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On acyclic and unicyclic graphs whose minimum rank equals the diameter

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Abstract

The minimum rank of a graph G is defined as the smallest possible rank over all symmetric matrices governed by G. It is well known that the minimum rank of a connected graph is at least the diameter of that graph. In this paper, we investigate the graphs for which equality holds between minimum rank and diameter, and completely describe the acyclic and unicyclic graphs for which this equality holds.

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1. Introduction

The minimum rank of a graph has now become an important research stream within the realm of combinatorial matrix theory – especially arising in an inverse eigenvalue problem for graphs [7,8].

Additionally, minimum rank of graphs has been developed as a separate topic from the aforementioned inverse eigenvalue problem, and this notion continues to be of interest (see, for

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example [1,4,5,9]). The reader is also encouraged to consult the recent survey paper [6] for current information regarding many aspects of this minimum rank problem for graphs.

To this end, if $G = (V, E)$ is a graph without loops, then we let $S(G)$ denote the collection of real symmetric matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if $\{i, j\} \in E$. Evidently, a_{ii} plays no role relative to the graph G. Then we define the *minimum rank of a graph G* as

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) : A \in S(G)\}.$$

The number of vertices (order) of G will be denoted by $|G|$. It is obvious that $1 \leqslant \mathrm{mr}(G) \leqslant |G| - 1$, whenever G does not consist of isolated vertices. Also note that, since the minimum rank of a disconnected graph is equal to the sum of the minima ranks of its connected components, we will restrict our study to connected graphs.

A related dual notion is that of the *maximum multiplicity of a graph*. For a given graph G, we let $M(G)$ denote the maximum possible multiplicity of an eigenvalue λ of A with $A \in S(G)$. It is also clear that for any graph G on n vertices

$$M(G) + \mathrm{mr}(G) = n. \tag{1}$$

It is because of (1) that the minimum rank and maximum multiplicity of a graph can be viewed as dual notions. In fact, in the case of trees (connected acyclic graphs) this duality was exploited, to some degree, to establish an important equation (see Theorem 1 below) for $M(T)$, where T is a tree.

For a graph G, a collection Π of vertex-disjoint paths, each of which is an induced subgraph of G, that covers all the vertices of G is referred to as a *path cover* of G. The *cardinality* of a path cover is the number of paths in the path cover. The *path cover number* of a graph G, denoted by $P(G)$, is the minimum cardinality of all path covers of G, and a path cover is called *minimal* if it contains $P(G)$ paths. We note that an isolated vertex counts as a path in $P(G)$. A vertex v in G is called *doubly terminal* if there exists a minimal path cover having a path consisting of the single vertex v. Alternatively, v is called *simply terminal* if v is not doubly terminal while being the endpoint of a path in some minimal path cover of G.

The next result, proved in [8], has been the genesis of numerous works on minimum rank (see [2,3]).

Theorem 1. *For a tree T, $M(T) = P(T)$.*

Consequently, the minimum rank of a tree is easy to calculate. However, for general graphs beyond trees this is no longer the case. Of course many properties about minimum rank are known and the minimum rank of many classes of graphs are known, but a complete understanding of minimum rank is nowhere near complete (see [1–4,6,7]). In the papers [2,3] one quantity that turned out to be useful was the notion of the rank-spread of a vertex. Given a graph G and an induced subgraph H of G, we let $G - H$ denote the subgraph of G induced by the vertices in G that are not in H. Specifically, if v is a vertex of G, then $G - v$ is the subgraph obtained from G by removing vertex v. Then, for a fixed vertex v in G, the *rank-spread of v in G* is given by $r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$. It is not difficult to verify that $0 \leqslant r_v(G) \leqslant 2$.

We include, for clarity of exposition, some necessary terms from graph theory. If G is a graph, then $\deg_G(v)$ is the *degree of v in G* (i.e., the number of edges in G incident with v). If P is a path, then the *length of P* is the number of edges in P. The *distance* between two vertices u, v in a connected graph G is defined as the length of the shortest path joining u and v, and is denoted by $d(u, v)$. For a connected graph G, the *diameter of G* is given by

$$\operatorname{diam}(G) = \max_{u,v} d(u, v).$$

Observe that $\operatorname{diam}(G)$ is at most the length of the longest induced path in G, and if G is acyclic, then $\operatorname{diam}(G)$ is the length of the longest induced path in G. Along similar lines, the *girth of a graph* G is defined as the number of edges in the smallest induced cycle in G.

It is well known and not difficult to verify that $\operatorname{mr}(G) \geqslant \operatorname{diam}(G)$, and that $\operatorname{mr}(G)$ is at least the girth of G minus two. Our main purpose is to characterize the graphs for which $\operatorname{mr}(G) = \operatorname{diam}(G)$.

We close the introduction with an outline of the paper. In the next section, we focus on a special class of trees for which the minimum rank equals the diameter. From this, we move on to more general graphs and special path covers, and finish with a complete characterization of the trees and unicyclic graphs for which minimum rank coincides with diameter.

