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Duality for a Class of Multiobjective Control Problems with Generalized Invexity

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B. Mond and I. Smart (*J. Math. Anal. Appl.* **136** (1988), 325–333) defined a kind of invexity and discussed the duality and sufficiency in scalar control problems with such invexity. D. Bhatia and P. Kumar (*J. Math. Anal. Appl.* **189** (1995), 676–692) defined another kind of invexity, corresponding generalized invexity, and discussed the duality for multiobjective control problems with such generalized invexity. In this paper, the duality results for multiobjective control problems with Mond and Smart's generalized invexity are discussed. © 2001 Academic Press

Key Words: invexity; generalized invexity; multiobjective control problems; efficient solution; duality.

1. INTRODUCTION

Recently, quite a few authors discussed duality for multiobjective variational problems with different generalized convexity or generalized invexity, such as [1, 3, 6–8, 10–13, 16]. Most of them considered the Wolfe type and Mond–Weir type duals for multiobjective variational problems.

The General dual concept or an equivalent was introduced in some papers, such as [5, 14, 15, 17] for conventional multiobjective mathematical programming and in [10] for multiobjective variational problems.

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Bhatia and Kumar [2] discussed multiobjective control problems with ρ-pseudoinvexity, ρ-strictly pseudoinvexity, ρ-quasiinvexity, or ρ-strictly quasiinvexity. Nahak and Nanda [12] discussed efficiency and duality for multiobjective variational control problems with $(F\text{–}\rho)$-convexity. The objective functionals and constraint functionals in both papers were different. In the present paper, we discuss duality for multiobjective control problems with the same objective functionals and constraint conditions as in [2], but with the invexity defined in [9].

2. NOTATIONS AND PRELIMINARIES

Let $I = [t_0, t_f]$ be a real interval, and let $f_i: I \times R^n \times R^m \to R(i = 1, 2, \dots, p)$, $g_j: I \times R^n \times R^m \to R(j = 1, 2, \dots, l)$, and $h_k: I \times R^n \times R^m \to R(k = 1, 2, \dots, n)$ be continuously differentiable functions. Denote by X the space of piecewise smooth functions $x: I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ and by U the space of piecewise continuous control functions $u: I \to R^m$ with the norm $\|u\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = x(t_0) + \int_{t_0}^{t} u(s)ds,$$

where $x(t_0)$ is a given boundary value. Denote the partial derivatives of f_i with respect to t, x, and u, respectively, by f_{it}, f_{ix}, and f_{iu} such that

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \dots, \frac{\partial f_i}{\partial x_n}\right)^T, \qquad f_{iu} = \left(\frac{\partial f_i}{\partial u_1}, \frac{\partial f_i}{\partial u_2}, \dots, \frac{\partial f_i}{\partial u_n}\right)^T,$$

$i = 1, 2, \dots, p$, where T denotes the transpose operator. The partial derivatives of the vector functions g and h are similarly defined, using $n \times l$ matrix and $n \times n$ matrix, respectively.

Consider the multiobjective control problem (VCP)

$$\min \int_{t_0}^{t_f} f(t, x, u)dt = \left(\int_{t_0}^{t_f} f_1(t, x, u)dt, \dots, \int_{t_0}^{t_f} f_p(t, x, u)dt\right)$$

$$\text{subject to } x(t_0) = \alpha, x(t_f) = \beta, \tag{1}$$

$$\dot{x} = h(t, x, u), \qquad t \in I, \tag{2}$$

$$g(t, x, u) = (g_1(t, x, u), \dots, g_1(t, x, u))^T \leq 0, \quad t \in I. \tag{3}$$

For any partition $\{\Sigma, \Sigma'\}$ of $\{1, 2, \ldots, l\}$, i.e., $\Sigma \bigcup \Sigma' = \{1, 2, \ldots, l\}$, $\Sigma \bigcap \Sigma' = \phi$, we propose two types of general duals for (VCP):

$(VCD1)$

$$\max\left(\int_{t_0}^{t_f}[f_1(t, y, \nu) + \mu(t)_\Sigma^T g_\Sigma(t, y, \nu)]\,dt, \ldots, \int_{t_0}^{t_f}[f_p(t, y, \nu) + \mu(t)_\Sigma^T g_\Sigma(t, y, \nu)]\,dt\right) \quad \text{subject to } y(t_0) = \alpha, y(t_f) = \beta, \tag{4}$$

$$\sum_{i=1}^{p}\lambda_i f_{iy}(t, y, \nu) + g_y(t, y, \nu)\mu(t) + h_y(t, y, \nu)\gamma(t) + \dot{\gamma}(t) = 0, \quad t \in I, \tag{5}$$

$$\sum_{i=1}^{p}\lambda_i f_{i\nu}(t, y, \nu) + g_\nu(t, y, \nu)\mu(t) + h_\nu(t, y, \nu)\gamma(t) = 0, \quad t \in I, \tag{6}$$

$$\int_{t_0}^{t_f}\gamma(t)^T[h(t, y, \nu) - \dot{y}]\,dt \geq 0, \tag{7}$$

$$\int_{t_0}^{t_f}\mu(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\,dt \geq 0, \tag{8}$$

$$\mu(t) \geq 0, \qquad t \in I, \tag{9}$$

$$\lambda_i \geq 0, \quad i = 1, 2, \ldots, p, \sum_{i=1}^{p}\lambda_i = 1, \tag{10}$$

where $\mu(t)_\Sigma$ denotes the $|\Sigma|$ column vector function with the component indices in Σ, and similar notations have the same meanings.

