional simplicial complexes, one for each constant curvature model space. They were able to show that the flows

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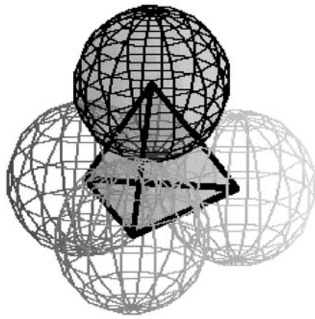

Fig. 1. Tetrahedron with balls at the vertices.

converge to constant curvature if a circle packing whose nerve is the one-skeleton of the triangulation exists. The reader is also directed to some later work of Luo on how these flows evolve the conformal structure [20]. We shall use Cooper and Rivin's combinatorial scalar curvature to define combinatorial Yamabe flow on three-dimensional simplicial complexes which is a three-dimensional analogue of Chow and Luo's work when the triangles are modeled on Euclidean triangles. We shall look at the evolution of curvature from a geometric viewpoint, understanding the heat equation on curvature which is induced by the flow. The flow turns out not to be parabolic in the usual sense of Laplacians on graphs, an analytic property which we study in a related paper [10]. The geometric flow perspective is very much inspired by Richard Hamilton's works on the Ricci flow, e.g. [14].

The combinatorial Yamabe flow is a way of studying prescribed scalar curvature on simplicial complexes, which we might call the combinatorial Yamabe problem. The Yamabe problem has been studied in great detail (see [19] for a good overview). The Riemannian case has been solved by the work of Aubin [3] and Schoen [28]. Yamabe flow in the smooth category has been studied by Hamilton and others. We refer the reader to Hamilton [13] and Ye [31].

2. Geometric structures and combinatorial manifolds

We essentially take our formalism from Cooper and Rivin in [8]. We shall use the notation f_i to denote evaluation of a function *f* at *i* in a finite set and $f(t)$ to denote evaluation at *t* in an interval. Let $\mathscr{S} = \{\mathscr{S}_0, \mathscr{S}_1, \ldots, \mathscr{S}_n\}$ be a simplicial complex of dimension n, where $\mathscr{S}_i$ is the *i*-dimensional skeleton. We define a metric structure as a map

$$r : \mathscr{S}_0 \to (0, \infty)$$

such that for every edge $\{i, j\} \in \mathscr{S}_1$ between vertices i and j, the length of the edge is $\ell_{ij} = r_i + r_j$. Any such metric structure or a particular tetrahedron within such a structure, is called conformal and the set of all is called the conformal class. We can think of this as having an *n*-dimensional sphere packing whose nerve is the collection of edges $\mathscr{S}_1$, although it is not necessarily an actual sphere packing. One such conformal tetrahedron is shown in Fig. 1.

Conformal tetrahedra are also called circumscriptible tetrahedra, and the condition on the edges is equivalent to the condition that there exists a sphere tangent to each of the edges of the tetrahedron

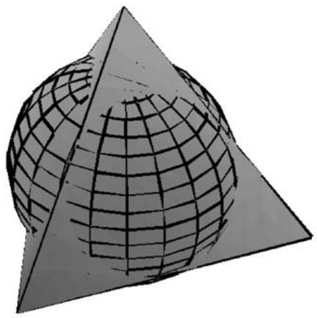

Fig. 2. Tetrahedron with circumscripted sphere.

[1, Chapter 9B.1] as seen in Fig. 2. (We call this sphere the *circumscripted sphere* since it is circumscripted by the tetrahedron.)

The function r determines the two-dimensional faces since there is a one-to-one correspondence between triples (r_i, r_j, r_k) and triples of sides for Euclidean triangles given by

$$r_i = \tfrac{1}{2}(\ell_{ij} + \ell_{ik} - \ell_{jk})$$

and so forth. We shall also put the restriction that each higher-dimensional simplex can be realized as a Euclidean simplex. We shall return to this condition a little later.

Each metric of this type is in some sense conformal to the metric where all $r_i = 1$, since they can be gotten by rescaling the function r at each point. This is similar to multiplying a Riemannian metric g by a function f^2 at every point to get a new metric $f^2 g$ which is conformal to the metric g. The metric structure $\{r_i\}_{i \in \mathscr{S}_0}$ determines the geometry, which comes from the lengths of the edges, similar to the way a Riemannian metric determines the metric space structure of a Riemannian manifold.

In the sequel, we shall limit ourselves primarily to three dimensions. Cooper and Rivin [8, Section 3] observe that for a collection $\{r_i, r_j, r_k, r_\ell\}$ to determine a Euclidean tetrahedron, we can use Descartes' circle theorem, also called Soddy's theorem, which says that four circles in the plane of radii r_i, r_j, r_k, r_ℓ are externally tangent if

$$Q_{ijk\ell} \doteqdot \left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell}\right)^2 - 2\left(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_\ell^2}\right) = 0.$$

For a nice proof of Soddy's theorem, see [26]. We also direct the reader to the interesting article [18] on Descartes' circle theorem. Looking at the proof, it is clear that if this quantity is negative, then we get three circles which are mutually tangent and a circle in the middle which cannot be tangent to all the others, as seen in Fig. 3, and hence we cannot form a Euclidean tetrahedron from spheres of these radii. If $Q_{ijk\ell}$ is positive, we can form a Euclidean tetrahedron corresponding to $\{r_i, r_j, r_k, r_\ell\}$. So our condition for nondegeneracy of the tetrahedron is

$$Q_{ijk\ell} = \left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell}\right)^2 - 2\left(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_\ell^2}\right) > 0.$$

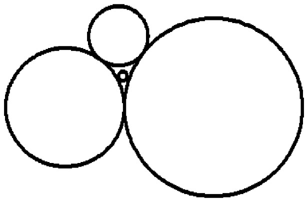

Fig. 3. Failure of four circles to be mutually tangent.