2. Centipedes

A tree T is called a *centipede* if T consists of a path P plus (possibly) one or more edges (*legs*) attached to the nonterminal vertices (*joints*) of P (see Fig. 1). A path is thus a "leg-less" centipede. Note that the diameter of a centipede is simply the length of P. We will call P a *diametrical path* of the centipede. In general, an induced path Q is called a *diametrical path for G* if its length is equal to the diameter of G. We say that a centipede is *regular* if no two consecutive joints both have legs, and *irregular* otherwise.

We begin with a characterization of regular centipedes that will be vital to our understanding of trees for which the minimum rank equals the diameter.

Proposition 2. *Let C be a centipede with diametrical path P. Then the following statements are equivalent*:

(i) *C is regular*;
(ii) *P belongs to some minimal path cover of C*;
(iii) $P(C) = |C| - \operatorname{diam}(C)$;
(iv) $\operatorname{mr}(C) = \operatorname{diam}(C)$.

Proof. (i) implies (ii): Let Π be a minimal path cover for a regular centipede C whose associated diametrical path P has vertices $u_1, u_2, \ldots, u_m$, where u_1 and u_m are terminal vertices. Since u_1 is a terminal vertex, there is some path P_1 in the collection Π that starts at u_1 and includes the vertices $u_1, u_2, \ldots, u_k$. Now if $k = m$, then we are done, otherwise $k < m$ and we deduce that the vertex u_{k+1} does not belong to P_1. Since C is regular, at most one of u_k, u_{k+1} has legs, and, bearing in mind that Π is a minimal path cover, it follows that exactly one of u_k, u_{k+1} has legs. Assume first that u_k has legs with endpoints $x_1, x_2, \ldots, x_r$, and that x_r belongs to P_1. In this case (see Fig. 2), since u_{k+1} does not have any legs, it follows that u_{k+1} is the endpoint of some other path P_2 in Π. If we let P_1' be the path obtained from P_1 by deleting the vertex x_r, and P' is the path obtained by joining P_1' and P_2 with the edge u_k, u_{k+1} (Fig. 3), then, by replacing P_1 and P_2 with

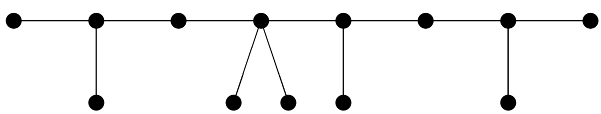

Fig. 1. A centipede.

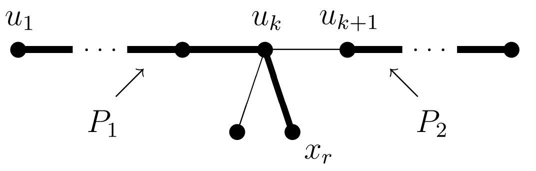

Fig. 2. The path cover with P_1 and P_2.

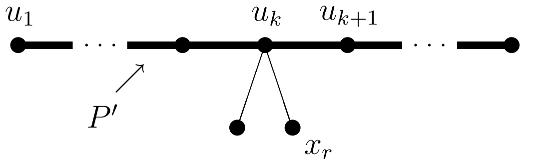

Fig. 3. The path cover with P' and x_r.

P' and x_r, we obtain another minimal path cover of C. However, this new path cover contains a path which includes more vertices (namely, at least $u_1, u_2, \ldots, u_k, u_{k+1}$) than Π did. Similar reasoning applies if u_{k+1} has legs while u_k does not. Repeating this process, we eventually arrive at a minimal path cover of C that contains the path P.

(ii) implies (i): Assume that P belongs to a minimal path cover, that C is irregular, and that the terminal vertices of P are a and b. Since P belongs to a minimal path cover, it is clear that the remaining paths in this specified cover must be simply vertices. Now since C is irregular, there are two adjacent joints u and v on P. Suppose that $\{u, x\}$ and $\{v, y\}$ are two legs of C and that P_1 (P_2) is the part of P from from a to u (from b to v). Then replace the paths P, x, y in the original path cover by the paths $P_1 + \{u, x\}$ and $P_2 + \{v, y\}$ to yield another path cover with a fewer number of paths. Since this is a contradiction, we conclude that C must have been regular.

(ii) implies (iii): If the diametrical path P belongs to a minimal path cover, then, counting the remaining vertices as paths of length zero, we have

$$P(C) = 1 + (|C| - (\mathrm{diam}(C) + 1)) = |C| - \mathrm{diam}(C).$$

(iii) implies (ii): If P does not belong to a minimal path cover, then $P(C)$ is strictly less than the cardinality of the path cover consisting of P and the isolated vertices that correspond to the endpoints of the legs. Then $P(C) < |C| - \mathrm{diam}(C)$.