$(VCD2)$

$$\max\left(\int_{t_0}^{t_f}\{f_1(t,y,\nu)+\mu(t)_\Sigma^T g_\Sigma(t,y,\nu)+\gamma(t)^T[h(t,y,\nu)-\dot{y}]\}\,dt,\ldots, \int_{t_0}^{t_f}\{f_p(t,y,\nu)+\mu(t)_\Sigma^T g_\Sigma(t,y,\nu)+\gamma(t)^T[h(t,y,\nu)-\dot{y}]\}\,dt\right)$$

$$\text{subject to } y(t_0)=\alpha, y(t_f)=\beta, \tag{11}$$

$$\sum_{i=1}^{p}\lambda_i f_{iy}(t,y,\nu)+g_y(t,y,\nu)\mu(t)+h_y(t,y,\nu)\gamma(t)+\dot{\gamma}(t)=0, \quad t\in I, \tag{12}$$

$$\sum_{i=1}^{p}\lambda_i f_{i\nu}(t,y,\nu)+g_\nu(t,y,\nu)\mu(t)+h_\nu(t,y,\nu)\gamma(t)=0, \quad t\in I, \tag{13}$$

$$\int_{t_0}^{t_f} \mu(t)_{\Sigma'}^T g_{\Sigma'}(t,y,\nu)\,dt \ge 0, \tag{14}$$

$$\mu(t) \ge 0, \qquad t \in I \tag{15}$$

$$\lambda_i \ge 0, \qquad i=1,2,\ldots,p, \qquad \sum_{i=1}^{p} \lambda_i = 1. \tag{16}$$

Remark. When $\Sigma = \{1, 2, \ldots, l\}$, $\Sigma' = \phi$, (VCD1) is just (WVCD) in [2], and when $\Sigma = \phi$, $\Sigma' = \{1, 2, \ldots, l\}$, (VCD2) is just (MVCD) in [2].

DEFINITION 1 [4]. A feasible solution (x^*, u^*) for (VCP) is said to be an efficient solution for (VCP) if for all feasible solutions (x, u),

$$\int_{t_0}^{t_f} f_i(t, x, u)\,dt \le \int_{t_0}^{t_f} f_i(t, x^*, u^*)\,dt, \forall i \in \{1, 2, \ldots, p\}$$
$$\Rightarrow \int_{t_0}^{t_f} f_i(t, x, u)\,dt = \int_{t_0}^{t_f} f_i(t, x^*, u^*)\,dt, \forall i \in \{1, 2, \ldots, p\}.$$

DEFINITION 2 [9]. If there exist vector functions $\eta(t, x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R^n$, with $\eta = 0$ at t if $x(t) = x^*(t)$, and $\zeta(t, x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R^m$ such that for the scalar function $h(t, x, \dot{x}, u)$ the functional $H(x, \dot{x}, u) = \int_{t_0}^{t_f} h(t, x, \dot{x}, u)\,dt$ satisfies

$$\begin{aligned} &H(x,\dot{x},u) - H(x^*,\dot{x}^*,u^*) \\ &\quad \ge \int_{t_0}^{t_f} \left[\eta^T h_x(t,x^*,\dot{x}^*,u^*) + \frac{d\eta^T}{dt} h_{\dot{x}}(t,x^*,\dot{x}^*,u^*) + \zeta^T h_u(t,x^*,\dot{x}^*,u^*) \right] dt, \end{aligned}$$

then H is said to be invex in x^*, $\dot{x}^*$, and u^* on $[t_0, t_f]$ with respect to η and ζ.

DEFINITION 3. If for all $x \in X$ and $u \in U$,

$$\begin{aligned} &\int_{t_0}^{t_f} \left[\eta^T h_x(t,x^*,\dot{x}^*,u^*) + \frac{d\eta^T}{dt} h_{\dot{x}}(t,x^*,\dot{x}^*,u^*) + \zeta^T h_u(t,x^*,\dot{x}^*,u^*) \right] dt \ge 0 \\ &\qquad \Rightarrow H(x,\dot{x},u) \ge H(x^*,\dot{x}^*,u^*), \end{aligned}$$

then H is said to be pseudoinvex in x^*, $\dot{x}^*$, and u^* on $[t_0, t_f]$ with respect to η and ζ.

DEFINITION 4. If for all $x \in X$, $x \ne x^*$, and $u \in U$,

$$\begin{aligned} &\int_{t_0}^{t_f} \left[\eta^T h_x(t,x^*,\dot{x}^*,u^*) + \frac{d\eta^T}{dt} h_{\dot{x}}(t,x^*,\dot{x}^*,u^*) + \zeta^T h_u(t,x^*,\dot{x}^*,u^*) \right] dt \ge 0 \\ &\qquad \Rightarrow H(x,\dot{x},u) > H(x^*,\dot{x}^*,u^*), \end{aligned}$$

then H is said to be strictly pseudoinvex in x^*, $\dot{x}^*$, and u^* on $[t_0, t_f]$ with respect to η and ζ.

DEFINITION 5. If for all $x \in X$ and $u \in U$,

$$H(x, \dot{x}, u) \leq H(x^*, \dot{x}^*, u^*) \Rightarrow \int_{t_0}^{t_f} \left[\eta^T h_x(t, x^*, \dot{x}^*, u^*) + \frac{d\eta^T}{dt} h_{\dot{x}}(t, x^*, \dot{x}^*, u^*) + \zeta^T h_u(t, x^*, \dot{x}^*, u^*) \right] dt \leq 0,$$

then H is said to be quasiinvex in $x^*, \dot{x}^*$, and u^* on $[t_0, t_f]$ with respect to η and ζ.

DEFINITION 6. If for all $x \in X$, $x \neq x^*$, and $u \in U$,

$$H(x, \dot{x}, u) \leq H(x^*, \dot{x}^*, u^*) \Rightarrow \int_{t_0}^{t_f} \left[\eta^T h_x(t, x^*, \dot{x}^*, u^*) + \frac{d\eta^T}{dt} h_{\dot{x}}(t, x^*, \dot{x}^*, u^*) + \zeta^T h_u(t, x^*, \dot{x}^*, u^*) \right] dt < 0,$$

then H is said to be strictly quasiinvex in $x^*, \dot{x}^*$, and u^* on $[t_0, t_f]$ with respect to η and ζ.

