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Asymptotic Analogs of the Rogers–Ramanujan Identities

CHARLES H. BRENNER

Department of Mathematics, University of California, Los Angeles, California 90024

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Let $p(n \mid S)$ be the number of partitions of n with parts belonging to the set S; let $q(n \mid S)$ be the number of partitions of n with parts distinct and belonging to the set S; let $q_d(n)$ be the number of partitions of n with parts differing by at least d. Asymptotic formulas for $p(n \mid S)$, $q(n \mid S)$, and $q_d(n)$ are derived. Using these formulas necessary and/or sufficient conditions are obtained on sets S and S' for the various asymptotic relations $p(n \mid S) \sim q(n \mid S')$, $q(n \mid S) \sim q(n \mid S')$, and $q_d(n) \sim q(n \mid S)$. The last case leads to a nonexistence theorem analogous to those of Lehmer for equality. The other comparisons lead to infinite families of cases of asymptotic equality without strict equality. These new formulas can be interpreted as asymptotic analogs of classical Rogers–Ramanujan identities.

0. Introduction

By a *partition* of n we mean a non-increasing sequence of positive integers, called *parts*, which sum to n. The number of partitions of n is denoted by $p(n)$. We shall also use notation as follows for the number of partitions of n where the parts obey additional restrictions as indicated:

$p(n \mid S)$	parts belong to a set S
$q(n)$	parts are distinct
$q(n \mid S)$	parts distinct and belong to S
$q_d(n)$	parts differ by at least d
$q_{d,m}(n)$	parts $\geqslant m$ and differ by at least d

The case where S is a union of arithmetic progressions is of particular interest, so we shall adopt the notation

$$S = \{r_1, r_2, \ldots \bmod k\}$$

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to mean the set

$$\{r_i + kn \mid n \geqslant 0\},$$

with the r_i positive and distinct mod k. The possibility $r_i \geqslant k$ is permitted.

For example

$$q_2(n) = p(n \mid 1, 4 \bmod 5) \tag{1}$$

$$q_{2,2}(n) = p(n \mid 2, 3 \bmod 5) \tag{2}$$

are the Rogers–Ramanujan identities. Other examples of such identities have been given by Schur [28], Slater [29], Gordon [12–14], Göllnitz [11], and Andrews [3]. A unified understanding of some of these identities can be obtained using Lie algebras [20].

Non-existence results have been given by Lehmer [19] and Alder [1].

It is natural to ask for a formula to calculate arbitrary values of a partition function, but usually no simple one is available. We may settle then for an asymptotic formula; for example, Hardy and Ramanujan [15] proved that

$$p(n) \sim e^{\pi\sqrt{2n/3}}/4n\sqrt{3} \tag{3}$$

by exploiting the theory of elliptic modular functions. Their method has been refined by Rademacher [24] and others to give convergent series for certain partition functions.

Ingham [17], on the other hand, showed that (3) can be obtained by less refined results—using a Tauberian theorem—and Meinardus has given other conditions under which formulas like (3) can be obtained. We shall apply Meinardus' and Ingham's methods to all the partition functions listed above (Sect. 1). Similar results under different conditions have been obtained elsewhere—see, for example, Andrew [2].

Andrew [2] has suggested using asymptotics to narrow the search for identities of the Rogers–Ramanujan type. We shall exploit this idea, proving asymptotic inequality between classes of partition functions which Lehmer has shown to be not identically equal. On the other hand, we will show that asymptotic equality without equality is possible, and give some examples (Sect. 2).

1. Some Asymptotic Formulae

Let S be a set of positive integers. Let

$$P(\tau \mid S) = \sum_{n=0}^{\infty} p(n \mid S)\, e^{-n\tau}$$

and

$$Q(\tau \mid S) = \sum_{n=0}^{\infty} q(n \mid S)\, e^{-n\tau}.$$

We shall sometimes refer to $P(\tau \mid S)$ as the generating function of $p(n \mid S)$ although strictly speaking the generating function is $P(-\ln \tau \mid S)$. Define

$$D(s) = \sum_{\nu \in S} \nu^{-s},$$

where $s = \sigma + it$ is a complex variable. The series converges absolutely for $\sigma > 1$, and defines a holomorphic function of s there.