We call $Q_{ijk\ell}$ the nondegeneracy quadratic. As noted in [1, Section 793], $Q_{ijk\ell}$ is really 4 divided by the square of the radius of the sphere tangent to each of the edges (the circumscripted sphere), and is related to the volume in the following way:

$$V_{ijk\ell}^2 = \tfrac{1}{9} r_i^2 r_j^2 r_k^2 r_\ell^2 Q_{ijk\ell}.$$

Thus, if we consider the formula for the square of the volume as formal, the nondegeneracy condition is that $V_{ijk\ell}^2 > 0$.

Now, we shall define a quantity K called the curvature. For a Euclidean tetrahedron with vertices $\{i, j, k, \ell\}$ we define the solid angle $\alpha_{ijk\ell}$ at a vertex i as the area of the triangle on the unit sphere cut out by the planes determined by $\{i, j, k\}$, $\{i, j, \ell\}$, $\{i, k, \ell\}$ where i is the center of the sphere. Note that the solid angle is also sometimes called the trihedral angle. If we define $\beta_{ijk\ell}$ as the dihedral angle in the tetrahedron $\{i, j, k, \ell\}$ along the edge $\{i, j\}$, which is also an angle of the aforementioned spherical triangle, the Gauss–Bonnet theorem gives the formula for the solid angle as

$$\alpha_{ijk\ell} = \beta_{ijk\ell} + \beta_{ikj\ell} + \beta_{i\ell jk} - \pi.$$

Note that solid angles $\alpha_{ijk\ell}$ are symmetric in the last three indices and dihedral angles $\beta_{ijk\ell}$ are symmetric in the first two and in the last two indices. We can now define the curvature K_i at a vertex i as

$$K_i \doteqdot 4\pi - \sum_{\{i,j,k,\ell\} \in \mathscr{S}_3} \alpha_{ijk\ell}.$$

Note that the sum is over j, k, ℓ since the vertex i is fixed.

The curvature K_i can be thought of as a scalar curvature since it measures the difference at a given vertex between the total angles in Euclidean space and the total angles of the complex. It was initially looked at by Cooper and Rivin in [8]. They found that a metric structure cannot be deformed (staying conformal) while keeping the scalar curvature constant.

Constant scalar curvature is a critical point of the total curvature functional

$$T = \sum K_i r_i.$$

Cooper and Rivin showed that the space of nondegenerate simplices is not convex, but that the function T is weakly concave as a function of the r_i and strongly concave if the condition $\sum r_i = 1$ is imposed. We cannot use this to show that there is a unique constant scalar curvature metric on a given conformal class, but any constant scalar curvature metric is a local minimum of T.

Another way to prove this is to look at the functional T as a function of $s_i = 1/r_i$. Then the set of nondegenerate simplices is convex since it is the intersection of a cone with a half-space. However, upon computing the Hessian of T, we can only prove that the function is concave in a neighborhood of constant curvature. Much like scalar curvature in the smooth case, there may be several constant scalar curvature metrics in a given conformal class. However, we have yet to find a complex admitting two constant scalar curvature metrics.

Remark 1. Just because a topological manifold admits a constant sectional curvature 0 metric, a given triangulation of that manifold may not admit a metric with curvature 0. For any vertex transitive triangulation, i.e. a triangulation such that the same number of tetrahedra meet at every vertex, the triangulation with $r_i = 1$ for all $i \in \mathscr{S}_0$ is constant curvature. If d is the degree of the vertex, i.e. the number of tetrahedra meeting at each vertex, the curvature must be $4\pi - d(3\cos^{-1}(\frac{1}{3}) - \pi)$. Hence in order for the curvature to be zero, $d = (4\pi)/(3\cos^{-1}(\frac{1}{3}) - \pi) \approx 22.795$. Thus no vertex transitive triangulation of the torus admits a zero curvature metric. This observation is similar to the one noted in [2] about dynamical triangulations.

Remark 2. The curvature considered here is different than the one considered in the Regge calculus by Regge [27] and others (for instance [11,12,9]). Our curvature is concentrated at the vertices while Regge's curvature is concentrated at the edges in dimension 3. The solid angle which we use is reminiscent of the interpretation of Ricci curvature as solid angle deficit (see, for instance, the introduction of [4]).

3. Combinatorial Yamabe flow

We now define combinatorial Yamabe flow on the metric structure as

$$\frac{\mathrm{d}r_i}{\mathrm{d}t} = -K_i r_i \tag{1}$$

for each $i \in \mathscr{S}_0$. Note how similar this looks to the Yamabe flow on Riemannian manifolds, which is

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij},$$

where g_{ij} is the Riemannian metric and R is its scalar curvature. In particular, both preserve their respective conformal class. We use the term 'combinatorial' since this is used by Chow and Luo, but it is really more of a piecewise linear or piecewise Euclidean flow because it depends on the geometry of the triangulation and not just the topological, or combinatorial, structure. Much like the Yamabe flow and Ricci flow, the evolution of curvature will play a key role in understanding the behavior of this equation. Next we shall compute this evolution.

