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SELF-SIMILAR SOLUTIONS AND SEMI-LINEAR WAVE EQUATIONS IN BESOV SPACES

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ABSTRACT. – We prove that the initial value problem for semi-linear wave equations is well-posed in the Besov space $\dot{B}_2^{s_p,\infty}(\mathbb{R}^n)$, where the nonlinearity is of type u^p, with $p \in \mathbb{N}$ and $s_p = \frac{n}{2} - \frac{2}{p} > \frac{1}{2}$. This allows to obtain self-similar solutions as well as to recover previous results under weaker smallness assumptions on the data. © 2000 Éditions scientifiques et médicales Elsevier SAS

Introduction

We are interested in the Cauchy problem for the following semi-linear wave equation

$$\begin{cases} \Box u = \pm u^p, \\ u(x,0) = u_0(x), \\ \partial_t u(x,0) = u_1(x), \end{cases} \tag{1}$$

for $n \geqslant 2$. Given our main interest, scaling will play an important role: it reads:

$$\begin{cases} u_0(x) \longrightarrow u_{0,\lambda}(x) = \lambda^{\frac{2}{p-1}} u_0(\lambda x), \\ u_1(x) \longrightarrow u_{1,\lambda}(x) = \lambda^{\frac{2}{p-1}+1} u_1(\lambda x), \\ u(x,t) \longrightarrow u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda t). \end{cases} \tag{2}$$

Let s_p be such that $s_p - \frac{n}{2} = -\frac{2}{p-1}$. The homogeneous Sobolev space $\dot{H}^{s_p}$ is expected to be the "critical" space for well-posedness as its norm is invariant by the scaling (2). Indeed, well-posedness holds for initial data $(u_0, u_1) \in H^{s_p} \times H^{s_p-1}$ ([10]), for $p \geqslant p^\star = \frac{n+3}{n-1}$, or equivalently $s_p \geqslant 1/2$. Below $p^\star$, concentration effects take over scaling, and (1) is ill-posed below some critical value above s_p ([10] or [20] for recent results). It should be noted that for radially symmetric data well-posedness holds up to the scaling for $s_p > \frac{1}{2n}$ ([10]), but we will not generalize such results here. Thus, the Cauchy problem for (1) with data in Sobolev spaces appears to be well-understood (we focus here on local in time theory or global for small data). However, it seems interesting to look for solutions which would be invariant by rescaling, and the Sobolev theory fails to provide such solutions, as the initial data is required to be homogeneous but fails to be in the correct Sobolev space. This failure was overcome recently

by the use of suitable functional spaces, modeled out of the linear wave operator, and which allow homogeneous Cauchy data to be chosen. This approach was initiated in [5,6] for the Schrödinger equation, and subsequently used for (1) in [17,16,12]. However, we lack a good understanding of the functional spaces introduced by these authors, and known admissible data providing self-similar solutions have to be a lot more regular than one would expect if one considers the correct Sobolev space where (1) is well-posed. We intend to bridge the gap between the classical Sobolev theory and these recent works, in the range $p > p^\star$, as we did in the context of the Schrödinger equation in [15,13].

In the same way as [13], a natural extension to $\dot{H}^{s_p}$ is the homogeneous Besov space $\dot{B}_2^{s_p,\infty}$, and unlike its Sobolev counterpart, it contains homogeneous functions. Let us recall that

$$f(x) \in \dot{H}^{s_p} \Leftrightarrow \int |\xi|^{2s_p} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \approx \sum_j 2^{2s_p j} \int\limits_{2^j<|\xi|<2^{j+1}} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi < +\infty$$

and one can weaken this requirement to

$$f(x) \in \dot{B}_2^{s_p,\infty} \Leftrightarrow \sup_j 2^{2s_p j} \int\limits_{2^j<|\xi|<2^{j+1}} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi < +\infty. \tag{3}$$

From this definition, we obtain immediately $1/|x|^{\frac{2}{p-1}} \in \dot{B}_2^{s_p,\infty}$, and thus solving the Cauchy problem in such a space will, among other things, provide self-similar solutions.

It should be noted that previous works on self-similar solutions allow for lower values of $p > p_0$, and that (1) is known to admit global (weak) solutions for small compactly supported smooth data ([7]) for $p_c < p < p^\star$, with $p_c < p_0$. Nevertheless none of these solutions are known to preserve the regularity of the initial data, and while our techniques fail, regularity questions below $p^\star$ remain of interest and probably require more than just the Strichartz estimates we will use. The exact form of the non-linearity in (1) is relevant only with respect to the methods which will be used. Essentially one can deal with more general non-linearities, but this requires a lot more technicalities, which are irrelevant to the equation itself, and have to do with composition in Besov spaces. Thus, by restricting ourselves to non-linearities of type $\bar{u}^{p_1}u^{p_2}$ where p_1 and p_2 are integers, we don't have to worry about further regularity assumption on the non-linearity, and evaluating the non-linearity u^p or the difference $u^p - v^p$ is equivalent. It should be noted however that one expects the same results to hold for non-integer values of p, or generic nonlinearities which behave like u^p. We refer to [18] for definition of such suitable classes of functions $F_p(u)$.