3. THE DUALITY BETWEEN (VCP) AND (VCD1)

THEOREM 1 (Weak Duality). *Assume that for all feasible $(\bar{x}, \bar{u})$ for (VCP) and all feasible $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ for (VCD1), $\int_{t_0}^{t_f} [f_i(t, y, \nu) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma} \times (t, y, \nu)]\, dt$ is strictly quasiinvex at $(\bar{y}, \bar{\nu})$ on $X \times U$ with respect to η and ζ, and $\int_{t_0}^{t_f} \bar{\gamma}(t)^T [h(t, y, \nu) - \dot{y}]\, dt$ and $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ are quasiinvex at $(\bar{y}, \bar{\nu})$ on $X \times U$ with respect to the same η and ζ, then the following cannot hold simultaneously,*

$$\int_{t_0}^{t_f} f_i(t, \bar{x}, \bar{u})\, dt \leq \int_{t_0}^{t_f} \left[f_i(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, \bar{y}, \bar{\nu}) \right] dt, \quad \forall i \in \{1, 2, \ldots, p\}, \tag{17}$$

$$\int_{t_0}^{t_f} f_j(t, \bar{x}, \bar{u})\, dt \underset{\neq}{\leq} \int_{t_0}^{t_f} \left[f_j(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, \bar{y}, \bar{\nu}) \right] dt, \quad \textit{for some} \quad j \in \{1, 2, \ldots, p\}. \tag{18}$$

Proof. Suppose to the contrary that (17) and (18) hold for some feasible $(\bar{x}, \bar{u})$ for (VCP) and some feasible $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ for (VCD1), then by (3), (9), and (17),

$$\int_{t_0}^{t_f} \left[f_i(t, \bar{x}, \bar{u}) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, \bar{x}, \bar{u})\right] dt \\ \leq \int_{t_0}^{t_f} \left[f_i(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, \bar{y}, \bar{\nu})\right] dt, \qquad \forall i \in \{1, 2, \ldots, p\}.$$

By the strictly quasiinvexity of $\int_{t_0}^{t_f} [f_i(t, y, \nu) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, y, \nu)]\, dt$, when $(\bar{x}, \bar{u}) \neq (\bar{y}, \bar{\nu})$,

$$\int_{t_0}^{t_f} \{\eta^T(t, \bar{x}, \bar{u}, \bar{y}, \bar{\nu})[f_{iy}(t, \bar{y}, \bar{\nu}) + g_{\Sigma y}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_\Sigma] \\ + \zeta^T(t, \bar{x}, \bar{u}, \bar{y}, \bar{\nu})[f_{i\nu}(t, \bar{y}, \bar{\nu}) + g_{\Sigma\nu}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_\Sigma]\} dt < 0, \\ i \in \{1, 2, \ldots, p\}. \tag{19}$$

Multiply each equation of (19) by $\bar{\lambda}_i \geq 0$, $i = 1, 2, \ldots, p$, and add them together;

$$\int_{t_0}^{t_f} \left\{\eta^T(t, \bar{x}, \bar{u}, \bar{y}, \bar{\nu})\left[\sum_{i=1}^{p} \bar{\lambda}_i f_{iy}(t, \bar{y}, \bar{\nu}) + g_{\Sigma_y}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_\Sigma\right]\right. \\ \left. + \zeta^T(t, \bar{x}, \bar{u}, \bar{y}, \bar{\nu})\left[\sum_{i=1}^{p} \bar{\lambda}_i f_{i\nu}(t, \bar{y}, \bar{\nu}) + g_{\Sigma\nu}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_\Sigma\right]\right\} dt < 0. \tag{20}$$

By (2) and (7),

$$\int_{t_0}^{t_f} \bar{\gamma}(t)^T[h(t, \bar{x}, \bar{u}) - \dot{\bar{x}}]\, dt \leq \int_{t_0}^{t_f} \bar{\gamma}(t)^T[h(t, \bar{y}, \bar{\nu}) - \dot{\bar{y}}]\, dt.$$

The quasiinvexity of $\int_{t_0}^{t_f} \bar{\gamma}(t)^T[h(t, y, \nu) - \dot{y}]\, dt$ implies that

$$\int_{t_0}^{t_f} \left[\eta^T(t, \bar{x}, \bar{u}, \bar{y}, \bar{\nu})h_y(t, \bar{y}, \bar{\nu})\bar{\gamma}(t) + \frac{d\eta^T}{dt}(-E_{n\times n})\bar{\gamma}(t)\right. \\ \left. + \zeta^T(t, \bar{x}, \bar{u}, \bar{y}, \bar{\nu})h_\nu(t, \bar{y}, \bar{\nu})\bar{\gamma}(t)\right] dt \leq 0. \tag{21}$$

Integrate $\int_{t_0}^{t_f} (d\eta^T/dt)(-E_{n\times n})\bar{\gamma}(t)\, dt$ by parts;

$$-\int_{t_0}^{t_f} \frac{d\eta^T}{dt}\bar{\gamma}(t)\, dt = -\eta^T\bar{\gamma}(t)\big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \eta^T\dot{\bar{\gamma}}(t)\, dt = \int_{t_0}^{t_f} \eta^T\dot{\bar{\gamma}}(t)\, dt.$$

By (21), we obtain

$$\int_{t_0}^{t_f} \{\eta^T(t, \bar{x}, \bar{u}, \bar{y}, \bar{\nu})[h_y(t, \bar{y}, \bar{\nu})\bar{\gamma}(t) + \dot{\bar{\gamma}}(t)] + \zeta^T(t, \bar{x}, \bar{u}, \bar{y}, \bar{\nu})h_\nu(t, \bar{y}, \bar{\nu})\bar{\gamma}(t)\}\, dt \le 0 \tag{22}$$