Lemma 1. *Suppose $D(s)$ can be analytically continued to some half plane $\sigma \geqslant -c_0$, $0 < c_0 < 1$, and has a pole of residue A at $s = 1$. Suppose that the estimate*

$$D(s) = O(|t|^{c_1})$$

holds uniformly in σ as $|t| \to \infty$, where c_1 is some positive constant. Then

$$P(\tau \mid S) \sim e^{D'(0)} \tau^{-D(0)} e^{\pi^2 A/6\tau}, \tag{4}$$

and

$$Q(\tau \mid S) \sim 2^{D(0)} e^{\pi^2 A/12\tau} \tag{5}$$

as $\tau \to 0$ in any fixed Stolz wedge $|\arg \tau| \leqslant \Delta$ where $\Delta < \pi/2$.

Proof. Meinardus [22] proved (4) (with slightly different conditions) in the following way. We have

$$P(\tau \mid S) = \prod_{\nu \in S} (1 - e^{-\nu\tau})^{-1},$$

convergent for complex τ inside the unit disc, so that

$$\begin{aligned} \ln P(\tau \mid S) &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\nu \in S} e^{-\nu k \tau} \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\nu \in S} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (\nu k \tau)^{-s} \Gamma(s)\, ds, \end{aligned}$$

by the Mellin inversion formula

$$e^{-\tau} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \tau^{-s} \Gamma(s)\, ds.$$

Then

$$\ln P(\tau \mid S) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \tau^{-s} \Gamma(s)\, D(s)\, \zeta(s+1)\, ds.$$

The integrand $f(s)$ is analyzed as follows. Within a Stolz wedge $|\arg \tau| \leqslant \Delta < \pi/2$,

$$\begin{aligned} |\tau^{-s}| &= |e^{-(\sigma+it)(\ln|\tau| + i \arg \tau)}| \\ &= |\tau|^{-\sigma} e^{-t \arg \tau} \\ &< |\tau|^{-\sigma} e^{\Delta |t|}. \end{aligned} \tag{6}$$

In the strip $-c_0 \leqslant \sigma \leqslant 2$, the estimates

$$\Gamma(s) = O(e^{-(\pi/2)|t|} |t|^{c_1}), \qquad D(s) = O(|t|^{c_2}), \qquad \zeta(s+1) = O(|t|^{c_3}), \tag{7}$$

hold uniformly in σ as $|t| \to \infty$ for appropriate c_1, c_2, c_3 [6, hypothesis, 32]. This shows that $f(s) \to 0$ as $|t| \to \infty$. The only singularities of $f(s)$ in the strip $-c_0 \leqslant \sigma \leqslant 2$ are those at $s = 0$ and $s = 1$, so

$$\ln P(\tau \mid S) = \frac{1}{2\pi i} \int_{-c_0+i\infty}^{-c_0+i\infty} f(s)\, ds + \operatorname*{Res}_{s=0} f(s) + \operatorname*{Res}_{s=1} f(s).$$

Clearly,

$$\operatorname*{Res}_{s=1} f(s) = \tau^{-1} \Gamma(1)\, A \zeta(2) = \pi^2 A / 6\tau.$$

Near $s = 0$, on the other hand, the factors of $f(s)$ have Laurent series

$$\begin{aligned} \tau^{-s} &= 1 - s \ln \tau + O(s^2), \\ \Gamma(s) &= 1/s - \gamma + O(s), \\ D(s) &= D(0) + sD'(0) + O(s^2), \\ \zeta(s+1) &= 1/s + \gamma + O(s), \end{aligned}$$

(where γ is Euler's constant) so that

$$\operatorname*{Res}_{s=0} f(s) = D'(0) - D(0) \ln \tau.$$

Consequently,

$$\ln P(\tau | S) = \frac{\pi^2 A}{6\tau} - D(0) \ln \tau + D'(0) + \varepsilon,$$

where

$$\varepsilon = \frac{1}{2\pi i} \int_{-c_0 - i\infty}^{-c_0 + i\infty} \tau^{-s} \Gamma(s)\, D(s)\, \zeta(s+1)\, ds.$$

By virtue of the estimates (6) and (7)

$$\varepsilon = O(|\tau|^{c_0})$$

and the estimate (4) for $P(\tau \mid S)$ follows. The estimate (5) for $Q(\tau \mid S)$ can be obtained by a similar calculation, or by noting that

$$Q(\tau \mid S) = \prod_{v \in S} (1 + e^{-ev\tau}) = P(\tau \mid S)/P(2\tau \mid S). \quad \blacksquare$$

The asymptotic formula for $Q(\tau \mid S)$ does not depend on $D'(0)$.