In the next section, we will state and prove various existence results for data in Besov spaces. Then, in the last section we will deal with the specific case of self-similar solutions, and the related problem of long term asymptotics and scattering.

To end this section let us recall definitions and results on Besov spaces and their characterizations via frequency localization ([1] for details).

DEFINITION 1.– *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\phi} = 1$ for $|\xi| \leqslant 1$ and $\widehat{\phi} = 0$ for $|\xi| > 2$, $\phi_j(x) = 2^{nj}\phi(2^j x)$, $S_j = \phi_j * \cdot$, $\Delta_j = S_{j+1} - S_j$. Let f be in $\mathcal{S}'(\mathbb{R}^n)$.*

- *If $s < n/p$, or if $s = n/p$ and $q = 1$, f belongs to $\dot{B}_p^{s,q}$ if and only if the following two conditions are satisfied*:
  - *The partial sum $\sum_{-m}^{m} \Delta_j(f)$ converges to f as a tempered distribution.*
  - *The sequence $\varepsilon_j = 2^{js}\|\Delta_j(f)\|_{L^p}$ belongs to l^q.*

- *If $s > n/p$, or $s = n/p$ and $q > 1$, let us denote $m = E(s - n/p)$, the integer part of $s - n/p$. Then $\dot{B}_p^{s,q}$ is the space of distributions f, modulo polynomials of degree less than $m+1$, such that*:
  - *We have $f = \sum_{-\infty}^{\infty} \Delta_j(f)$ for the quotient topology.*
  - *The sequence $\varepsilon_j = 2^{js}\|\Delta_j(f)\|_{L^p}$ belongs to l^q.*

Another closely related type of space will be of help:

DEFINITION 2. – *Let $u(x,t) \in \mathcal{S}'(\mathbb{R}^{n+1})$, Δ_j be a frequency localization with respect to the x variable. We will say that $u \in \mathcal{L}_t^\rho(\dot{B}_p^{s,q})$ iff*

$$2^{js}\|\Delta_j u\|_{L_t^\rho(L_x^p)} = \varepsilon_j \in l^q, \tag{4}$$

and other requirements are the same as in the previous definition.

To end this section, we recall two lemmas, which allow for an easy characterization of Besov spaces, depending on the sign of s.

LEMMA 1. – *Let $s > 0$, E a Banach functional space, $q \in [1, +\infty]$, and define $\dot{B}_E^{s,q}$ by the following condition*:

$$f \in \dot{B}_E^{s,q} \quad \textit{iff} \quad 2^{js}\|\Delta_j f\|_E = \varepsilon_j \in l^q.$$

Then, if $f = \sum_j f_j$, where $\operatorname{supp}\hat{f}_j \in B(0, 2^j)$ and $(2^{js}\|f_j\|_E)_j \in l^q$, we have $f \in \dot{B}_E^{s,q}$ with appropriate norm control.

Its counterpart for $s < 0$ reads

LEMMA 2. – *Let $s < 0$, $\dot{B}_E^{s,q}$ defined as in Lemma* 1. *Then, an equivalent characterization of $f \in \dot{B}_E^{s,q}$ is*

$$2^{js}\|S_j u\|_E = \varepsilon_j \in l^q. \tag{5}$$

We omit both proofs, which involve a summation over large or small frequencies along with Young inequality for discrete sequences. The interest of Lemma 2 stems from that, for $s < 0$, controlling a $\Delta_j u$ piece gives the same control over $S_j u$, while Lemma 1 allows for considering sums of pieces localized in balls rather than in annuli when $s > 0$.

1. Existence and regularity theorems

1.1. Statements of results

We intend to prove the following theorem:

THEOREM 1. – *Let $p \in \mathbb{N}$, $p > p^\star$, $(u_0, u_1) \in (\dot{B}_2^{s_p,\infty}, \dot{B}_2^{s_p-1,\infty})$, such that*:

$$\|u_0\|_{\dot{B}_2^{s_p,\infty}} + \|u_1\|_{\dot{B}_2^{s_p-1,\infty}} < \varepsilon_0(p,n). \tag{6}$$

Then there exists a global solution of (1) *such that*:

$$u(x,t) \in L_t^\infty\big(\dot{B}_2^{s_p,\infty}\big) \quad \textit{and} \quad \partial_t u(x,t) \in L_t^\infty\big(\dot{B}_2^{s_p-1,\infty}\big), \tag{7}$$

$$u(x,t) \underset{t\to 0}{\longrightarrow} u_0(x) \quad \textit{weakly}. \tag{8}$$