By (3), (8), and (9),

$$\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, \bar{x}, \bar{\nu})\, dt \le \int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, \bar{y}, \bar{\nu})\, dt.$$

It follows from the quasiinvexity of $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ that

$$\int_{t_0}^{t_f} [\eta^T g_{\Sigma' y}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_{\Sigma'} + \zeta^T g_{\Sigma' \nu}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_{\Sigma'}\, dt \le 0. \tag{23}$$

Add (20), (22), and (23);

$$\begin{aligned}
&\int_{t_0}^{t_f} \Bigg\{\eta^T(t,\bar{x},\bar{u},\bar{y},\bar{\nu})\Bigg[\sum_{i=1}^{p}\lambda_i f_{iy}(t,\bar{y},\bar{\nu}) + g_{\Sigma y}(t,\bar{y},\bar{\nu})\bar{\mu}(t)_\Sigma \\
&\quad + g_{\Sigma' y}(t,\bar{y},\bar{\nu})\bar{\mu}(t)_{\Sigma'} + h_y(t,\bar{y},\bar{\nu})\bar{\gamma}(t) + \dot{\bar{\gamma}}(t)\Bigg] \\
&\quad + \zeta^T(t,\bar{x},\bar{u},\bar{y},\bar{\nu})\Bigg[\sum_{i=1}^{p}\bar{\lambda}_i f_{i\nu}(t,\bar{y},\bar{\nu}) + g_{\Sigma\nu}(t,\bar{y},\bar{\nu})\bar{\mu}(t)_\Sigma \\
&\qquad + g_{\Sigma'\nu}(t,\bar{y},\bar{\nu})\bar{\mu}(t)_{\Sigma'} + h_\nu(t,\bar{y},\bar{\nu})\bar{\gamma}(t)\Bigg]\Bigg\}\, dt \\
&= \int_{t_0}^{t_f} \Bigg\{\eta^T(t,\bar{x},\bar{u},\bar{y},\bar{\nu})\Bigg[\sum_{i=1}^{p}\lambda_i f_{iy}(t,\bar{y},\bar{\nu}) + g_y(t,\bar{y},\bar{\nu})\bar{\mu}(t) \\
&\qquad + h_y(t,\bar{y},\bar{\nu})\bar{\gamma}(t) + \dot{\bar{\gamma}}(t)\Bigg] + \zeta^T(t,\bar{x},\bar{u},\bar{y},\bar{\nu}) \\
&\quad \times \Bigg[\sum_{i=1}^{p}\bar{\lambda}_i f_{i\nu}(t,\bar{y},\bar{\nu}) + g_\nu(t,\bar{y},\bar{\nu})\bar{\mu}(t) + h_\nu(t,\bar{y},\bar{\nu})\bar{\gamma}(t)\Bigg]\Bigg\}\, dt < 0,
\end{aligned}$$

which contradicts (5) and (6).

Remark. From the proof of Theorem 1 we can obtain that (17) and (18) cannot hold simultaneously if $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ is strictly quasiinvex at $(\bar{y}, \bar{\nu})$ on $X \times U$ with respect to the same η and ζ and $\int_{t_0}^{t_f}[f_i(t, y, \nu) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, y, \nu)]\, dt$ and $\int_{t_0}^{t_f} \bar{\gamma}(t)^T[h(t, y, \nu) - \dot{y}]\, dt$ are quasiinvex at $(\bar{y}, \bar{\nu})$ on $X \times U$ with respect to the same η and ζ, or $\int_{t_0}^{t_f} \bar{\gamma}(t)^T[h(t, y, \nu) - \dot{y}]\, dt$ is strictly quasiinvex at $(\bar{y}, \bar{\nu})$ on $X \times U$ with respect to the same η and ζ and $\int_{t_0}^{t_f}[f_i(t, y, \nu) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, y, \nu)]\, dt$ and $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ are quasiinvex at $(\bar{y}, \bar{\nu})$ on $X \times U$ with respect to the same η and ζ.

THEOREM 2 (Weak Duality). *Assume that for all feasible* $(\bar{x}, \bar{u})$ *for* (*VCP*) *and all feasible* $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ *for* (*VCD*1), *one of the functionals* $\int_{t_0}^{t_f} [\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, y, \nu) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, y, \nu)]\, dt$, $\int_{t_0}^{t_f} \bar{\gamma}(t)^T [h(t, y, \nu) - \dot{y}]\, dt$, *on* $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, v)\, dt$ *is strictly quasiinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to* η *and* ζ *and the other two are quasiinvex, then* (17) *and* (18) *cannot hold simultaneously.*

Proof. Assume to the contrary that (17) and (18) hold simultaneously for some feasible $(\bar{x}, \bar{u})$ for (VCP) and some feasible $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ for (VCD1). Multiply each equation of (17) by $\bar{\lambda}_i \geq 0$, $i = 1, 2 \ldots, p$, and add them together,

$$\int_{t_0}^{t_f} \sum_{i=1}^{p} \bar{\lambda}_i f_i(t, \bar{x}, \bar{u})\, dt \leq \int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, \bar{y}, \bar{\nu}) \right] dt.$$

By (3) and (9),

$$\int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, \bar{x}, \bar{u}) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, \bar{x}, \bar{u}) \right] dt$$
$$\leq \int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, \bar{y}, \bar{\nu}) \right] dt.$$

The left part of the proof is similar to the proof of Theorem 1.