The next two lemmas are results from the literature.

LEMMA 2 [23]. *Let*

$$Q_{d,m}(\tau) = \sum_{n=0}^{\infty} q_{d,m}(n)\, e^{-n\tau}.$$

Then

$$Q_{d,m}(\tau) \sim \{\alpha_d^{d+1-2m}(d\alpha_d^{d-1} + 1)\}^{-1/2}\, e^{A_d/\tau} \tag{8}$$

where $\tau = x + iy$, uniformly in y as $x \to 0+$ with $|y| < x^{35/24}$, and

$$q_{d,m}(\tau) \sim C(d, m)\, n^{-3/4}\, e^{2\sqrt{A_d n}} \qquad \text{as} \quad n \to \infty. \tag{9}$$

Here

$$C(d, m) = \frac{1}{2\sqrt{\pi}} A_d^{1/4} \{\alpha_d^{d+1-2m}(d\alpha_d^{d-1} + 1)\}^{-1/2},$$

$$A_d = \frac{d}{2} \ln^2 \alpha_d + \sum_{r=1}^{\infty} \frac{\alpha_d^{rd}}{r^2}, \tag{10}$$

and

$$\alpha_d > 0 \qquad \alpha_d^d + \alpha_d - 1 = 0. \tag{11}$$

LEMMA 3 [17]. *Let $f(\tau) = \int_0^\infty e^{-u\tau}\, dr(u)$. Suppose*

(i) $f(\tau) \sim \chi(\tau)\, e^{\phi(\tau)}$ *as* $\tau = 0$ *in every Stolz wedge* $|\arg \tau| \leqslant \Delta$, $\Delta < \pi/2$, *with*

$$\chi(\tau) = C(M/\tau)^{m\beta - 1/2},$$
$$\phi(\tau) = \beta^{-1}(M/\tau)^{\beta} \qquad (\beta, M, C > 0, m \text{ real});$$

(ii) $r(u)$ *is increasing (in the wide sense) for* $u \geqslant 0$.

Then

$$r(u) \sim \left(\frac{1-\alpha}{2\pi}\right)^{1/2} C(Mu)^{m\alpha - 1/2}\, e^{\alpha^{-1}(Mu)^{\alpha}}, \qquad (\alpha = \beta/(\beta+1)).$$

LEMMA 4. *Suppose the set S satisfies the conditions of Lemma* 1. *Then*

$$p(n \mid S) \sim \frac{e^{D'(0)}}{2\sqrt{\pi}} \left(\frac{\pi^2 A}{6}\right)^{1/4 - D(0)/2} n^{D(0)/2 - 3/4}\, e^{\pi\sqrt{2An/3}} \tag{12}$$

provided $p(n \mid S)$ is eventually increasing;

$$q(n \mid S) \sim 2^{D(0) - 3/2} \left(\frac{A}{3}\right)^{1/4} n^{-3/4} e^{\pi\sqrt{An/3}} \tag{13}$$

provided $q(n \mid S)$ is eventually increasing.

Proof. To show (12), let

$$r(u) = \max_{n \leqslant u} p(n \mid S).$$

Then put

$$\begin{aligned} f(\tau) &= \int_0^{\infty} e^{-u\tau}\, dr(u) \\ &= \int_0^{\infty} e^{-u\tau}\, d\left[\sum^{u} r(n) - \sum^{u-1} r(n)\right] \\ &\sim (1 - e^{-\tau})\, P(\tau \mid S) \\ &\sim \tau P(\tau \mid S) \end{aligned}$$

Referring to (4), we take

$$\chi(\tau) = e^{D'(0)} \tau^{1 - D(0)},$$
$$\phi(\tau) = \pi^2 A/6\tau,$$

and apply Lemma 3 with $\beta=1$, $\alpha=\frac{1}{2}$, $m=D(0)-\frac{1}{2}$, $M=\pi^2A/6$, and $CM^{D(0)-1}=e^{D'(0)}$. This gives

$$p(n\mid S)\sim r(n)\sim\frac{1}{2\sqrt{\pi}}\,e^{D'(0)}M^{1-D(0)}(Mn)^{D(0)/2-3/4}\,e^{2\sqrt{\pi^2An/6}}$$

which is equivalent to (12). The formula (13) is derived in a similar fashion from (5) with

$$\chi(\tau)=2^{D(0)}\tau,$$
$$\phi(\tau)=\pi^2A/12\tau,$$

so $\beta=1$, $\alpha=\frac{1}{2}$, $m=-\frac{1}{2}$, $M=\pi^2A/12$, and $CM^{-1}=2^{D(0)}$. Consequently,

$$q(n\mid S)\sim\frac{1}{2\sqrt{\pi}}\,2^{D(0)}M(Mn)^{-3/4}e^{2\sqrt{\pi^2An/12}}. \quad \blacksquare$$

If the asymptotic formula (13) for $q(n\mid S)$ is written in the form $\exp(a+b\ln n+c\sqrt{n})$, a and c depend on S but b does not, which is curious.