Moreover, this solution is unique under an additional assumption $u \in \mathcal{L}_t^{\bar{p}}(\dot{B}_{\bar{q}}^{\bar{s},\infty})$ *and*

$$\sup_j \left(2^{j\bar{s}} \|u\|_{L_t^{\bar{p}}(L_x^{\bar{q}})}\right) < \varepsilon_1, \tag{9}$$

with $\eta = 1 - \frac{p^\star}{p}$, $\bar{s} = s_p - \frac{1-\eta}{2}$, $\bar{p} = \frac{2(n+1)}{(n-1)(1-\eta)}$ *and* $\bar{q} = \frac{2(n+1)}{(n-1+2\eta)}$.

The uniqueness condition is, as usual in such problems for which solutions are obtained by a fixed-point argument, related to the auxiliary space needed for such an argument. Condition (8) relates to the Besov spaces we consider (see [3,14] for discussions on such problems). Indeed strong continuity at $t = 0$ is forbidden, and therefore we obtain a somewhat weaker result than what is usually meant for "well-posedness". However, if one has some additional regularity on the initial data, then this regularity is preserved for the solution, namely we obtain:

THEOREM 2. – *Let* $(u_0, u_1) \in (\dot{H}^{s_p}, \dot{H}^{s_p-1})$ *verify the hypothesis of Theorem* 1. *Then the global solution obtained by Theorem* 1 *is such that*:

$$u(x,t) \in C_t\left(\dot{H}^{s_p}\right), \qquad \partial_t u(x,t) \in C_t\left(\dot{H}^{s_p-1}\right). \tag{10}$$

This result can be seen as an extension of global well-posedness in Sobolev spaces for small data, as one can construct initial data with an arbitrary norm in the Sobolev space, but a small one in the Besov space. It should be noted however that the smallness constant $\varepsilon(p,n)$ tends to zero as p gets close to $p^\star$, so that in such case there is no easy way to compare Theorem 2 and results from [11]. Theorem 2 can be extended to replace $\dot{H}^{s_p}$ by $\dot{B}_2^{s_p,q}$ for any $1 \leqslant q < \infty$. We restricted ourselves to $q = 2$ to recover the usual Sobolev spaces.

We can as well construct a local in time theory for $\dot{B}_2^{s_p,q}$, where $1 \leqslant q < \infty$, obtaining:

THEOREM 3. – *Let* $(u_0, u_1) \in (\dot{B}_2^{s_p,q}, \dot{B}_2^{s_p-1,q})$. *Then there exists a local in time solution of* (1) *such that*:

$$u(x,t) \in C\left([0,T], \dot{B}_2^{s_p,q}\right), \qquad \partial_t u(x,t) \in C\left([0,T], \dot{B}_2^{s_p-1,q}\right). \tag{11}$$

Moreover, this solution is unique under the additional assumption:

$$u \in \mathcal{L}_{[0,T]}^{\bar{p}}\left(\dot{B}_{\bar{q}}^{\bar{s},q}\right). \tag{12}$$

In addition, one has further regularity improvements: let

$$u(x,t) = u_L(x,t) + w(x,t), \tag{13}$$

where u_L is the solution to the linear wave equation. Then we have:

THEOREM 4. – *Let* $(u_0, u_1) \in (\dot{B}_2^{s_p,q}, \dot{B}_2^{s_p-1,q})$, *and* u *be the corresponding* (*local or global*) *solution to* (1). *Then*:

$$w(x,t) \in C_t\left(\dot{B}_2^{s_p,\sup(1,\frac{q}{p})}\right). \tag{14}$$

Thus, the term coming from the non-linear part of the equation has slightly more regularity than the free solution.

1.2. Proofs

We now prove Theorem 1, following a classical strategy: we rely on estimates for the linear part in order to chose appropriate functional spaces for which a fixed point can be set up. Recall the fundamental solution, which gives the solution to the linear equation:

$$u_L(t) = \dot{W}(t)u_0 + W(t)u_1, \tag{15}$$

where $W(t)$ (resp. $\dot{W}(t)$) is a Fourier multiplier with symbol $\sin(t|\xi|)/|\xi|$ (resp. $\cos(t|\xi|)$). We consider now the integral equation

$$u(x,t) = u_L(t) + \int_0^t W(t-s)u^p(x,s)\,\mathrm{d}s. \tag{16}$$

Let us recall briefly how (16) can be solved for initial data in Sobolev spaces ([10] and references therein). Let $\alpha = \frac{2(n+1)}{n-1}$, $\beta = \frac{2(n+1)}{n+3}$. We have the so-called Strichartz estimate:

PROPOSITION 1. – *The following space-time estimate* ([19]) *holds*

$$\|u\|_{C_t(\dot{H}^{\frac{1}{2}})} + \|u\|_{L^\alpha_{t\times x}} \lesssim \|\Box u\|_{L^\beta_{t\times x}} + \|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|u_1\|_{\dot{H}^{-\frac{1}{2}}}. \tag{17}$$

One can prove well-posedness in H^{s_p} for (1), letting $u \in C_t(H^{s_p}) \cap L^\alpha_t(H^{s_p-\frac{1}{2}}_\alpha)$. Then $u^{p-1} \in L^{\frac{n+1}{2}}_{t\times x}$ by Sobolev embedding and interpolation. This leads to $u^p \in L^\beta_t(H^{s_p-\frac{1}{2}}_\beta)$ by Kato–Ponce type estimates ("rule for fractional derivatives") and the result follows from the Strichartz estimates. From here, and since the Besov spaces we are interested in are defined via frequency localization, it seems obvious to require for a solution u to (16):

$$2^{js_p}\Delta_j u \in L^\infty_t\big(L^2_x\big), \tag{18}$$

$$2^{j(s_p-\frac{1}{2})}\Delta_j u \in L^\alpha_t\big(L^\alpha_x\big), \tag{19}$$

as the linear part u_L verifies these estimates for initial data in the appropriate Besov spaces, thanks to a localized (in frequency) version of (17). If we are able to prove

$$2^{j(s_p-\frac{1}{2})}\Delta_j\big(u^p\big) \in L^\beta_{t\times x} \tag{20}$$

when u verifies (18) and (19), this will allow to apply Picard fixed point theorem using the Strichartz estimate. Recall

$$\bar{s} = s_p - \frac{1-\eta}{2}, \quad \frac{1}{\bar{p}} = \frac{1-\eta}{\alpha} \quad \text{and} \quad \frac{1}{\bar{q}} = \frac{1-\eta}{\alpha} + \frac{\eta}{2},$$

and remark that by complex interpolation (with parameter η)

$$u \in L^\infty_t\big(\dot{B}^{s_p,\infty}_2\big) \cap \mathcal{L}^\alpha_t\big(\dot{B}^{s_p-\frac{1}{2},\infty}_\alpha\big) \hookrightarrow u \in \mathcal{L}^{\bar{p}}_t\big(\dot{B}^{\bar{s},\infty}_{\bar{q}}\big).$$

To get (20) we prove:

PROPOSITION 2. – *Let* $u \in \mathcal{L}_t^{\bar{p}}(\dot{B}_{\bar{q}}^{\bar{s},\infty})$. *Then* $u^p \in \mathcal{L}_t^{\beta}(\dot{B}_{\beta}^{s_p-\frac{1}{2},\infty})$, *and*

$$\sup_j 2^{j(s_p-\frac{1}{2})}\left\|\Delta_j\left(u^p\right)\right\|_{L^{\beta}_{t\times x}} \lesssim \sup_j 2^{j\bar{s}}\|\Delta_j u\|^p_{L_t^{\bar{p}}(L_x^{\bar{q}})}. \tag{21}$$

In order to prove the proposition, we will make use of Lemma 1, since $s_p - \frac{1}{2} > 0$. Namely, writing u as a telescopic sum:

$$u^p = \sum_j (S_{j+1}u)^p - (S_j u)^p = \sum_j \Delta_j u\big((S_{j+1}u)^{p-1} + \cdots + (S_j u)^{p-1}\big), \tag{22}$$

we are left to consider p pieces, each being an infinite sum of functions localized at frequencies $|\xi| \lesssim 2^j$. All terms are essentially the same up to shifts in indices, and we will only deal with the last one,

$$v = \sum_j \Delta_j u (S_j u)^{p-1} = \sum_j v_j.$$

Note that decomposition (22) is a paraproduct type formula ([2]) in its most simple version. The difficulty here arises from getting an estimate on $S_j u$. It doesn't seem possible to get a uniform estimate, but on the other hand since we know that $\Delta_j u$ has regularity $\bar{s}$ rather than just $s_p - \frac{1}{2}$, we will be allowed to lose some regularity in estimating the $S_j u$ piece. In order to recover (20) we would like to get

$$2^{j\frac{\eta}{2(p-1)}} S_j u \in L_t^{\tilde{p}(p-1)}\big(L_x^{\frac{(n+1)(p-1)}{2-\eta}}\big), \tag{23}$$

with

$$\frac{1}{\bar{p}} + \frac{1}{\tilde{p}} = \frac{1}{\bar{q}} + \frac{2-\eta}{(n+1)} = \frac{1}{\beta},$$

and we would like to have $\tilde{p}(p-1) = \bar{p}$. All together these equalities lead to the choice of $\eta = 1 - \frac{p^\star}{p}$. Then, using Bernstein inequality,

$$\|\Delta_j u\|_{L_x^r} \lesssim 2^{nj(\frac{1}{\bar{q}}-\frac{1}{r})}\|\Delta_j u\|_{L_x^{\bar{q}}}$$

with $\bar{s} - \frac{n}{\bar{q}} = -\frac{\eta}{2(p-1)} - \frac{n}{r}$. Computing r gives $\frac{(n+1)(p-1)}{2-\eta}$ (as expected as scaling ties up all exponents):

$$\frac{1}{r} = \frac{2}{(n+1)(p-1)} + \frac{\eta}{p-1}\left(\frac{1}{\alpha} - \frac{1}{2}\right).$$

Thus, we get (23) up to the replacement of Δ_j by S_j; using Lemma 2 we can replace Δ_j by S_j and obtain the desired estimate. This completes the proof of Proposition 2.