THEOREM 3 (Weak Duality). *Assume that for all feasible* $(\bar{x}, \bar{u})$ *for* (*VCP*) *and for all feasible* $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ *for* (*VCD*1), $\int_{t_0}^{t_f} \{\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, y, \nu) + \bar{\mu}(t)^T g(t, y, \nu) + \bar{\gamma}(t)^T [h(t, y, \nu) - \dot{y}]\}\, dt$ *is strictly pseudoinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to* η *and* ζ, *then* (17) *and* (18) *cannot hold simultaneously.*

Proof. Multiply (5) from the left-hand side with η^T; then take integration from t_0 to t_f on both sides,

$$\int_{t_0}^{t_f} \eta^T \left[\sum_{i=1}^{p} f_{iy}(t, \bar{y}, \bar{\nu}) \bar{\lambda}_i + g_y(t, \bar{y}, \bar{\nu}) \bar{\mu}(t) + h_y(t, \bar{y}, \bar{\nu}) \bar{\gamma}(t) + \dot{\bar{\gamma}}(t) \right] dt = 0.$$

Integrate $\int_{t_0}^{t_f} \eta^T \dot{\bar{\gamma}}(t)\, dt$ by parts;

$$\int_{t_0}^{t_f} \left\{ \eta^T \left[\sum_{i=1}^{p} f_{iy}(t, \bar{y}, \bar{\nu}) \bar{\lambda}_i + g_y(t, \bar{y}, \bar{\nu}) \bar{\mu}(t) + h_y(t, \bar{y}, \bar{\nu}) \bar{\gamma}(t) \right] \right.$$
$$\left. - \frac{d\eta^T}{dt} \bar{\gamma}(t) \right\} dt = 0,$$

$$\int_{t_0}^{t_f}\left\{\eta^T\left[\sum_{i=1}^{p}\bar{\lambda}_i f_{iy}(t,\bar{y},\bar{\nu})+g_y(t,\bar{y},\bar{\nu})\bar{\mu}(t)+h_y(t,\bar{y},\bar{\nu})\bar{\gamma}(t)\right]\right.$$
$$+\frac{d\eta^T}{dt}\left[\sum_{i=1}^{p}\bar{\lambda}_i f_{i\dot{y}}(t,\bar{y},\bar{\nu})+g_{\dot{y}}(t,\bar{y},\bar{\nu})\bar{\mu}(t)\right.$$
$$\left.\left.+[h(t,\bar{y},\bar{\nu})-\dot{\bar{y}}]_{\dot{y}}\bar{\gamma}(t)\right]\right\}dt=0. \tag{24}$$

Multiply (6) from the left-hand side with ζ^T, and then take integration from t_0 to t_f on both sides;

$$\int_{t_0}^{t_f}\zeta^T\left[\sum_{i=1}^{p}\bar{\lambda}_i f_{i\nu}(t,\bar{y},\bar{\nu})+g_v(t,\bar{y},\bar{\nu})\bar{\mu}(t)+h_\nu(t,\bar{y},\bar{\nu})\bar{\gamma}(t)\right]dt=0. \tag{25}$$

Add (24) and (25);

$$\int_{t_0}^{t_f}\left\{\eta^T\left[\sum_{i=1}^{p}\bar{\lambda}_i f_{iy}(t,\bar{y},\bar{\nu})+g_y(t,\bar{y},\bar{\nu})\bar{\mu}(t)+h_y(t,\bar{y},\bar{\nu})\bar{\gamma}(t)\right]\right.$$
$$+\frac{d\eta^T}{dt}\left[\sum_{i=1}^{p}\bar{\lambda}_i f_{i\dot{y}}(t,\bar{y},\bar{\nu})+g_{\dot{y}}(t,\bar{y},\bar{\nu})\bar{\mu}(t)+[h(t,\bar{y},\bar{\nu})-\dot{\bar{y}}]_{\dot{y}}\bar{\gamma}(t)\right]$$
$$\left.+\zeta^T\left[\sum_{i=1}^{p}\bar{\lambda}_i f_{i\nu}(t,\bar{y},\bar{\nu})+g_\nu(t,\bar{y},\bar{\nu})\bar{\mu}(t)+h_\nu(t,\bar{y},\bar{\nu})\bar{\gamma}(t)\right]\right\}dt=0 \tag{26}$$

By assumption,

$$\int_{t_0}^{t_f}\left\{\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,\bar{x},\bar{u})+\bar{\mu}(t)^T g(t,\bar{x},\bar{u})+\bar{\gamma}(t)^T[h(t,\bar{x},\bar{u})-\dot{\bar{x}}]\right\}dt$$
$$>\int_{t_0}^{t_f}\left\{\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,\bar{y},\bar{\nu})+\bar{\mu}(t)^T g(t,\bar{y},\bar{\nu})+\bar{\gamma}(t)^T[h(t,\bar{y},\bar{\nu})-\dot{\bar{y}}]\right\}dt.$$

By (2), (3), (7), and (8),

$$\int_{t_0}^{t_f}\left[\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,\bar{x},\bar{u})\right]dt>\int_{t_0}^{t_f}\left\{\sum_{i=1}^{p}\bar{\lambda}_i f_i(t,\bar{y},\bar{\nu})+\bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t,\bar{y},\bar{\nu})\right\}dt.$$

It follows obviously that (17) and (18) cannot hold simultaneously.

THEOREM 4 (Weak Duality). *Assume that for all feasible $(\bar{x},\bar{u})$ for (VCP) and for all feasible $(\bar{y},\bar{\nu},\bar{\lambda},\bar{\mu},\bar{\gamma})$ for (VCD1), $\int_{t_0}^{t_f}[f_i(t,y,\nu)+\bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t,y,\nu)]\,dt$ is strictly pseudoinvex at $(\bar{y},\bar{\nu})$ on $X\times U$ with respect to η and ζ and $\int_{t_0}^{t_f}\bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t,y,\nu)\,dt$ and $\int_{t_0}^{t_f}\bar{\gamma}(t)^T[h(t,y,\nu)-\dot{y}]\,dt$ are quasiinvex at $(\bar{y},\bar{\nu})$ on $X\times U$ with respect to η and ζ, then (17) and (18) cannot hold simultaneously.*