LEMMA 5. *Let $S=\{r_1,\ldots,r_a \bmod k\}$. Then*

$$A=a/k, \tag{14}$$

$$D(0)=\sum_{j=1}^{a}\left(\frac{1}{2}-\frac{r_j}{k}\right), \tag{15}$$

$$e^{D'(0)}=k^{-D(0)}\prod_{j=1}^{a}\frac{\Gamma(r_j/k)}{\sqrt{2\pi}}. \tag{16}$$

If $\{r_1,\ldots,r_a\}=\{k-r_1,\ldots,k-r_a\}$, we have

$$D(0)=0, \tag{17}$$

$$e^{D'(0)}=1\Big/2^{a/2}\prod_{r_j\leqslant k/2}\sin\frac{\pi r_j}{k}. \tag{18}$$

Proof. Equation (14) is well known. Since $\sum_{n=0}^{\infty}(r+nk)^{-s}=k^{-s}\zeta(s,r/k)$, (15) and (16) follow from corresponding properties of Hurwitz' zeta function; (17) is immediate from (15); (18) follows from (16) because

$$\Gamma(a)\,\Gamma(1-a)=\frac{\pi}{\sin\pi a}\qquad\text{and}\qquad\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\Big/\sin\frac{\pi}{2};$$

so

$$\Gamma\left(\frac{r_i}{k}\right)\Gamma\left(\frac{k-r_i}{k}\right)\Big/2\pi = 1\Big/2\sin\frac{\pi r_i}{k},$$

and

$$\Gamma\left(\frac{1}{2}\right)\Big/\sqrt{2\pi} = 1\Big/\sqrt{2}\sin\frac{\pi}{2}. \quad \blacksquare$$

THEOREM 6. *Suppose* $S = \{r_1, \ldots, r_a \bmod k\}$ *has no common divisor. Then* (12) *and* (13) *hold, with* A, $D(0)$, *and* $e^{D'(0)}$ *given by Lemma* 5.

Proof. The Dirichlet series can be continued to the entire complex plane in this case. The only difficulty is to establish the necessary monotonicity.

Bateman and Erdös [7] proved that $p(n \mid S)$ is monotonic if and only if either

(i) $1 \in S$, or

(ii) whatever element is deleted from S, the remaining set has no common divisor.

Obviously (ii) holds here. Roth and Szekeres [27] give the following condition for eventual monotonicity of $q(n \mid S)$, $S = \{s_1, s_2, \ldots\}$:

I. $\lim_{j \to \infty} \log_j s_j$ exists, and

II. $J_j = \inf_{1/2s_j \leqslant \beta < 1/2} (\ln j)^{-1} \sum_{i=1}^{j} \|\beta s_i\|^2 \to \infty$ as $j \to \infty$, where $\|x\|$ is the distance from x to the nearest integer.

When S is a union of arithmetic progressions the first limit is 1, so I holds.

To examine the growth of J_j, suppose j is large and consider two cases, depending on β. If $\|\beta k\| \geqslant 1/2sj$, then $\|\beta s_i\| \geqslant 1/4$ for nearly 2/5 of the s_i up to s_j. So the sum in II grows linearly with j. If $\|\beta k\| < 1/2sj$, then k is close to some integer t, $0 < t < k$. Since S has no common divisor, the $\|\beta r_i\|$ cannot also all be small—by a straightforward computation, if j is sufficiently large we must have $\|\beta r_i\| > \beta/2k^2 > 1/4k^3$ for some i. It follows that nearly 1/3 of the numbers $\|\beta r_i\|$, $\|\beta(r_i + k)\|, \ldots, \|\beta(r_i + [j/a]k)\|$ exceed $1/8k^3$, and again there is a lower bound for the sum that is linear in j. Hence J_j exceeds a positive linear function of j, so II holds. $\blacksquare$

In particular we note that n occurs to the power $-\frac{3}{4}$ in formula (13). This gives an affirmative answer to part of a conjecture of Andrews [2].