We shall need later a similar estimate for a product of p functions: let us define:

$$E_q = L_t^\infty\big(\dot{B}_2^{s_p,q}\big), \tag{24}$$

$$F_q^{(\gamma)} = \mathcal{L}_t^\gamma\left(\dot{B}_\gamma^{s_p-\frac{1}{2},q}\right) = \left\{u(x,t) \mid \left(2^{j(s_p-\frac{1}{2})}\|\Delta_j u\|_{L^\gamma_{t,x}}\right)_{j\in\mathbb{Z}} \in l^q\right\} \tag{25}$$

and

$$G_q^{(\eta)} = \mathcal{L}_t^{\bar{p}}\big(\dot{B}_{\bar{q}}^{\bar{s},q}\big) = \big\{u(x,t) \mid \big(2^{j\bar{s}}\|\Delta_j u\|_{L_t^{\bar{p}}(L_x^{\bar{q}})}\big)_{j\in\mathbb{Z}} \in l^q\big\}. \tag{26}$$

Then we have:

PROPOSITION 3. – *Let $f_k \in G_\infty^{(\eta)}$ for $1 \leqslant k \leqslant p$. Then $\Pi_1^p f_k \in F_\infty^{(\beta)}$ and*

$$\big\|\Pi_1^p f_k\big\|_{F_\infty^{(\beta)}} \lesssim \Pi_1^p \|f_1\|_{G_\infty^{(\eta)}} \cdots \|f_k\|_{G_\infty^{(\eta)}} \cdots \|f_p\|_{G_\infty^{(\eta)}}. \tag{27}$$

The proof of Proposition 3 is a simple modification of the proof of Proposition 2, and we will therefore omit it.

We are now in a position to prove Theorem 1. Setting up a fixed point for (16) is essentially straightforward: we chose $G_\infty^{(\eta)} = G$ as our Banach space, and denote $E = E_\infty$, $F = F_\infty^{(\alpha)}$, and $F' = F_\infty^{(\beta)}$. Recall that localizing (17) in frequency gives:

$$2^{\frac{j}{2}}\|\Delta_j u\|_{L_t^\infty(L_x^2)} + \|\Delta_j u\|_{L_{t,x}^\alpha} \lesssim \|\Box\Delta_j u\|_{L_{t,x}^\beta} + 2^{\frac{j}{2}}\|\Delta_j u_0\|_{L^2} + 2^{-\frac{j}{2}}\|\Delta_j u_1\|_{L^2}. \tag{28}$$

Therefore we control the linear part (recall $\|u\|_G \leqslant \|u\|_{E\cap F}$ by interpolation),

$$\big\|u_L(x,t)\big\|_{E\cap F} \lesssim \|u_0\|_{\dot{B}_2^{sp,\infty}} + \|u_1\|_{\dot{B}_2^{sp-1,\infty}}, \tag{29}$$

and combining (27) and (28), we get for u and $\tilde{u}$ two solutions of (16):

$$\|u - \tilde{u}\|_{E\cap F} \lesssim \big\|u^p - \tilde{u}^p\big\|_{F'} \lesssim \big(\|u\|_G^{p-1} + \|\tilde{u}\|_G^{p-1}\big)\|u - \tilde{u}\|_G. \tag{30}$$

Then, applying Banach fixed point theorem in G, we get a solution $u \in G$ which in addition belongs to $E \cap F$, for $\|u_L\|_G$ small enough. This is implied by the smallness condition on the initial data, thanks to (29). The uniqueness condition (which is really uniqueness in a ball of G) follows from the fixed point. We have therefore proved Theorem 1, except for condition (8).

Before proceeding, let us make a remark: one might naturally ask why uniqueness in the whole space G couldn't be obtained a posteriori by an inductive argument on small time intervals. This relates intimately to the continuity with respect to the initial data, which we cannot obtain in our situation, due to the nature of the Besov spaces involved. Indeed, even for the linear part $u_L(x,t)$ we only have weak convergence to the initial data (as illustrated by homogeneous initial data).