Proof. Similar to the first part of the proof of Theorem 3, we obtain (26). It follows from (26) that

$$\int_{t_0}^{t_f} \Bigg\{ \eta^T \Bigg[\sum_{i=1}^{p} \bar{\lambda}_i f_{iy}(t, \bar{y}, \bar{\nu}) + g_{\Sigma\, y}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_\Sigma \Bigg] + \frac{d\eta^T}{dt} \Bigg[\sum_{i=1}^{p} \bar{\lambda}_i f_{i\dot{y}}(t, \bar{y}, \bar{\nu}) + g_{\Sigma\dot{y}}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_\Sigma \Bigg] + \zeta^T \Bigg[\sum_{i=1}^{p} \bar{\lambda}_i f_{i\nu}(t, \bar{y}, \bar{\nu}) + g_{\Sigma\nu}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_\Sigma \Bigg] \Bigg\} dt$$

$$= - \int_{t_0}^{t_f} \Bigg[\eta^T g_{\Sigma'\, y}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_{\Sigma'} + \frac{d\eta^T}{dt} g_{\Sigma'\dot{y}}(t, \bar{y}, \nu)\bar{\mu}(t)_{\Sigma'} + \zeta^T g_{\Sigma'\nu}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_{\Sigma'} \Bigg] dt - \int_{t_0}^{t_f} \Bigg\{ \eta^T h_y(t, \bar{y}, \bar{\nu})\bar{\gamma}(t) + \frac{d\eta^T}{dt} [h(t, \bar{y}, \bar{\nu}) - \dot{\bar{y}}]_{\dot{y}} \bar{\gamma}(t) + \zeta^T h_\nu(t, \bar{y}, \bar{\nu})\bar{\gamma}(t) \Bigg\} dt \tag{27}$$

By (2), (3), (7), (8), and (9),

$$\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, \bar{x}, \bar{u})\, dt \le \int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, \bar{y}, \bar{\nu})\, dt,$$

$$\int_{t_0}^{t_f} \bar{\gamma}(t)^T [h(t, \bar{x}, \bar{u}) - \dot{x}]\, dt \le \int_{t_0}^{t_f} \bar{\gamma}(t)^T [h(t, \bar{y}, \bar{\nu}) - \dot{\bar{y}}(t)]\, dt$$

By the quasiinvexity of $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ and $\int_{t_0}^{t_f} \bar{\gamma}(t)^T [h(t, y, \nu) - \dot{y}]\, dt$,

$$\int_{t_0}^{t_f} \Bigg[\eta^T g_{\Sigma'\, y}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_{\Sigma'} + \frac{d\eta^T}{dt} g_{\Sigma'\, \dot{y}}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_{\Sigma'} + \zeta^T g_{\Sigma'\nu}(t, \bar{y}, \bar{\nu})\bar{\mu}(t)_{\Sigma'} \Bigg] dt \le 0 \tag{28}$$

and

$$\int_{t_0}^{t_f} \Bigg\{ \eta^T h_y(t, \bar{y}, \bar{\nu})\bar{\gamma}(t) + \frac{d\eta^T}{dt} [h(t, \bar{y}, \bar{\nu}) - \dot{\bar{y}}]_{\dot{y}} \bar{\gamma}(t) + \zeta^T h_\nu(t, \bar{y}, \bar{\nu})\bar{\gamma}(t) \Bigg\} dt \le 0. \tag{29}$$

Substitute (28) and (29) into (27);

$$\int_{t_0}^{t_f} \left\{ \eta^T \left[\sum_{i=1}^{p} \bar{\lambda}_i f_{iy}(t, \bar{y}, \bar{\nu}) + g_{\Sigma y}(t, \bar{y}, \bar{\nu}) \bar{\mu}(t)_{\Sigma} \right] \right.$$

$$+ \frac{d\eta^T}{dt} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_{i\dot{y}}(t, \bar{y}, \bar{\nu}) + g_{\Sigma \dot{y}}(t, \bar{y}, \bar{\nu}) \bar{\mu}(t)_{\Sigma} \right]$$

$$\left. + \zeta^T \left[\sum_{i=1}^{p} \bar{\lambda}_i f_{i\nu}(t, \bar{y}, \bar{\nu}) + g_{\Sigma \nu}(t, \bar{y}, \bar{\nu}) \bar{\mu}(t)_{\Sigma} \right] \right\} dt \geq 0.$$

By assumption,

$$\int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, \bar{x}, \bar{u}) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, \bar{x}, \bar{u}) \right] dt$$

$$> \int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, \bar{y}, \bar{\nu}) \right] dt.$$

By (3) and (9),

$$\int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, \bar{x}, \bar{u}) \right] dt > \int_{t_0}^{t_f} \left[\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, \bar{y}, \bar{\nu}) \right] dt.$$

It follows obviously that (17) and (18) cannot hold simultaneously.

THEOREM 5 (Weak Duality). *Assume that for all feasible* $(\bar{x}, \bar{u})$ *for* (*VCP*) *and for all feasible* $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ *for* (*VCD*1),

(i) $\bar{\lambda}_i > 0,\ i = 1, 2, \ldots, p$;

(ii) *for each* $i \in \{1, 2, \ldots, p\}$, $\int_{t_0}^{t_f} [f_i(t, y, \nu) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, y, \nu)]\, dt$ *is pseudoinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to* η *and* ζ;

(iii) $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ *and* $\int_{t_0}^{t_f} \bar{\gamma}(t)^T h[(t, y, \nu) - \dot{y}]\, dt$ *are quasi-invex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to* η *and* ζ,

then (17) *and* (18) *cannot hold simultaneously.*

Proof. It is similar to the proof of Theorem 4.

For the strong duality theorem, some results about scalar optimal control will be needed. Consider one scalar optimal control problem as follows:

(CP)

$$\min \int_{t_0}^{t_f} f(t, x, u)\, dt$$
$$\text{s.t. } \dot{x} = h(t, x, u)$$
$$g(t, x, u) \leq 0,$$

where f, g, and h are as defined earlier.