Recall the Schläfli formula, which, for a Euclidean tetrahedron denoted by the complex $\{\mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3\}$, gives that

$$\sum_{\{i,j\} \in \mathscr{T}_1} \ell_{ij} \mathrm{d}\beta_{ijk\ell} = 0$$

(see Milnor [25] for a proof). We can reorganize this as Cooper and Rivin [8] do to get

$$\sum_{i \in \mathscr{T}_0} r_i (\mathrm{d}\beta_{ijk\ell} + \mathrm{d}\beta_{ikj\ell} + \mathrm{d}\beta_{i\ell jk}) = \sum_{i \in \mathscr{T}_0} r_i \mathrm{d}\alpha_{ijk\ell} = 0$$

or

$$r_i \frac{\partial \alpha_{ijk\ell}}{\partial r_i} + r_j \frac{\partial \alpha_{jik\ell}}{\partial r_i} + r_k \frac{\partial \alpha_{kij\ell}}{\partial r_i} + r_\ell \frac{\partial \alpha_{\ell ijk}}{\partial r_i} = 0.$$

Since there are only four vertices in $\mathcal{T}_0$, we may denote the solid angle at vertex i by α_i without fear of confusion. Then, we consider

$$A \doteqdot \sum_{i \in \mathcal{T}_0} r_i \alpha_i, \tag{2}$$

so

$$\begin{aligned} \mathrm{d}A &= \sum_{i \in \mathcal{T}_0} \alpha_i \mathrm{d}r_i + \sum_{i \in \mathcal{T}_0} r_i \mathrm{d}\alpha_i \\ &= \sum_{i \in \mathcal{T}_0} \alpha_i \mathrm{d}r_i \end{aligned}$$

by the Schläfli formula. We thus have

$$\frac{\partial A}{\partial r_i} = \alpha_i$$

and hence

$$\frac{\partial \alpha_i}{\partial r_j} = \frac{\partial^2 A}{\partial r_i \partial r_j} = \frac{\partial \alpha_j}{\partial r_i}$$

by commuting the partial derivatives. In our expanded notation, which we shall use for complexes larger than one simplex, this says

$$\frac{\partial \alpha_{ijk\ell}}{\partial r_j} = \frac{\partial \alpha_{jik\ell}}{\partial r_i}.$$

Using our derivation from the Schläfli formula we also have

$$r_i \frac{\partial \alpha_{ijk\ell}}{\partial r_i} + r_j \frac{\partial \alpha_{ijk\ell}}{\partial r_j} + r_k \frac{\partial \alpha_{ijk\ell}}{\partial r_k} + r_\ell \frac{\partial \alpha_{ijk\ell}}{\partial r_\ell} = 0. \tag{3}$$

This equality has a much more geometric interpretation; it says that the directional derivative of the angle $\alpha_{ijk\ell}$ in the direction of scaling (r_i, r_j, r_k, r_ℓ) is zero, since if these are scaled equally, the new tetrahedron is similar to the original and hence all angles remain the same.

The evolution of curvature is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}K_i &= -\sum_{\{i,j,k,\ell\}\in\mathscr{S}_3} \frac{\mathrm{d}}{\mathrm{d}t}\alpha_{ijk\ell} \\
&= -\sum_{\{i,j,k,\ell\}\in\mathscr{S}_3} \left(\frac{\partial\alpha_{ijk\ell}}{\partial r_i}\frac{\mathrm{d}r_i}{\mathrm{d}t} + \frac{\partial\alpha_{ijk\ell}}{\partial r_j}\frac{\mathrm{d}r_j}{\mathrm{d}t} + \frac{\partial\alpha_{ijk\ell}}{\partial r_k}\frac{\mathrm{d}r_k}{\mathrm{d}t} + \frac{\partial\alpha_{ijk\ell}}{\partial r_\ell}\frac{\mathrm{d}r_\ell}{\mathrm{d}t}\right) \\
&= \sum_{\{i,j,k,\ell\}\in\mathscr{S}_3} \left(\frac{\partial\alpha_{ijk\ell}}{\partial r_i}K_i r_i + \frac{\partial\alpha_{ijk\ell}}{\partial r_j}K_j r_j + \frac{\partial\alpha_{ijk\ell}}{\partial r_k}K_k r_k + \frac{\partial\alpha_{ijk\ell}}{\partial r_\ell}K_\ell r_\ell\right) \\
&= \sum_{\{i,j,k,\ell\}\in\mathscr{S}_3} \left(\frac{\partial\alpha_{ijk\ell}}{\partial r_j}r_j(K_j - K_i) + \frac{\partial\alpha_{ijk\ell}}{\partial r_k}r_k(K_k - K_i) + \frac{\partial\alpha_{ijk\ell}}{\partial r_\ell}r_\ell(K_\ell - K_i)\right)
\end{aligned}$$

using (3). We call the coefficients

$$\Omega_{ijk\ell} \doteqdot \frac{\partial\alpha_{ijk\ell}}{\partial r_j} r_j.$$

In this notation, we see that the evolution of curvature is

$$\frac{\mathrm{d}}{\mathrm{d}t}K_i = \sum_{\{i,j,k,\ell\}\in\mathscr{S}_3} [\Omega_{ijk\ell}(K_j - K_i) + \Omega_{ikj\ell}(K_k - K_i) + \Omega_{i\ell jk}(K_\ell - K_i)].$$

In order to compute the coefficients $\Omega_{ijk\ell}$, we need to compute the partial derivatives of the solid angles. We do this computation using the following formulas from Euclidean geometry. Recall that $\alpha_{ijk\ell}$ refers to the solid angle of tetrahedron $\{i, j, k, \ell\}$ at i and that $\beta_{ijk\ell}$ refers to the dihedral angle of tetrahedron $\{i, j, k, \ell\}$ along the edge $\{i, j\}$. We also need the face angles. Denote the angle of the triangle $\{i, j, k\}$ at the vertex i by γ_{ijk}. We can then use the law of cosines and the expression for area in terms of sines to compute γ_{ijk} as

$$\begin{aligned}
\cos\gamma_{ijk} &= \frac{\ell_{ij}^2 + \ell_{ik}^2 - \ell_{jk}^2}{2\ell_{ij}\ell_{ik}} = \frac{r_i^2 + r_i r_j + r_i r_k - r_j r_k}{(r_i + r_j)(r_i + r_k)}, \\
\sin\gamma_{ijk} &= \frac{2A_{ijk}}{\ell_{ij}\ell_{ik}} = \frac{2\sqrt{r_i r_j r_k (r_i + r_j + r_k)}}{(r_i + r_j)(r_i + r_k)},
\end{aligned} \tag{4}$$

where $\ell_{ij} = r_i + r_j$ is the length of edge $\{i, j\}$ and $A_{ijk} = \sqrt{r_i r_j r_k (r_i + r_j + r_\ell)}$ is the area of triangle $\{i, j, k\}$ by Heron's formula.