Let us now prove (8); thanks to the remark above, we are left to deal with the non-linear part. From $u^p \in F'$, applying Bernstein inequality we get:

$$\big\|\Delta_j(u^p)\big\|_{L_x^2} \lesssim 2^{nj(\frac{1}{\beta}-\frac{1}{2})}\big\|\Delta_j(u^p)\big\|_{L_x^\beta},$$

so that

$$\big\|W(t-s)\Delta_j\big(u^p\big)(s)\big\|_{L_x^2} \lesssim 2^{j(\frac{1}{n+1}-1)}\big\|\Delta_j\big(u^p\big)(s)\big\|_{L_x^\beta}.$$

Integrating over time and using Hölder inequality gives the following estimate for the non-linear part w:

$$\|\Delta_j w\|_{L_t^\infty(L_x^2)} \lesssim 2^{j(\frac{-n}{n+1})}\big\|\Delta_j\big(u^p\big)(s)\big\|_{L_{t,x}^\beta} \times t^{\frac{2(n+1)}{n-1}},$$

$$\|\Delta_j w\|_{L_t^\infty(L_x^2)} \lesssim 2^{j(\frac{2}{p-1}-\frac{n}{\beta})}\big\|u^p\big\|_F \times t^\alpha,$$

which gives the convergence to zero when $t \to 0$ in $\dot{B}_2^{-\frac{2}{p-1}+\frac{n}{\beta},\infty}$ and thus in the weak sense. This concludes the proof of Theorem 1.

We proceed with the proofs of Theorems 2 and 3. They are essentially routine modifications of the previous argument, and therefore we only sketch the proofs. The key ingredient is the following proposition:

PROPOSITION 4. – *Let* $f_1 \in G_2^{(\eta)}$ *and* $f_k \in G_\infty^{(\eta)}$ *for* $2 \leqslant k \leqslant p$. *Then* $\Pi_1^p f_k \in F_2^{(\beta)}$ *and*

$$\left\| \Pi_1^p f_k \right\|_{F_2^{(\beta)}} \lesssim \Pi_1^p \| f_1 \|_{G_2^{(\eta)}} \cdots \| f_k \|_{G_\infty^{(\eta)}} \cdots \| f_p \|_{G_\infty^{(\eta)}}, \tag{31}$$

with the same constant as in (27).

Indeed, looking at the proof of Proposition 2, we remark that replacing $l^\infty(j)$ by $l^2(j)$ for one factor in the product can be carried along all estimates, thus giving (31). This completes the proof of Proposition 4.

We now proceed with the proof of Theorem 2. As obviously for initial data in the Sobolev spaces the linear part $v_L \in E_2 \cap F_2^\alpha$, the property $u^{(n)} \in E_2 \cap F_2^\alpha$ can be carried along all iterates $u^{(n)}$ of the fixed point argument in the proof of Theorem 1, thus in effect proving Theorem 2 (continuity in time follows in the same way). Note that since the constant in (31) is the same as in (27), we do not need to further restrict the size of the initial data in the (large) Besov space in order to remain in the Sobolev space. Of course the same argument applies as well if we replace 2 by any $1 \leqslant q < \infty$.

In order to obtain the local in time result, Theorem 3, we have to carry along two modifications: the first one is replacing the time interval, which was up to now $[0, \infty)$, by $[0, T]$ in all estimates. This amounts to a simple rewriting. Next, we have to check why for small T we can meet the smallness requirement, which would write

$$\| u_L \|_{G_\infty^\eta(T)} < \varepsilon_0. \tag{32}$$

This in turn follows from a combination of two facts: as $u_L \in G_q^\eta(T)$, the characterizing dyadic block has the following property:

$$\lim_{|j| \to \infty} 2^{j\bar{s}} \| \Delta u_L \|_{L_t^{\bar{p}}(L_x^{\bar{q}})} = 0. \tag{33}$$

On the other hand, at fixed j we have:

$$\lim_{T \to 0} 2^{j\bar{s}} \| \Delta u_L \|_{L_{[0,T]}^{\bar{p}}(L_x^{\bar{q}})} = 0. \tag{34}$$

Splitting $\| u_L \|_{G_\infty^\eta(T)}$ in two parts gives us smallness for small time, and continuity in time is again carried along the iterates. This ends the proof of Theorem 3.

Let us make two additional comments. The last argument actually proves that one could indeed state a local in time result for the completion of the Schwartz class in the Besov spaces with $q = \infty$, and get strong continuity in such a case. Another feature of the proofs concerns the use of Strichartz estimates: unlike previous work, we only made use of the main Strichartz estimate, together with a frequency localization argument. Thus, generalized Strichartz estimates ([8]) are not really needed to construct solutions, and only provide some extra amount of information on such solutions.