2. Some Comparisons

2.1.1. *The Case $q \sim q$*

Suppose that S and S' are two sets satisfying the hypotheses of Theorem 6. Then from (13), (14), and (15)

$$q(n \mid S) \sim q(n \mid S') \tag{19}$$

if and only if S and S' have representations such that

$$k = k', \qquad a = a', \qquad \sum_{i=1}^{a} (r_i - r_i') = 0.$$

On the other hand, suppose S and S' merely satisfy the hypotheses of Lemma 4, and that the symmetric difference $S \Delta S'$ is finite. Then (19) holds if and only if S and S' omit the same number of elements of each other. These two observations suggest the following assertion.

Theorem 7. *Let $S = \{r_1, r_2, \ldots\}$ and $S' = \{r_1', r_2', \ldots\}$ be two increasing sequences of positive integers, and suppose*

$$\left| \sum_{i=1}^{N} (r_i - r_i') \right| \leqslant B$$

for some B and all N. Then (19) *holds provided both functions satisfy $q(n) \sim q(n+1)$.*

Proof. Let $Y(S, n, m)$ be the set of partitions of integers from n to m inclusive into distinct parts from S.

If we substitute for each part r_i in a partition in Y the corresponding part r_i' from S', the sum of each partition changes by at most B. Consequently,

$$|Y(S, n, n + kB)| \leqslant |Y(S', n - B, n + (k+1)B)|.$$

Since $q(n) \sim q(n+1)$,

$$(1 + kB)\, q(n \mid S) \lesssim [(k+2)B + 1]\, q(n \mid S'),$$

where $\lesssim$ means "asymptotically less than or equal to." Since this is true for any k, and the roles of S and S' may be reversed, the result follows. ∎

Since any partition function grows more slowly than exponentially, the condition $q(n) \sim q(n+1)$ is satisfied provided $q(n)$ is eventually monotonic. Here is an example indicating that the hypothesis cannot be omitted. Let $S = \{1, 2, 3, 7, 14, 28, 56, \ldots\}$ and $S' = \{1, 2, 4, 7, 14, 28, 56, \ldots\}$. Then $B = 1$.

But $q(n \mid S)$ is 2 when $n \equiv 3 \bmod 7$, 1 otherwise; while $q(n \mid S')$ is 2 for multiples of 7, and 1 otherwise. So they are not asymptotically equal.

2.1.2. *The Case* $q = q$

It is trivial that $q(n \mid S) = q(n \mid S') \Leftrightarrow S = S'$, whether S and S' are unions of arithmetic progressions or not.

2.2.1. *The Case* $p = q$

The possibility $p(n \mid S) = q(n \mid S')$ has two well-known instances. Euler observed that $p(n \mid 1 \bmod 2) = q(n)$, and Schur [28] that $p(n \mid 1, 5 \bmod 6) = q(n \mid 1, 2, 4, 5 \bmod 6)$. One can easily generalize:

THEOREM 8. *$p(n \mid S) = q(n \mid S')$ if and only if S' is the disjoint union $\dot{\bigcup}_{j=0}^{\infty} 2^j S$.*

Proof.

$$p(n \mid S) = q(n \mid S') \Leftrightarrow \prod_{v \in S'} (1-x)^{2v} \prod_{\mu \in S} (1-x)^{\mu} = \prod_{v \in S'} (1-x)^{v}.$$

Thus $2S' \dot{\cup} S = S'$, from which the claim easily follows. ∎

Bachmann [5] attributes this result to J. Schur, circa 1910.

EXAMPLE. Let H be any collection of odd integers,

$$S = \{\text{odd positive integers not divisible by any } h \in H\},$$

$$S' = \{\text{positive integers not divisible by any } h \in H\}.$$

Euler's is the case $H = \varnothing$ and Schur's corresponds to $H = \{3\}$.

EXAMPLE.

$$S = \{\text{positive integers prime to 30, or} \equiv \pm 5 \bmod 30\},$$

$$S' = \{\text{positive integers prime to 15, or} \equiv \pm 5 \bmod 15\}.$$

2.2.2. *The Case* $p \sim q$

Examples with $p(n \mid S) \sim q(n \mid S')$ can of course be constructed using Theorem 7 and 8, but there are also possibilities not depending on that idea.