Using the law of cosines for the face angles (4), we can compute the evolution of the face angles, which turns out to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_{ijk} = -2\,\frac{A_{ijk}}{P_{ijk}}\left(\frac{K_j - K_i}{\ell_{ij}} + \frac{K_k - K_i}{\ell_{ik}}\right),$$

where we have introduced the notation $P_{ijk} = 2(r_i + r_j + r_k)$ for the perimeter of the triangle $\{i, j, k\}$. It should be noted that this computation was entirely formal, and is thus the same formula derived by Chow and Luo [5] for simplicial surfaces. If we define the curvature of a surface to be $k_i = 2\pi - \sum \gamma_{ijk}$ then the formula for evolution of the face angles implies that the evolution of curvature is in fact parabolic in the usual sense of Laplacians on graphs (this is studied in greater detail in [10]). The curvature evolution turns out to be

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = (\Delta k)_i,$$

where the Laplacian is the one defined by He in [16]. We shall explore this aspect in the next section.

The face angles are used to compute the dihedral angles and solid angles via spherical geometry. If we consider the solid angle formed by three planes, say those determined by $\{i, j, k\}$, $\{i, j, \ell\}$, and $\{i, k, \ell\}$, we see that the planes intersect the sphere and form a spherical triangle. It is clear that the angles of this triangle are the dihedral angles $\beta_{ijk\ell}, \beta_{ikj\ell}, \beta_{i\ell jk}$ and that the length of the sides of this triangle are the face angles $\gamma_{ijk}, \gamma_{ij\ell}, \gamma_{ik\ell}$. Hence the relationship between the dihedral angles and the face angles can be expressed in terms of the spherical law of cosines, which says

$$\cos\beta_{ijk\ell} = \frac{\cos\gamma_{ik\ell} - \cos\gamma_{ijk}\cos\gamma_{ij\ell}}{\sin\gamma_{ijk}\sin\gamma_{ij\ell}}. \tag{5}$$

We use formula (5) for the cosine of the dihedral angle and the following expression for the volume of simplex $\{i, j, k, \ell\}$:

$$V_{ijk\ell} = \frac{2A_{ijk}A_{ij\ell}\sin\beta_{ijk\ell}}{3\ell_{ij}}$$

to compute

$$\begin{aligned}\frac{\partial\beta_{ijk\ell}}{\partial r_i} = \frac{2r_ir_jr_k^2r_\ell^2}{3P_{ijk}P_{ij\ell}V_{ijk\ell}}\Bigg[&-\frac{1}{r_k^2} - \frac{1}{r_\ell^2} - 2\frac{r_j}{r_i}\left(\frac{1}{r_ir_k} + \frac{1}{r_ir_\ell} + \frac{1}{r_kr_\ell}\left(2 + \frac{r_j}{r_i}\right)\right)\\ &+ \left(\frac{1}{r_j} - \frac{1}{r_i}\right)\left(\frac{2}{r_i} + \frac{1}{r_k} + \frac{1}{r_\ell}\right)\Bigg],\end{aligned}$$

$$\frac{\partial\beta_{ijk\ell}}{\partial r_k} = \frac{r_i^2r_j^2r_\ell}{3P_{ijk}V_{ijk\ell}}\left[\left(\frac{1}{r_i} + \frac{1}{r_j}\right)\left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} - \frac{1}{r_\ell}\right)\right].$$

Now, compute the evolution of the solid angles using the formula for the area of a spherical triangle, $\alpha_{ijk\ell} = \beta_{ijk\ell} + \beta_{ikj\ell} + \beta_{i\ell jk} - \pi$. We get

$$\begin{aligned}\frac{\partial\alpha_{ijk\ell}}{\partial r_i} = -\frac{8r_j^2r_k^2r_\ell^2}{3P_{ijk}P_{ij\ell}P_{ik\ell}V_{ijk\ell}}\Bigg[&\left(\frac{2}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell}\right) + \frac{r_j}{r_i}\left(\frac{1}{r_i} + \frac{1}{r_k} + \frac{1}{r_\ell}\right)\\ &+ \frac{r_k}{r_i}\left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_\ell}\right) + \frac{r_\ell}{r_i}\left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k}\right) + (2r_i + r_j + r_k + r_\ell)Q_{ijk\ell}\Bigg],\end{aligned} \tag{6}$$

which we see is always negative if the tetrahedron is nondegenerate, i.e. $Q_{ijk\ell} > 0$. The other partial derivatives look like

$$\frac{\partial\alpha_{ijk\ell}}{\partial r_j} = \frac{4r_ir_jr_k^2r_\ell^2}{3P_{ijk}P_{ij\ell}V_{ijk\ell}}\left(\frac{1}{r_i}\left(\frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell}\right) + \frac{1}{r_j}\left(\frac{1}{r_i} + \frac{1}{r_k} + \frac{1}{r_\ell}\right) - \left(\frac{1}{r_k} - \frac{1}{r_\ell}\right)^2\right) \tag{7}$$

which we would like to say is positive, but is not always (although in the case of most “good” tetrahedra, it is positive).