2. Asymptotics, self-similar solutions and scattering

2.1. Statements of results

Recall that one of our initial motivations was to obtain self-similar solutions: these solutions will be invariant by the rescaling (2), and therefore have the form:

$$u(x,t) = \frac{1}{t} U\left(\frac{x}{t}\right), \tag{35}$$

and this requires for initial data to be homogeneous of degree $\frac{-2}{p-1}$, in other words

$$u_0(x) = \frac{\phi_0(x/|x|)}{|x|^{\frac{2}{p-1}}}$$

and $u_0 \in \dot{B}_2^{s_p,\infty}$ translates into $\phi_0 \in \dot{H}^{s_p}(S^{n-1})$ ([4]) (and the same applies to u_1 by shifting the regularity). Then we have:

THEOREM 5. – *Let* $(\phi_0, \phi_1) \in (\dot{H}^{s_p}(S^{n-1}), \dot{H}^{s_p-1}(S^{n-1}))$ *such that*

$$\|\phi_0\|_{\dot{H}^{s_p}(S^{n-1})} + \|\phi_1\|_{\dot{H}^{s_p-1}(S^{n-1})} \lesssim \varepsilon_0,$$

then, defining $u_0(x) = \phi_0(x/|x|)/|x|^{\frac{2}{p-1}}$ *and* $u_1(x) = \phi_1(x/|x|)/|x|^{\frac{2}{p-1}+1}$ (1) *has a unique self-similar solution*

$$u(x,t) = \frac{1}{t} U\left(\frac{x}{t}\right),$$

such that

$$\|U\|_{\dot{B}_{\tilde{q}}^{\tilde{s},\tilde{p}}} \lesssim \varepsilon_1.$$

Moreover the profile U *verifies*

$$U \in \dot{B}_{\tilde{q}}^{s,\tilde{p}}$$

for all $(s, \tilde{p}, \tilde{q})$ *such that* $s \leqslant s_p$, $\tilde{p} \geqslant 2$, $\frac{1}{\tilde{p}} + \frac{n-1}{2\tilde{q}} = \frac{n-1}{4}$ *and* $s - \frac{n}{\tilde{q}} - \frac{1}{\tilde{p}} = -\frac{2}{p-1}$ (*we exclude the end-point* $\tilde{p} = 2$ *when* $n = 3$).

We remark that we therefore obtain more general initial data than considered in [16], where the requirement is $\phi_0 \in C^n(S^{n-1})$ and $\phi_1 \in C^{n-1}(S^{n-1})$. Here $s_p - \frac{n-1}{2} < n$ and $\dot{H}^{s_p}(S^{n-1}) \in C^{\frac{1}{2}-\frac{2}{p-1}}(S^{n-1})$, and when $p < 5$ we even have unbounded ϕ_0.

The next interesting question to address is asymptotics. Essentially, one expects asymptotic completeness and scattering to hold in the range $q < \infty$ (or, again, in the completion of $\mathcal{S}$ for $q = \infty$), while for $q = \infty$ strange (with respect to the usual Sobolev case) things may happen. Indeed, we have:

THEOREM 6. – *Let* $(u_0, u_1) \in (\dot{B}_2^{s_p,q}, \dot{B}_2^{s_p-1,q})$ *verify the hypothesis of Theorem* 1 *and let* $u(x,t)$ *be the solution of* (1) *with these data. Then there exist* $(u_0^+, u_1^+) \in (\dot{B}_2^{s_p,q}, \dot{B}_2^{s_p-1,q})$ *such that, if* u_L^+ *is the solution to the free wave equation with these data,*

$$\lim_{t\to+\infty} \left\|u(t) - u_L^+(t)\right\|_{\dot{B}_2^{s_p,q}} = 0. \tag{36}$$

Moreover, if $(u_0^-, u_1^-) \in (\dot{B}_2^{s_p,q}, \dot{B}_2^{s_p-1,q})$ *with a small norm in the sense of Theorem* 1*, there exists a solution* u *to* (1) *such that*:

$$\lim_{t\to-\infty} \left\| u(t) - u_L^-(t) \right\|_{\dot{B}_2^{s_p,q}} = 0. \tag{37}$$

Lastly, the map (u_0^-, u_1^-) *to* (u_0^+, u_1^+) *is continuous.*

We remark that, as in Theorem 2, we only require smallness of the data in the largest possible Besov space, $(\dot{B}_2^{s_p,\infty}, \dot{B}_2^{s_p-1,\infty})$. Indeed, scattering is known to follow from the finiteness of global space-time norms, say $\mathcal{L}_t^{\bar{p}}(\dot{B}_{\bar{p}}^{\bar{s},q})$ in the present case. Such norms are proved to be finite by the same argument which preserves the solution $u \in \dot{B}_2^{s_p,q}$ in the proof of Theorem 2 (where we restricted ourselves to $q = 2$ for sake of simplicity).