THEOREM 9. *Let $R = \{r_1, ..., r_a\}$ with $1 \leqslant r_i < k$ be chosen such that $R = \{k - r_1, ..., k - r_a\}$ (so R is symmetrical about $k/2$) and $S = \{r_1, ..., r_a \bmod k\}$ has no common factor, and let*

$$R' = \{r'_1, ..., r'_{2a}\}, \qquad 1 \leqslant r'_i \tag{20}$$

be a collection of representatives of distinct congruence classes mod k *such that* $S' = \{r'_1, \ldots, r'_{2a} \bmod k\}$ *has no common factor. Then* $p(n \mid S) \sim q(n \mid S')$ *if and only if*

$$\sum_{i=1}^{2a} r'_i = k\left(\frac{3a}{2} + \log_2 \prod_{r_j \leqslant k/2} \sin\frac{\pi r_j}{k}\right). \tag{21}$$

Proof. Let $D(s)$ and $E(s)$ be the Dirichlet series associated with S and S' respectively; let A and B be the corresponding residues at 1. Then by Lemma 1 we have

$$P(\tau \mid S) \sim e^{D'(0)}\tau^{-D(0)}e^{\pi^2 A/6\tau},$$
$$Q(\tau \mid S') \sim 2^{E(0)}e^{\pi^2 B/12\tau},$$

as $\tau \to 0$ with $|\arg \tau| \leqslant \Delta < \tau/2$.

We have $A = a/k$, $B = 2a/k$, and $D(0) = 0$ by Lemma 5. Therefore $P(\tau \mid S) \sim Q(\tau \mid S')$ as $\tau \to 0$ if and only if $e^{D'(0)} = 2^{E(0)}$. Now

$$e^{D'(0)} = \left(2^{a/2} \prod_{r_j \leqslant k/2} \sin\frac{\pi r_j}{k}\right)^{-1}$$

by (18) and $E(0) = a - (1/k)\sum_{i=1}^{2a} r'_i$ by (15). Hence $P(\tau \mid S) \sim Q(\tau \mid S')$ as $\tau \to 0$ if and only if (21) holds. By (12) and (13) the asymptotics of $p(n \mid S)$ and $q(n \mid S')$ are determined by the same parameters, so $p(n \mid S) \sim q(n \mid S')$ if and only if (21) holds. ▮

EXAMPLES. In order to satisfy (21), we need

$$\prod_i \sin\frac{\pi r_i}{k} \tag{22}$$

to be a rational power of 2. $k = 12$ gives two obvious possibilities for R; the first of these is

$$R = \{2, 3, 9, 10\}.$$

Since $\sin(\pi/6) \cdot \sin(\pi/4) = 2^{-3/2}$, by the theorem we require a solution to

$$\sum_{i=1}^{8} r'_i = 12\left(6 - \tfrac{3}{2}\right) = 54.$$

Thus, for example,

$$p(n \mid 2, 3, 9, 10 \bmod 12) \sim q(n \mid 2, 3, 5, 6, 8, 9, 10, 11 \bmod 12).$$

However a shortcoming of this example is that it can be proved without Theorem 9, for it follows from Theorem 8 that

$$p(n \mid 2, 3, 9, 10 \bmod 12) = q(n \mid 2, 3, 4, 6, 8, 9, 10, 12 \bmod 12)$$

and from Theorem 7 that

$$q(n \mid 2, 3, 4, 6, 8, 9, 10, 12 \bmod 12) \sim q(n \mid 2, 3, 5, 6, 8, 9, 10, 11 \bmod 12).$$

The other obvious possibility with $k = 12$ is

$$R = \{2, 3, 6, 9, 10\}.$$

This time R and $2R$ are not disjoint, so Theorem 8 will not apply. We require $\sum_{i=1}^{10} r_i' = 12(\frac{15}{2} - \frac{3}{2}) = 72$, so that

$$p(n \mid 2, 3, 6, 9, 10 \bmod 12) \sim q(n \mid 2, 3, 4, 6, 7, 8, 9, 10, 11, 12 \bmod 12),$$

among various possibilities. None of these possibilities gives strict equality.