LEMMA 1 (Kuhn–Tucker Necessary Optimality Conditions). *If* $(x^0, u^0) \in X \times U$, *$X$ being the space of continuously differentiable state function* $x : I \to R^n$ *such that* $x(t_0) = \alpha$, $x(t_f) = \beta$ *and is equipped with the norm* $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ *and U being the space of piecewise continuous control functions* u: $I \to R^m$ *with the uniform* $\|u\|_\infty$ *solves* (CP), *if the Frechet derivative* $[D - H_x(x^0, u^0)]$ *is surjective, and if the optimal solution* (x^0, u^0) *is normal, then there exist piecewise smooth* μ^0: $I \to R^l$ *and* $\gamma^0 : I \to R^n$ *satisfying, for all* $t \in I$,

$$f_x(t, x^0, u^0) + g_x(t, x^0, u^0)\mu^0(t) + h_x(t, x, u)\gamma^0(t) + \dot{\gamma}^0(t) = 0$$
$$f_u(t, x^0, u^0) + g_u(t, x^0, u^0)\mu^0(t) + h_u(t, x, u)\gamma^0(t) = 0.$$
$$\mu^0(t)^T g(t, x^0, u^0) = 0$$
$$\mu^0(t) \geq 0.$$

LEMMA 2 (Chankong and Haimes). (x^0, u^0) *is an efficient solution for* (VCP) *if and only if* (x^0, u^0) *solves* $P_k(x^0, u^0)$ *for all* $k = 1, 2, \ldots, p$, *where* $P_k(x^0, u^0)$ *is defined as*

$$\min \int_{t_0}^{t_f} f_k(t, x, u)\, dt$$
$$\text{s.t. } x(t_0) = \alpha,\ x(t_f) = \beta$$
$$\dot{x} = h(t, x, u)$$
$$g(t, x, u) \leq 0$$
$$\int_{t_1}^{t_f} f_j(t, x, u)\, dt \leq \int_{t_1}^{t_f} f_j(t, x^0, u^0)\, dt$$

for all $j \in \{1, 2, \ldots, p\}$, $j \neq k$.

THEOREM 6 (Strong Duality). *Let* (x^0, u^0) *be an efficient solution for* (VCP) *and if all the constraint qualifications of Lemma* 1 *for* $P_k(x^0, u^0)$ *for at least one* $k \in \{1, 2, \dots, p\}$ *hold, then there are nonnegative* $\lambda^0 \in R^p$, *piecewise smooth* $\mu^0 : I \to R^l$ *and* $\gamma^0 : I \to R^n$ *such that* $(x^0, u^0, \lambda^0, \mu^0, \gamma^0)$ *is feasible for* $(VCD1)$ *and* $\mu^0(t)^T g(t, x^0, u^0) = 0$, *and if any one of the weak duality theorems holds between* (VCP) *and* $(VCD1)$, *then* $(x^0, u^0, \lambda^0, \mu^0, \gamma^0)$ *is efficient for* $(VCD1)$.

Proof. As pointed out in [8], $P_k(x^0, u^0)$ is a hybrid constrained optimal control problem and the Lagrangian multipliers with respect to $\int_{t_1}^{t_f} f_j(t, x, u)\, dt \le \int_{t_1}^{t_f} f_j(t, x^0, u^0)\, dt$, $j \in \{1, 2, \dots, p\}$, $j \ne k$ are constants in the corresponding optimal control problem without constraint. By Lemma 1 and the assumption in the theorem, there are nonnegative $\bar{\lambda} \in R^{p-1}$ and piecewise smooth $\bar{\mu}^0 : I \to R^l$ and $\bar{\gamma}^0 : I \to R^n$, such that

$$
\begin{aligned}
& f_{k\,x}(t, x^0, u^0) + \sum_{\substack{j=1 \\ j \ne k}}^{p} \bar{\lambda}_j f_{jx}(t, x^0, u^0) + g_x(t, x^0, u^0)\bar{\mu}^0(t) \\
& \quad + h_x(t, x^0, u^0)\bar{\gamma}^0(t) + \dot{\bar{\gamma}}^0(t) = 0, \\
& f_{k\,u}(t, x^0, u^0) + \sum_{\substack{j=1 \\ j \ne k}}^{p} \bar{\lambda}_j f_{ju}(t, x^0, u^0) + g_u(t, x^0, u^0)\bar{\mu}^0(t) \\
& \quad + h_u(t, x^0, u^0)\bar{\gamma}^0(t) = 0, \\
& \bar{\mu}^0(t)^T g(t, x^0, u^0) = 0, \\
& \bar{\mu}^0(t) \ge 0.
\end{aligned}
$$

Dividing the above formulas by $1 + \sum_{i=1,\, i \ne k}^{p} \bar{\lambda}_i$, we obtain

$$
\begin{aligned}
& \sum_{i=1}^{p} \lambda_i f_{ix}(t, x^0, u^0) + g_x(t, x^0, u^0)\mu^0(t) + h_x(t, x^0, u^0)\gamma^0(t) + \dot{\gamma}^0(t) = 0, \\
& \sum_{i=1}^{p} \lambda_i f_{iu}(t, x^0, u^0) + g_u(t, x^0, u^0)\mu^0(t) + h_u(t, x^0, u^0)\gamma^0(t) = 0, \\
& \mu^0(t)^T g(t, x^0, u^0) = 0, \\
& \mu^0(t) \ge 0,
\end{aligned}
$$

where

$$\lambda_i = \bar{\lambda}_i \Bigg/ \left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i\right), i \neq k, \lambda_k = 1 \Bigg/ \left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i\right),$$

$$\mu^0(t) = \bar{\mu}^0(t) \Bigg/ \left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i\right), \gamma^0(t) = \bar{\gamma}^0(t) \Bigg/ \left(1 + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i\right).$$

Then $(x^0, u^0, \lambda^0, \mu^0, \gamma^0)$ is a feasible solution for (VCD1) and with $\mu^0(t)^T g(t, x^0, u^0) = 0$. It is easy to show that if any of weak duality theorems holds between (VCP) and (VCD1), then $(x^0, u^0, \lambda^0, \mu^0, \gamma^0)$ is efficient for (VCD1).