Finally, we sum cyclically in the last three indices and find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\beta_{ijk\ell}+\beta_{ikj\ell}+\beta_{i\ell jk})=\Omega_{ijk\ell}(K_i-K_j)+\Omega_{ikj\ell}(K_i-K_k)+\Omega_{i\ell jk}(K_i-K_\ell),$$

with

$$\Omega_{ijk\ell}=\frac{4r_ir_j^2r_k^2r_\ell^2}{3P_{ijk}P_{ij\ell}V_{ijk\ell}}\left(\frac{1}{r_i}\left(\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)+\frac{1}{r_j}\left(\frac{1}{r_i}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)-\left(\frac{1}{r_k}-\frac{1}{r_\ell}\right)^2\right).$$

Thus, the evolution of curvature is

$$\begin{aligned}\frac{\mathrm{d}K_i}{\mathrm{d}t}&=-\sum_{\{i,j,k,\ell\}\in\mathcal{S}_3}\left(\frac{\mathrm{d}}{\mathrm{d}t}\beta_{ijk\ell}+\frac{\mathrm{d}}{\mathrm{d}t}\beta_{ikj\ell}+\frac{\mathrm{d}}{\mathrm{d}t}\beta_{i\ell jk}\right)\\&=\sum_{\{i,j,k,\ell\}\in\mathcal{S}_3}[\Omega_{ijk\ell}(K_j-K_i)+\Omega_{ikj\ell}(K_k-K_i)+\Omega_{i\ell jk}(K_\ell-K_i)].\end{aligned}$$

We can define our Laplacian as

$$(\Delta f)_i=\sum_{\{i,j,k,\ell\}\in\mathcal{S}_3}[\Omega_{ijk\ell}(f_j-f_i)+\Omega_{ikj\ell}(f_k-f_i)+\Omega_{i\ell jk}(f_\ell-f_i)] \tag{8}$$

in order to write the evolution of curvature as

$$\frac{\mathrm{d}K_i}{\mathrm{d}t}=(\Delta K)_i.$$

Unfortunately the coefficients $\Omega_{ijk\ell}$ are not always positive. We notice that Δ is self-adjoint with respect to the inner product

$$\langle f,g\rangle=\sum_{i\in\mathcal{S}_0}f_ig_ir_i$$

and satisfies

$$\sum_{i\in\mathcal{S}_0}\Delta f_ir_i=0,$$

which is analogous in the smooth category to $\int\Delta f\,\mathrm{d}x=0$ by the divergence theorem.

4. Combinatorial Laplacians

In this section, we investigate the Laplacian defined in (8) and similar operators. In [16], He looked at variations of the type we are studying, that is

$$\frac{\mathrm{d}r_i}{\mathrm{d}t}=-f_ir_i$$

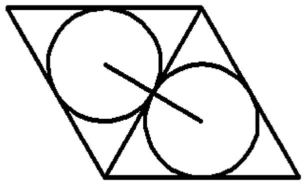

Fig. 4. Two triangles with inscribed circles and dual edge.

in two dimensions. He then derived a curvature evolution which involved a combinatorial Laplacian. The work is similar to the smooth derivation of the evolution of curvature under a conformal flow on a Riemannian manifold given by

$$\frac{\partial}{\partial t} g_{ij} = -f g_{ij}.$$

This was studied by Hamilton in two dimensions for the Ricci flow, and the result is

$$\frac{\partial R}{\partial t} = \Delta_g f + Rf$$

where Δ_g is the Laplacian with respect to the metric g (see [13]).

He's Laplacian is the same Laplacian derived by Chow and Luo in [5], which can be written as

$$\Delta f_i = \sum_{\{i,j\}\in\mathscr{S}_1} \frac{\ell_{ij}^*}{\ell_{ij}}(f_j - f_i) \tag{9}$$

using the relation

$$\frac{\partial \theta_{ijk}}{\partial r_j} r_j = \frac{r_{ijk}}{\ell_{ij}}, \tag{10}$$

where θ_{ijk} is the angle at vertex i in triangle $\{i, j, k\}$ and r_{ijk} is the length of the radius of the circle inscribed in triangle $\{i, j, k\}$. Hence to get (9), we simply add

$$\frac{\partial \theta_{ijk}}{\partial r_j} r_j + \frac{\partial \theta_{ij\ell}}{\partial r_j} r_j = \frac{\ell_{ij}^*}{\ell_{ij}},$$

where $\ell_{ij}^* = r_{ijk} + r_{ij\ell}$ is the length of the dual edge (see Fig. 4). The dual vertex $\star\{i, j, k\}$ to the triangle $\{i, j, k\}$ is the center of the inscribed circle, while the edge $\star\{i, j\}$ dual to $\{i, j\}$ is the edge which goes from the dual vertex $\star\{i, j, k\}$ to the dual vertex $\star\{i, j, \ell\}$ and is perpendicular to $\{i, j\}$. This Laplacian is similar to the Laplacians found in the image processing literature which we shall now describe.