2.2. Proofs

Theorem 5 is nothing but a restatement of Theorem 1 in the context of homogeneous data. The only part which is missing in the proof is the estimate on the profile U. Recall that Besov spaces can be defined via a continuous family of frequency localized operators, say Δ_μ, where μ has to be understood as 2^{-j}. Then, recalling that a solution will verify say:

$$\sup_\mu \mu^{-j\bar{s}} \left\| \Delta_\mu u(x,t) \right\|_{L_t^{\bar{p}}(L_x^{\bar{q}})} < \infty,$$

and taking advantage of the special form $u(x,t) = \frac{1}{t}U(\frac{x}{t})$, a change of variable (taking $\mu = 1$ and thinking of $\theta = 1/t$ as a frequency parameter) gives:

$$\int_0^\infty \left(\theta^{-\bar{s}} \left\| \Delta_\theta U(x) \right\|_{L_x^{\bar{q}}}\right)^{\bar{p}} \frac{\mathrm{d}\theta}{\theta} < \infty$$

which is nothing but the Besov norm of U in $\dot{B}_{\bar{q}}^{\bar{s},\bar{p}}$. All other values follow in the same way, directly from the proof of Theorem 1 in the range $\tilde{p} \geqslant \alpha$ and by the extra use of generalized Strichartz estimates ([8,9]) for the range $2 \leqslant \tilde{p} < \alpha$.

We proceed with the asymptotic completeness and scattering in the range $q < \infty$. The proof of Theorem 6 follows essentially line by line the proof of Theorem 2.2 in [10]. Therefore we remain sketchy and simply provide the case where we consider the completion of the Schwartz class in $(\dot{B}_2^{s_p,\infty}, \dot{B}_2^{s_p-1,\infty})$ for the initial data. The key point here is the following:

$$\sup_j 2^{j\bar{s}} \left\| \Delta_j \left(u^p\right) \right\|_{L_t^{\bar{p}}(L_x^{\bar{q}})} < \infty \tag{38}$$

and

$$\lim_{|j|\to\infty} 2^{j\bar{s}} \left\| \Delta_j \left(u^p\right) \right\|_{L_t^{\bar{p}}(L_x^{\bar{q}})} = 0. \tag{39}$$

Therefore there exists j_n such that for $|j| > j_n$

$$2^{j\bar{s}} \left\| \Delta_j \left(u^p\right) \right\|_{L_t^{\bar{p}}(L_x^{\bar{q}})} < \frac{1}{n}$$

and then there exists T_n such that

$$\sup_{|j|\leqslant j_n} 2^{j\bar{s}} \left\| \Delta_j \left(u^p\right) \right\|_{L^{\bar{p}}_{[T_n,\infty)}(L^{\bar{q}}_x)} < \frac{1}{n}.$$

Consider $u_{L,n}$ solution to the free wave equation with data $(u(T_n), \partial_t u(T_n))$, we have, since $\Box(u - u_{L,n}) = u^p$:

$$\sup_j 2^{js_p} \left\| u(t) - u_{L,n}(t) \right\|_2 \lesssim \sup_j 2^{j\bar{s}} \left\| \Delta_j \left(u^p\right) \right\|_{L^{\bar{p}}_{[T_n,t]}(L^{\bar{q}}_x)} \lesssim \frac{1}{n}. \tag{40}$$

In turn, since $u(T_m) = u_{L,m}(T_m)$, when $T_m > T_n$, then:

$$\sup_j 2^{js_p} \left\| u_{L,m}(T_m) - u_{L,n}(T_m) \right\|_2 \lesssim \frac{1}{n}$$

which yields a Cauchy sequence at $t = 0$ since by the energy inequality

$$\sup_j 2^{js_p} \left\| u_{L,m}(0) - u_{L,n}(0) \right\|_2 \lesssim \sup_j 2^{js_p} \left\| u_{L,m}(T_m) - u_{L,n}(T_m) \right\|_2.$$

Call (u_0^+, u_1^+) the limit, passing to the limit in (40) (splitting again in two terms) gives the desired result when $t \to +\infty$. The proof of existence for (u_0^-, u_1^-) follows exactly the same way after solving the Cauchy problem with data at $t = -\infty$. Continuity of the map from the – states to the + states is a direct consequence of the formula $u^p - \tilde{u}^p = (u - v)(u^{p-1} + \cdots + \tilde{u}^{p-1})$ which gives us (30) and provides the right estimates in the spirit of [10]. This completes the proof of Theorem 6.

To end this section, we would like to make a few remarks on the case $q = \infty$. Clearly the previous scattering argument fails completely and one may ask what could possibly happen. Asymptotic results involving self-similar solutions like in [14] do not seem within reach, as unlike the heat operator, the wave operator does not provide any time decay in the Sobolev norm. Therefore, it seems likely that one has to require additional assumptions, like more regularity or decay, to obtain asymptotics results. Such results indeed exist in [12,16] for very particular sets of initial data, like cut-off (in space) of smooth homogeneous data.

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