MATHEMATICS

ALMOST COMMUTING MATRICES ARE NEAR COMMUTING MATRICES

BY

W. A. J. LUXEMBURG [1]) AND R. F. TAYLOR

(Communicated by Prof. A. C. ZAANEN at the meeting of December 20, 1969)

In the "Research Problem" section of the October issue of the American Mathematical Monthly vol. 67 (1969), p. 925, P. ROSENTHAL raised the following question: "Are almost commuting matrices near commuting matrices?". More precisely, P. Rosenthal asked whether there exists a mapping f of the set of positive real numbers $R^+=\{x: x>0\}$ into R^+ with the property that $f(\varepsilon) \to 0$ as $0<\varepsilon \to 0$ and such that for any pair of $n \times n$-matrices A, B of norm less than one which almost commute in the sense that the norm of the commutator $AB-BA$ is less than ε, there exists a pair of commuting matrices A', B' of norm less than one and such that the norms of the matrices $A-A'$ and $B-B'$ are less than $f(\varepsilon)$.

The purpose of this note is to show that the answer is affirmative. The technique used in deriving the result is that of nonstandard analysis which for this purpose works exceedingly well. The details are as follows.

Let $E_n (n=1, 2, \ldots)$ denote the n-dimensional complex Euclidean vector space, and let $M(E_n)$ denote the Banach algebra of all $n \times n$-matrices with complex entries. If $A=(a_{ij}) \in M(E_n)$, then $\|A\|$ denotes the Euclidean norm $\left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$ of A. By $M_1(E_n)$ we shall denote the set of all elements $A \in M(E_n)$ such that $\|A\| \leqslant 1$.

We shall now prove the following theorem.

Theorem. *For every $n (n=1, 2, \ldots)$ there exists a mapping f_n of $R^+=\{x: x>0\}$ into R^+ with the following properties:* (i) *$f_n(\varepsilon) \to 0$ as $0<\varepsilon \to 0$, and* (ii) *for every pair of matrices $A, B \in M_1(E_n)$ with $\|AB-BA\| \leqslant \varepsilon$ there exists a pair of matrices $A', B' \in M_1(E_n)$ such that $\|A-A'\|<f_n(\varepsilon)$, $\|B-B'\|<f_n(\varepsilon)$ and $A'B'=B'A'$.*

Proof. As we indicated above the proof will be based on the techniques of nonstandard analysis. For terminology and notation concerning the nonstandard reals the reader is referred to [1] or [2].

---

[1]) Work on this paper was supported in part by NSF Grant GP-14133.

Let $\varepsilon > 0$ be given, then it is clear that there exists always a positive number $\delta = \delta(\varepsilon) > 0$ which has the following property:

(*) For every pair of matrices $A, B \in M_1(E_n)$ such that $\|AB - BA\| \leqslant \varepsilon$ there exists a pair of *commuting* matrices $A', B' \in M_1(E_n)$ such that $\|A - A'\| \leqslant \delta(\varepsilon)$ and $\|B - B'\| \leqslant \delta(\varepsilon)$.

Indeed, we may take $\delta = \delta(\varepsilon) = 1$ for all $\varepsilon > 0$.

More precisely, we shall proof that for every $\varepsilon > 0$ the number $\delta_0(\varepsilon) = \inf(\delta : \delta > 0$ and δ has the property $(\star))$ has itself property $(\star)$. To this end, let $A, B \in M_1(E_n)$ and let $\|AB - BA\| \leqslant \varepsilon$. Then from the definition of δ_0 it follows that for every $k = 1, 2, \ldots$ there exists a pair of matrices A_k, $B_k \in M_1(E_n)$ such that $A_k B_k = B_k A_k$ and $\|A - A_k\| \leqslant \delta_0 + 1/k$ and $\|B - B_k\| \leqslant \delta_0 + 1/k$. Since $A_k, B_k \in M_1(E_n)$ $(k = 1, 2, \ldots)$ there is no loss in generality to assume that $\|A_k - A_0\| \to 0$ as $k \to \infty$ and $\|B_k - B_0\| \to 0$ as $k \to \infty$. Then

$$A_0 B_0 - B_0 A_0 = A_0(B_0 - B_k) + (A_0 - A_k) B_k + B_k (A_k - A_0) + (B_k - B_0) A_0$$

shows that $A_0 B_0 = B_0 A_0$ and the norm inequalities $\|A - A_k\| \leqslant \delta_0 + 1/k$, $\|B - B_k\| \leqslant \delta_0 + 1/k$ $(k = 1, 2, \ldots)$ imply that $\|A - A_0\| \leqslant \delta_0$ and $\|B - B_0\| \leqslant \delta_0$. Hence, we conclude that $\delta_0(\varepsilon)$ has property $(\star)$ for all $\varepsilon > 0$, and so also $\delta_0(\varepsilon) > 0$ for all $\varepsilon > 0$.