Combinatorial Laplacians on piecewise linear surfaces are used quite a bit in image processing, for instance [24,30]. There is a very clear description by Hirani [17] which defines the Laplace–Beltrami

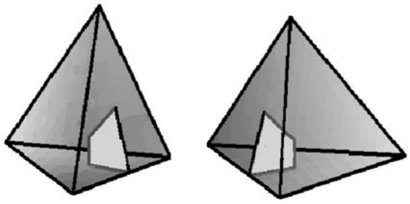

Fig. 5. Two views of the dual to an edge.

operator on functions defined at the vertices as

$$\Delta f_i = \frac{1}{|\star\{i\}|}\sum_{\{i,j\}\in\mathscr{S}_1}\frac{|\star\{i,j\}|}{|\{i,j\}|}(f_j - f_i)$$
$$= \frac{1}{V_i^*}\sum_{\{i,j\}\in\mathscr{S}_1}\frac{\ell_{ij}^*}{\ell_{ij}}(f_j - f_i), \tag{11}$$

where $\star\sigma^k$ is the $(n-k)$-dimensional dual of the k-dimensional simplex σ^k and $|\sigma^k|$ is the k-dimensional volume of the simplex. The second line uses our notation, where the dual of an edge ℓ_{ij}^* and the volume of a vertex V_i^* are defined appropriately. Note that He's Laplacian is exactly this, except for the volume factor in front, where duality comes from the inscribed circles. We also note that in most of the image processing literature, the two-dimensional dual comes from the center of the circumscribed circle instead of the inscribed circle described here.

It is interesting to note the geometric justification for the formula (11). If we consider the integral of the Laplace–Beltrami operator and use Stokes' theorem, we find

$$\int_U \Delta f \, \mathrm{d}V = \int_{\partial U} \frac{\mathrm{d}f}{\mathrm{d}n}\,\mathrm{d}S$$

where $\mathrm{d}f/\mathrm{d}n$ is the normal derivative and $\mathrm{d}S$ is the surface measure. We easily see that if we take U to be $\star\{i\}$ then the normal derivative is

$$\frac{f_j - f_i}{\ell_{ij}},$$

ℓ_{ij}^* is the surface measure, and V_i^* is the volume measure.

Recall the definition (8) of the Laplacian, we gave in the previous section. This Laplacian would be related to the Laplacian in this section (11), if we had an analogue of (10). The dual $\star\{i,j\}$ to an edge $\{i,j\}$ is a surface which goes through $\star T$ for any tetrahedron T containing $\{i,j\}$ and is perpendicular to $\{i,j\}$. Two pictures of the piece of $\star\{i,j\}$ in one tetrahedron can be seen in Fig. 5. We shall denote the area shown in one tetrahedron by $A_{ijk\ell}$, where the region is perpendicular to the edge $\{i,j\}$. The region whose area is $A_{ijk\ell}$ has four sides:

- the radius r_{ijk} of the circle inscribed in the face $\{i,j,k\}$ which intersects $\{i,j\}$,
- the radius $r_{ij\ell}$ of the circle inscribed in the face $\{i,j,\ell\}$ which intersects $\{i,j\}$,

- the line from the center of the circumscripted sphere to the face $\{i, j, k\}$ which is perpendicular to the plane determined by $\{i, j, k\}$, and
- the line from the center of the circumscripted sphere to the face $\{i, j, \ell\}$ which is perpendicular to the plane determined by $\{i, j, \ell\}$.

It can also be decomposed as two right triangles, each having hypotenuse coming from the radius $r_{ijk\ell}$ of the circumscripted sphere which intersects the edge $\{i, j\}$. One triangle has a leg coming from the radius r_{ijk} of the circle inscribed in $\{i, j, k\}$ and the other has a leg coming from the radius $r_{ij\ell}$ of the circle inscribed in $\{i, j, \ell\}$. The legs and hypotenuse meet at the same point in $\{i, j\}$. We shall call the lengths of the other respective legs $h_{ijk,\ell}$ and $h_{ij\ell,k}$ to denote that they are heights of tetrahedra which make up the tetrahedron $\{i, j, k, \ell\}$; i.e., if we let c denote the center of the circumscripted sphere, $h_{ijk,\ell}$ is the height of tetrahedron $\{i, j, k, c\}$ with base $\{i, j, k\}$. Thus the volume of tetrahedron $\{i, j, k, \ell\}$ is decomposed as

$$V_{ijk\ell} = \sum_{\{i,j,k\}\in\mathscr{T}_2} \tfrac{1}{3} A_{ijk} h_{ijk,\ell}. \tag{12}$$

If the center of the circumscripted sphere is outside the tetrahedron, we define $h_{ijk,\ell}$ to be negative if the center is on the opposite side of the plane determined by $\{i, j, k\}$ from the tetrahedron. This way (12) is still satisfied. The $h_{ijk,\ell}$ are symmetric in the first three indices, and the fourth indicates to which tetrahedron the h corresponds. The area $A_{ijk\ell}$ is then computed as

$$A_{ijk\ell} = \tfrac{1}{2} h_{ijk,\ell} r_{ijk} + \tfrac{1}{2} h_{ij\ell,k} r_{ij\ell},$$

where this may be negative. We can now show the analogue of (10).

Lemma 3.

$$\frac{\partial \alpha_{ijk\ell}}{\partial r_j} r_i r_j = \frac{A_{ijk\ell}}{\ell_{ij}},$$

where $A_{ijk\ell}$ is the (signed) area of the dual region to the side $\{i, j\}$ in the tetrahedron $\{i, j, k, \ell\}$ as described above.