4. DUALITY BETWEEN (VCP) AND (VCD2)

We state the following duality theorems 1′–6′ without proof that can be proved as in (VCD1).

THEOREM 1′ (Weak Duality). *Assume that for all feasible* $(\bar{x}, \bar{u})$ *for* (*VCP*) *and for all feasible* $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ *for* (*VCD*2), $\int_{t_0}^{t_f} \{f_i(t, y, \nu) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, y, \nu) + \bar{\gamma}(t)^T [h(t, y, \nu) - \dot{y}]\}\, dt$ *is strictly quasiinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to* η *and* ζ *and* $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ *are quasiinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to the same* η *and* ζ, *then the following cannot hold simultaneously*,

$$\int_{t_0}^{t_f} f_i(t, \bar{x}, \bar{u})\, dt \leq \int_{t_0}^{t_f} \{f_i(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, \bar{y}, \bar{\nu}) + \bar{\gamma}(t)^T [h(t, \bar{y}, \bar{\nu}) - \dot{\bar{y}}]\}\, dt, \quad \text{for all} \quad i \in \{1, 2, \ldots, p\}, \tag{30}$$

$$\int_{t_0}^{t_f} f_j(t, \bar{x}, \bar{u})\, dt \underset{\neq}{<} \int_{t_0}^{t_f} [f_j(t, \bar{y}, \bar{\nu}) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, \bar{y}, \bar{\nu}) + \bar{\gamma}(t)^T \times [h(t, \bar{y}, \bar{\nu}) - \dot{\bar{y}}]]\, dt, \quad \text{for some} \quad j \in i \in \{1, 2, \ldots, p\}. \tag{31}$$

THEOREM 2′ (Weak Duality). *Assume that for all feasible* $(\bar{x}, \bar{u})$ *for* (*VCP*) *and for all feasible* $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ *for* (*VCD*2), *one of the functionals* $\int_{t_0}^{t_f} \{\sum_{i=1}^{p} \bar{\lambda}_i f_i(t, y, \nu) + \bar{\mu}(t)_\Sigma^T g_\Sigma(t, y, \nu) + \bar{\gamma}(t)^T [h(t, y, \nu) - \dot{y}]\}\, dt$ *on* $\int_{t_0}^{t_f} \bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ *is strictly quasiinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to* η *and* ζ *and the other is quasiinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to the same* η *and* ζ, *then* (30) *and* (31) *cannot hold simultaneously.*

THEOREM 3′ (Weak Duality). *Assume that for all feasible* $(\bar{x}, \bar{u})$ *for* (*VCP*) *and all feasible* $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ *for* (*VCD*2), $\int_{t_0}^{t_f}\{\sum_{i=1}^{p}\bar{\lambda}_i f_i(t, y, \nu) + \bar{\mu}(t)^T g(t, y, \nu) + \bar{\gamma}(t)^T[h(t, y, \nu) - \dot{y}]\}\, dt$ *is strictly pseudoinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to* η *and* ζ, *then* (30) *and* (31) *cannot hold simultaneously.*

THEOREM 4′ (Weak Duality). *Assume that for all feasible* $(\bar{x}, \bar{u})$ *for* (*VCP*) *and for all feasible* $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ *for* (*VCD*2), $\int_{t_0}^{t_f}\{f_i(t, y, \nu) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, y, \nu) + \bar{\gamma}(t)^T[h(t, y, \nu) - \dot{y}]\}\, dt$ $(i = 1, 2, \ldots, p)$ *is strictly pseudoinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to* η *and* ζ *and* $\int_{t_0}^{t_f}\bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'} \times (t, y, \nu)\, dt$ *is quasiinvex at* $(\bar{y}, \bar{\nu})$ *on* $X \times U$ *with respect to the same* η *and* ζ, *then* (30) *and* (31) *cannot hold simultaneously.*

THEOREM 5′ (Weak Duality). *Assume that for all feasible* $(\bar{x}, \bar{u})$ *for* (*VCP*) *and for all feasible* $(\bar{y}, \bar{\nu}, \bar{\lambda}, \bar{\mu}, \bar{\gamma})$ *for* (*VCD*2),

(i) $\bar{\lambda}_i > 0, i = 1, 2, \ldots, p$;

(ii) *for each* $i \in \{1, 2, \ldots, p\}$, $\int_{t_0}^{t_f}\{f_i(t, y, \nu) + \bar{\mu}(t)_{\Sigma}^T g_{\Sigma}(t, y, \nu) + \bar{\gamma}(t)^T[h(t, y, \nu) - \dot{y}]\}\, dt$, *is pseudoinvex*;

(iii) $\int_{t_0}^{t_f}\bar{\mu}(t)_{\Sigma'}^T g_{\Sigma'}(t, y, \nu)\, dt$ *is quasiinvex,*

then (30) *and* (31) *cannot hold simultaneously.*

THEOREM 6′ (Strong Duality). *Let* (x^0, u^0) *be an efficient solution for* (*VCP*) *and let all the constraint qualifications of Lemma* 1 *for* $P_k(x^0, u^0)$ *for at least one* $k \in \{1, 2, \ldots, p\}$ *hold, then there are nonnegative* $\lambda^0 \in R^p$ *and piecewise smooth* $\mu^0 : I \to R^l$ *and* $\gamma^0 : I \to R^n$ *such that* $(x^0, u^0, \lambda^0, \mu^0, \gamma^0)$ *is feasible for* (*VCD*2) *and* $\mu^0(t)^T g(t, x^0, u^0,) = 0$, *and if any one of weak duality theorems holds between* (*VCP*) *and* (*VCD*2), *then* $(x^0, u^0, \lambda^0, \mu^0, \gamma^0)$ *is efficient for* (*VCD*2).

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