This suggests that for every $\varepsilon > 0$ we shall denote by $f_n(\varepsilon)$ the smallest positive number δ with property $(\star)$.

In order to prove the required result we have to show that $f_n(\varepsilon) \to 0$ as $0 < \varepsilon \to 0$. To this end, we shall use nonstandard analysis.

Let $^*R$ denote as usual the nonstandard reals and let $^*E_n$ denote the corresponding nonstandard n-dimensional Euclidean vector space. Then $^*M(^*E_n)$ is the nonstandard algebra of all the nonstandard $n \times n$-matrices whose entries are nonstandard complex numbers. To the function f_n there corresponds a function $^*f_n$, which with respect to $^*M_1(^*E_n)$ has the same properties as f_n has with respect to $M_1(E_n)$.

It is well-known (see [1] or [2]) that the statement $^*f_n(h)$ is infinitesimal for all positive infinitesimals $h \in {}^*R$ is equivalent to the required statement, namely $f_n(\varepsilon) \to 0$ as $0 < \varepsilon \to 0$.

For that reason assume that $A, B \in {}^*M_1(^*E_n)$ are such that $\eta = \|AB - BA\| =_1 0$, that is, the commutator's norm η of A and B is infinitesimal. Since $\|A\| \leqslant 1$ and $\|B\| \leqslant 1$ it follows immediately that the entries a_{ij} of A and the entries b_{ij} of B $(i, j = 1, 2, \ldots, n)$ are finite. Let $a_{ij}' = st(a_{ij})$ and $b_{ij}' = st(b_{ij})$ $(i, j = 1, 2, \ldots, n)$ be the standard parts of the numbers a_{ij} and b_{ij} respectively. These are uniquely determined standard numbers which are infinitely close to the original numbers. Thus $\|A - A'\| =_1 0$ and $\|B - B'\| =_1 0$. Furthermore, from $\|A'\| \leqslant \|A\| + \|A - A'\| \leqslant 1 + \|A - A'\| =_1 1$, and the fact that $\|A'\|$ is a standard number it follows that $\|A'\| \leqslant 1$, and similarly $\|B'\| \leqslant 1$. Since the norms $\|A - A'\|$ and $\|B - B'\|$ are infinitesimal the proof will be completed if we can show that A' commutes with B'. Indeed, then we have shown that $^*f_n(\eta) \leqslant \varepsilon$ for all positive infinitesimals

$0<\eta \in {}^*R$, and for all positive standard numbers $\varepsilon>0$, and so ${}^*f_n(\eta)=_1 0$ for all $0<\eta=_1 0$. To this end, observe that

$$A'B'-B'A'=A'(B'-B)+(A'-A)B+(AB-BA)+B(A-A')+ \\ +(B-B')A',$$

$\|A\|\leqslant 1$, $\|B\|\leqslant 1$, $\|A'\|\leqslant 1$ and $\|B'\|\leqslant 1$ imply that

$$\|A'B'-B'A'\|\leqslant\|B'-B\|+\|A'-A\|+\|AB-BA\|+\|A-A'\|+ \\ +\|B-B'\|=_1 0.$$

Since the norm $\|A'B'-B'A'\|$ is standard and at the same time infinitesimal it is equal to zero, and so it follows that $A'B'=B'A'$, which completes the proof.

Problems. There are a number of problems which are closely related to the result of the paper and which were also indicated by P. Rosenthal.

For instance, it is not known whether f_n is independent of n. If this would be the case then it would follow easily that any pair of commuting compact operators on a separable Hilbert space have a common proper invariant subspace. Whether this is indeed the case is still an open problem. Another interesting problem related to the one above is the following. Does every bounded operator on a separable Hilbert space which commutes with a compact operator has an proper invariant subspace?

For a discussion of these and related problems we refer the reader to the thesis [3] of the junior author.

California Institute of Technology
Pasadena, California

REFERENCES

1. Luxemburg, W. A. J., What is nonstandard analysis? To appear in the Proceeding of the conference on the Foundations of Mathematics, Annapolis 1968.
2. Robinson, Abraham, Non-standard Analysis, Studies in Logic and the Foundations of Mathematics, Amsterdam 1967.
3. Taylor, R. F., Invariant subspaces in Hilbert and normed spaces. Ph. D. Thesis, California Institute of Technology 1968.