Proof. We simply compute. Recall that P_{ijk} is the perimeter of $\{i, j, k\}$, A_{ijk} is the area of $\{i, j, k\}$, r_{ijk} is the radius of the circle inscribed in $\{i, j, k\}$, $r_{ijk\ell}$ is the radius of the sphere circumscripted by $\{i, j, k, \ell\}$, and $h_{ijk,\ell}$ is signed height of the tetrahedron defined by the center of the circumscripted sphere in $\{i, j, k, \ell\}$ and i, j, and k, with base $\{i, j, k\}$. The sign of $h_{ijk,\ell}$ is defined so that

$$3V_{ijk\ell} = h_{ijk,\ell} A_{ijk} + h_{ij\ell,k} A_{ij\ell} + h_{ik\ell,j} A_{ik\ell} + h_{jk\ell,i} A_{jk\ell}.$$

We have the following relations:

$$A_{ijk} = \tfrac{1}{2} r_{ijk} P_{ijk},$$
$$r_{ijk} = \frac{2A_{ijk}}{P_{ijk}} = \sqrt{\frac{r_i r_j r_k}{r_i + r_j + r_k}}.$$

We also know that

$$\frac{4}{r_{ijk\ell}^2} = Q_{ijk\ell} = \left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_\ell}\right)^2 - 2\left(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_\ell^2}\right).$$

The square of the height $h_{ijk,\ell}$ can be computed using the Pythagorean theorem:

$$\begin{aligned}h_{ijk,\ell}^2 &= r_{ijk\ell}^2 - r_{ijk}^2\\ &= \frac{r_{ijk\ell}^2 r_{ijk}^2}{4}\left(4\frac{r_i+r_j+r_k}{r_ir_jr_k} - \left(\left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)^2 - 2\left(\frac{1}{r_i^2}+\frac{1}{r_j^2}+\frac{1}{r_k^2}+\frac{1}{r_\ell^2}\right)\right)\right)\\ &= \frac{r_{ijk\ell}^2 r_{ijk}^2}{4}\left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}-\frac{1}{r_\ell}\right)^2.\end{aligned}$$

In order to determine $h_{ijk,\ell}$ correctly, we notice that if the center of the circumscripted sphere is on face $\{i, j, k\}$ then

$$r_{ijk}^2 = r_{ijk\ell}^2$$

and hence

$$\begin{aligned}&4\frac{r_i+r_j+r_k}{r_ir_jr_k} = \left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)^2 - 2\left(\frac{1}{r_i^2}+\frac{1}{r_j^2}+\frac{1}{r_k^2}+\frac{1}{r_\ell^2}\right),\\ &0 = \left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}-\frac{1}{r_\ell}\right)^2.\end{aligned}$$

We easily see that if the vector from the center to the side $\{i, j, k\}$ is in the same direction as the outward pointing normal of side $\{i, j, k\}$ then

$$\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k} > \frac{1}{r_\ell}$$

and if the vector is in the opposite direction than the outward pointing normal then

$$\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k} < \frac{1}{r_\ell}.$$

Hence the signed height $h_{ijk,\ell}$ is

$$h_{ijk,\ell} = \frac{r_{ijk\ell}r_{ijk}}{2}\left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}-\frac{1}{r_\ell}\right),$$

which may be negative. Notice that this assures

$$3V_{ijk\ell} = h_{ijk,\ell}A_{ijk} + h_{ij\ell,k}A_{ij\ell} + h_{ik\ell,j}A_{ik\ell} + h_{jk\ell,i}A_{jk\ell},$$

where $V_{ijk\ell}$ is the volume of $\{i, j, k, \ell\}$. The dual area is computed

$$\begin{aligned}A_{ijk\ell} &= \tfrac{1}{2} h_{ijk,\ell} r_{ijk} + \tfrac{1}{2} h_{ij\ell,k} r_{ij\ell} \\ &= \frac{r_{ijk\ell}}{4}\left(\frac{4A_{ijk}^2}{P_{ijk}^2}\left(\frac{1}{r_i}+\frac{1}{r_j}+\frac{1}{r_k}-\frac{1}{r_\ell}\right)+\frac{4A_{ij\ell}^2}{P_{ij\ell}^2}\left(\frac{1}{r_i}+\frac{1}{r_j}-\frac{1}{r_k}+\frac{1}{r_\ell}\right)\right) \\ &= \frac{4r_i^2 r_j^2 r_k^2 r_\ell^2 \ell_{ij}}{3V_{ijk\ell} P_{ijk} P_{ij\ell}}\left(\frac{1}{r_i}\left(\frac{1}{r_j}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)+\frac{1}{r_j}\left(\frac{1}{r_i}+\frac{1}{r_k}+\frac{1}{r_\ell}\right)-\left(\frac{1}{r_k}-\frac{1}{r_\ell}\right)^2\right). \quad \square\end{aligned}$$

Corollary 4. *We have that*

$$\Delta f_i = \frac{1}{r_i} \sum_{\{i,j\} \in \mathscr{S}_1} \frac{\ell_{ij}^*}{\ell_{ij}} (f_j - f_i),$$

where ℓ_{ij}^* *is the area dual to the side* $\{i, j\}$.

Proof. This follows from the fact that the dual area ℓ_{ij}^* is simply

$$\ell_{ij}^* = \sum_{\{i,j,k,\ell\} \in \mathscr{S}_3} A_{ijk\ell},$$

where the sum is over k and ℓ, i.e., all tetrahedra incident on the edge $\{i, j\}$. $\square$

Note the similarity to Hirani's definition (11). Also note that since ℓ_{ij}^* may be negative. This is not always a Laplacian on graphs in the usual sense (see, for instance [6]). Still, we can prove the maximum principle in more general circumstances, as seen in [10].

5. Convergence to constant curvature

In this section, we shall show that if the solution exists for all time, the curvatures converge to a constant. We restrict ourselves to well behaving solutions.

Definition 5. $\{r_i(t)\}_{i \in \mathscr{S}_0}$ is a nonsingular solution of the combinatorial Yamabe flow if there exists $\delta > 0$ such that for each for $t \in [0, \infty)$ it satisfies (1), $Q_{ijk\ell} > 0$ for all $\{i, j, k, \ell\} \in \mathscr{S}_3$, and

$$\frac{r_i}{\sum_{j \in \mathscr{S}_0} r_j} \geqslant \delta$$

for each $i \in \mathscr{S}_0$.