There is a rather limited repertory of angles we can use whose sines are rational powers of two. However, considering examples of equality as in Section 2.2.1 makes it clear that the product (22) can have the required property in other ways. For example,

$$\begin{aligned} &p(n \mid 1, 5, 7, 11, 13, 17, 19, 23 \bmod 24) \\ &\quad = q(n \mid 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23 \bmod 24) \end{aligned}$$

(Schur's identity), so from (21),

$$\sin\frac{\pi}{24} \cdot \sin\frac{5\pi}{24} \cdot \cdots \cdot \sin\frac{23\pi}{24} = 2^{-8},$$

a formula that is essentially due to Gauss [9]. Now using

$$\log_2\left(\sin\frac{6\pi}{24} \cdot \sin\frac{12\pi}{24}\right) = -\frac{1}{2},$$

we can modify the previous example to

$$\begin{aligned} &p(n \mid 1, 5, \mathbf{6}, 7, 11, \mathbf{12}, 13, 17, \mathbf{18}, 19, 23 \bmod 24) \\ &\quad \sim q(n \mid 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, \\ &\qquad 20, 21, 22, 23, 24 \bmod 24). \end{aligned}$$

More generally, for $4\,|\,k$, $k \geqslant 24$, and k not a power of 2, let

$$R = \{r \mid 1 \leqslant r < k \ \ (r, k) = 1\} \cup \left\{\frac{k}{4}, \frac{k}{2}, \frac{3k}{4}\right\}, \tag{23}$$

and choose R' so that

$$\sum_{i=1}^{2|R|} r_i' = k(|R| + 1). \tag{24}$$

If $12\,|\,k$, $k \geqslant 36$, further examples can be constructed by adjoining $\{k/6, 5k/6\}$ to the R of (23). Formula (24) remains valid. All of these cases lead to asymptotic equality between $p(n\,|\,r_1,..., r_a \bmod k)$ and $q(n\,|\,r_1',..., r_{2a}' \bmod k)$ without strict equality.

Examples with odd modulus are harder to come by. We can obtain an appropriate combination of sines. For example, using

$$\Pi_n = \prod_{j=1}^{n-1} \sin\frac{j\pi}{n} = 2^{1-n}n,$$

we easily have

$$\frac{\Pi_{15}}{\Pi_3 \Pi_5} = 2^{-8}$$

which suggests taking $S = \{1, 2, 4, 7, 8, 11, 13, 14 \bmod 15\}$. But the corresponding S' would have 16 residue classes mod 15 according to (20). Nonetheless, it is probably true that partitions into parts from the set S are asymptotically equinumerous with partitions into distinct parts not divisible by 15 and allowing two kinds of parts for each part size congruent to 7 or 8. The missing ingredient in the proof of such an assertion is a condition assuring the eventual monotonicity of the second partition function.

Each of the preceding examples corresponds in an obvious way to an asymptotic equality between generating functions. For example the last one corresponds to

$$\sum_{\substack{n=1 \\ (n,15)=1}}^{\infty} \frac{1}{1-x^n} \sim \prod_{n=1}^{\infty} \frac{(1+x^n)(1+x^{15n+7})(1+x^{15n+8})}{1+x^{15n}} \qquad \text{as} \quad x \to 1^-,$$

which can be proved using the results of Section 1. Such formulas may be thought of as examples of Rogers–Ramanujan type asymptotic identities. Of course, in each case above both sides will have a product representation. It would be interesting to find examples where one side is represented as a sum rather than a product, following the model of the Rogers–Ramanujan identities. For example, (1) stated in terms of generating functions is

$$\sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)(1-x^2)\cdots(1-x^m)} = \sum_{\substack{n \equiv \pm 1 \pmod 5 \\ n>0}} \frac{1}{1-x^n}.$$

To see that the coefficient of x^n on the left is $q_2(n)$, consider that a partition $n = n_1 + n_2 + \cdots + n_m$, $n_i \geqslant n_{i+1} + 2$, corresponds to a partition of $n - m^2$ into m or fewer parts by taking the $(m-j)$th part to be $n_{m-j} - (2j+1)$.

The next section contains a negative result in the direction of obtaining a new formula giving asymptotic equality between a sum and a product.

2.3.1. *The Case* $q \sim q_d$

Lehmer [19] proved that the number of partitions of n, with parts differing by at least d, $d \geqslant 2$, are not equinumerous with partitions into distinct parts taken from any set whatever. The following theorem is an attempt at an asymptotic version of Lehmer's result.

THEOREM 10. *Suppose S is a union arithmetic progressions, $d \geqslant 2$ an integer. Then*

$$q(n \mid S) \sim q_d(n) \tag{25}$$

is impossible.