Consider the average scalar curvature

$$k \doteq \frac{\sum_{i \in \mathscr{S}_0} K_i r_i}{\sum_{i \in \mathscr{S}_0} r_i}.$$

This can be thought of as an analogue of the average scalar curvature functional:

$$\frac{\int_M R \,\mathrm{d}V}{\int_M \,\mathrm{d}V}$$

on a Riemannian manifold. Note that this curvature really is an average in the sense that

$$K_{\min} \leqslant k \leqslant K_{\max},$$

if $K_{\min}$ and $K_{\max}$ are the minimal and maximal curvatures.

Proposition 6. *For any nonsingular solution $\{r_i(t)\}_{i\in\mathscr{S}_0}$ there is a number $k(\infty)$ such that the average scalar curvature $k(t)$ and all of the curvatures $K_i(t)$ converge to $k(\infty)$ as $t \to \infty$.*

Proof. It is easy to see that k is decreasing along the flow by a direct computation:

$$\begin{aligned}\frac{\mathrm{d}k}{\mathrm{d}t} &= -\frac{\sum_{i\in\mathscr{S}_0} K_i^2 r_i}{\sum_{i\in\mathscr{S}_0} r_i} + \frac{(\sum_{i\in\mathscr{S}_0} K_i r_i)^2}{(\sum_{i\in\mathscr{S}_0} r_i)^2}\\ &= -\frac{\sum_{i\in\mathscr{S}_0}\sum_{j\in\mathscr{S}_0}(K_i^2 r_i r_j - K_i K_j r_i r_j)}{(\sum_{i\in\mathscr{S}_0} r_i)^2}.\end{aligned}$$

Rearranging terms we get

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\frac{\sum_{i\in\mathscr{S}_0}\sum_{j\in\mathscr{S}_0}(K_i - K_j)^2 r_i r_j}{(\sum_{i\in\mathscr{S}_0} r_i)^2}. \tag{13}$$

Furthermore,

$$K_{\min} \leqslant k,$$

where $K_{\min}$ is the minimum of the curvature, which is bounded below by $4\pi - 2\pi d_{\max}$ if $d_{\max}$ is the maximum number of tetrahedra incident on any one vertex. Thus k is decreasing and bounded below, so it must converge to a limit $k(\infty)$. Moreover, the time derivative of k must go to zero or k would not be bounded below. Hence $\mathrm{d}k/\mathrm{d}t$ converges to zero. By formula (13) and the fact that

$$\frac{r_i}{\sum r_j} \geqslant \delta$$

we see that $(K_i - K_j)^2 \to 0$ for all pairs of vertices. Hence the curvatures becomes constant; this constant must be $k(\infty)$. □

6. Long-term existence

In this section, we will classify the possible long-term behavior.

Proposition 7. *All solutions to the combinatorial Yamabe flow on a maximal time interval* $[0, T)$ *must fit into one of the following categories*:

- *It is nonsingular* (*see Definition* 5).
- $T = \infty$ *and for some* $i \in \mathscr{S}_0$,
$$\frac{r_i}{\sum_{j \in \mathscr{S}_0} r_j} \to 0$$
as $t \to \infty$.
- $T < \infty$ *and for some* $\{i, j, k, \ell\} \in \mathscr{S}_3$, $Q_{ijk\ell} \to 0$ *as* $t \nearrow T$.

Solutions with an infinite time interval are covered, and so we need only look at finite time singularities. A priori, the following may also happen:

- there exists $i \in \mathscr{S}_0$ such that $r_i(t) \to 0$ as $t \nearrow T$,
- there exists $i \in \mathscr{S}_0$ such that $r_i(t) \to \infty$ as $t \nearrow T$.

We can consider $L_i(t) = \log r_i(t)$. Notice that

$$\frac{\mathrm{d}L_i}{\mathrm{d}t} = -K_i.$$

If $L_i \to \pm\infty$ in finite time, then $\mathrm{d}L_i/\mathrm{d}t = K_i \to \pm\infty$ in finite time. This is impossible, though, since

$$4\pi - 2\pi d_{\max} \leqslant K_i \leqslant 4\pi.$$

So finite time singularities occur only because $Q_{ijk\ell} \to 0$. The case of $T = \infty$ and there exists i such that

$$\frac{r_i}{\sum r_j} \to 0$$

is somehow analogous to collapse in the smooth case. Hence Proposition 6 is an analogue of Hamilton's theorem on nonsingular solutions [15].

7. Further remarks

It is quite easy to implement the combinatorial Yamabe flow numerically. With the help of Lutz's work (see [22,21]) on small triangulations of manifolds, the author has been able to run examples of the combinatorial Yamabe flow on manifolds homeomorphic to the sphere, torus, $S^2 \times S^1$, and $S^2 \tilde{\times} S^1$. In each case the combinatorial Yamabe flow found a constant curvature metric. However, these were all done with relatively small triangulations so many of the possible degeneracies coming from a poor triangulation cannot occur.

As noted in Remark 1, any metric which is vertex transitive, i.e., has the same number of tetrahedra incident on each vertex, will have a constant curvature metric where all r_i are equal. It is not clear that this is the only constant curvature metric in the conformal class. Numerical simulation has been unable to produce any examples of other constant curvature metrics, however. In the case of the torus, the metric where all r_i are equal is usually not one where the curvatures are zero. In all simulations of small triangulations of the torus tried, the combinatorial Yamabe flow converges to a positively curved constant curvature metric.

Constant curvature as indicated in this paper is a weak condition. A metric such that the sum of the dihedral angles along each edge is equal to 2π is a Euclidean structure on a manifold, but a metric where the solid angles equal 4π is not necessarily. The difference is something like the difference between constant sectional curvatures of zero and constant scalar curvature of zero; it is possible to have a metric where the latter is true but not the former.

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