Proof. We can assume S has no common divisor, otherwise (25) is clearly impossible. Then the asymptotics of the two sides of (25) are given by (13) and (9). However, it will be slightly more convenient to compare the formulas for the asymptotic behavior of the corresponding generating functions, (5) and (8), which after all depend upon the same sets of parameters. Assuming (25) then, and letting $\tau \to 0+$ through real values,

$$2^{D(0)} e^{\pi^2 A/12\tau} \sim \{\alpha_d^{d+1-2m}(d\alpha_d^{d-1} + 1)\}^{-1/2} e^{A_d/\tau}. \tag{26}$$

By taking logarithms, we obtain

$$\pi^2 A/12 = A_d, \tag{27}$$

which is probably already impossible for $d > 2$. (By (1) and Theorem 8, the case $d = 2$ is impossible.) But that would be hard to prove. Substituting (27) in (26) and putting $m = 1$ gives the condition

$$\alpha_d^{d-1}(d\alpha_d^{d-1} + 1) = 2^{-2D(0)} = \xi, \qquad \text{say}, \tag{28}$$

which simplifies to

$$(d-1-\xi)\,\alpha_d^2+(2d-1)\,\alpha_d+d=0 \tag{29}$$

by using the relation $\alpha_d^d+\alpha_d-1=0$ of (11).

Now (29) and (11) are inconsistent. We consider two cases:

Case I. $(d-1-\xi)\,x^2-(2d-1)x+d=\psi(x)$ is irreducible over $Q(\xi)$. The coefficients of ψ are not all divisible by d, so from (29), α_d is not a unit in this case, while by (11) it is.

Case II. ψ splits into linear factors over $\mathbf{Q}(\xi)$. Then $\alpha_d \in \mathbf{Q}(\xi)$. $\alpha_d \notin \mathbf{Q}$ by (11), so $\mathbf{Q}(\xi)=\mathbf{Q}(2^{1/h})$, for some integer $h \geqslant 2$ by (15). Then by (28) $2^{1/h} \in \mathbf{Q}(\alpha_d)$, which shows that 2 ramifies in any field containing α_d. In particular this would include the splitting field L of x^d+x-1, by (11). However the discriminant of the roots of x^d+x-1 is easily calculated as follows:

Let $f(x)=x^d+x-1$. Put $u=f'(\alpha)$, so $u=d\alpha^{d-1}+1$, and let α be a root of f. We eliminate α, using (11), and get

$$(1-u)(d-1+u)^{d-1}=d^d,$$

a monic polynomial in u of degree $d-1$ with constant term

$$d^d-(d-1)^{d-1}=c.$$

The product of the conjugates $f'(\alpha)$, and hence the discriminant of the conjugates of α, must therefore be $\pm c$. Since c is plainly odd, the discriminant of L must be odd. The prime 2 is therefore ramified in L, a contradiction. ▌

2.3.2. *Other Comparisons*

Lehmer [19] also proved a non-existence theorem with p in place of q, namely that $p(n \mid S)=q_d(n)$ only when $d=1$ (Euler's case) or $d=2$ (the first Rogers–Ramanujan case, (1)). The difficulty with adapting the above proof to this situation arises in Case II. With $\xi=e^{-2D'(0)}$, it is not so clear why its discriminant cannot divide c.

Alder [1] extended Lehmer's results by allowing $q_{d,m}$ in place of q_d, thus encompassing the second Rogers–Ramanujan case, (2), as the only further exception to the impossibility of equality. In terms of the method above, the new difficulty is that ψ may have degree higher than 2, so might split but not completely.

Hence in certain respects the proof is ad hoc. It would be more direct to prove that (27) cannot hold for $d \geqslant 3$. A is rational by (14) and by (10)

$$A_d=L(1-\alpha_d),$$

where

$$L(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}+\frac{1}{2}\ln x\ln(1-x)$$

is Rogers' [26] form of the dilogarithm function. This function has been studied by many (including Watson [31], van der Poorten [30], Richmond and Szekeres [25], Loxton [21]). The explicit evaluations

$$L(0)=0, \qquad L(1)=\frac{\pi^2}{6}, \qquad L\left(\frac{1}{2}\right)=A_1=\frac{\pi^2}{12},$$

$$L(\alpha_2)=\frac{\pi^2}{10}, \qquad L(\alpha_2^2)=A_2=\frac{\pi^2}{15}$$

are easily proved; Rogers suggests there are no others. Negative or related results by many authors, including various unavailing computer searches by the present author, tend to support Rogers assertion. This in turn lends credence to the assumption that (27) is impossible for $d\geqslant 3$.

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