(iii) $\Longleftrightarrow$ (iv): Since C is a tree, $P(C) = M(C) = |C| - \mathrm{mr}(C)$ and the result follows immediately. □

3. Particular path covers

First, we would like to mention a few specific path covers that will be of use. For a graph G, a path cover Π consisting of a diametrical path plus the remaining vertices as paths of length zero is called a *diametrical path cover* for G (see Fig. 4). Note that, from Proposition 2, a diametrical path cover for a centipede C is minimal if and only if C is regular. For a regular centipede C, with diametrical path P and corresponding diametrical path cover Π, we can obtain another minimal path cover Π' as follows: start at the left terminal vertex of the diametrical path P and, moving left-to-right on P, at each vertex u of P that has degree greater than two, cut the edge $\{u, v\}$, where v is the vertex to the right of u on P and add the edge $\{u, w\}$, where w is any vertex adjacent to u that does not lie on P, to the path constructed thus far (see Fig. 5). Continuing in this manner,

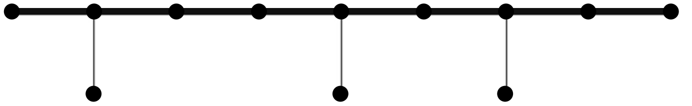

Fig. 4. Diametrical path cover for a centipede.

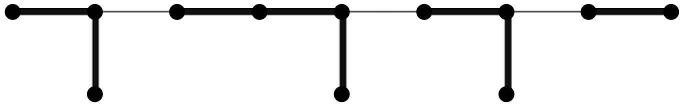

Fig. 5. LTR path cover.

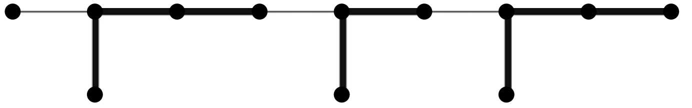

Fig. 6. RTL path cover.

it follows that Π' has the same number of paths as Π, and hence is also a minimal path cover for C. We call this minimal path cover the *left-to-right* (LTR) *path cover*, and, similarly, we have the *right-to-left* (RTL) *path cover* (Fig. 6).

We now examine more general graphs, but with an eye toward characterizing trees for which the minimum rank equals the diameter. We first establish the relationship between graphs in which the minimum rank equals the diameter and graphs in which a diametrical path cover is minimal. This is based upon the relationship between path cover number and maximal multiplicity of a graph.

Lemma 3. *For any graph G*

(i) *if $P(G) = M(G)$, then* $\mathrm{mr}(G) = \mathrm{diam}(G)$ *if and only if any diametrical path cover is a minimal path cover for G;*
(ii) *if $P(G) > M(G)$, then* $\mathrm{mr}(G) > \mathrm{diam}(G)$;
(iii) *if $P(G) < M(G)$, then any diametrical path cover is not a minimal path cover for G.*

Proof. First, note that the cardinality of any diametrical path cover is $|G| - \mathrm{diam}(G)$.

If $P(G) = M(G)$, then any diametrical path cover is a minimal path cover if and only if

$$|G| - \mathrm{diam}(G) = P(G) = M(G) = |G| - \mathrm{mr}(G),$$

that is, if and only if $\mathrm{diam}(G) = \mathrm{mr}(G)$, which proves (i).

If $P(G) > M(G)$, then (ii) follows by

$$|G| - \mathrm{diam}(G) \geqslant P(G) > M(G) = |G| - \mathrm{mr}(G).$$

Finally, let $P(G) < M(G)$, and suppose there is a diametrical path cover that is a minimal path cover. Then

$$|G| - \mathrm{diam}(G) = P(G) < M(G) = |G| - \mathrm{mr}(G),$$

which implies $\mathrm{mr}(G) < \mathrm{diam}(G)$, a contradiction. $\square$

For graphs whose minimum rank equals the diameter, we see that the property is, to a certain extent, hereditary. More precisely, deletion of vertices that do not lie on the diameter does not change the minimum rank nor the diameter.

Lemma 4. *Let P be a diametrical path of G where* $\operatorname{diam}(G) = \operatorname{mr}(G)$, *and let H be a set of vertices that do not lie on P. Then*

(i) $\operatorname{mr}(G - H) = \operatorname{mr}(G)$;
(ii) *if $G - H$ is connected, then* $\operatorname{mr}(G - H) = \operatorname{diam}(G - H)$.

Proof. Let G_1 be the connected component of $G - H$ that contains P. Then

$$\operatorname{diam}(G) \leqslant \operatorname{diam}(G_1) \leqslant \operatorname{mr}(G_1) \leqslant \operatorname{mr}(G - H) \leqslant \operatorname{mr}(G) = \operatorname{diam}(G) \tag{2}$$

and (i) follows.

If $G - H$ is connected, then $G_1 = G - H$, and hence (2) forces $\operatorname{diam}(G_1) = \operatorname{mr}(G_1)$, from which (ii) follows. □

An immediate consequence is the following corollary.

Corollary 5. *Suppose G is a graph for which* $\operatorname{mr}(G) = \operatorname{diam}(G)$ *and v is a vertex of G that does not lie on some diametrical path of G. Then $r_v(G) = 0$.*

4. Acyclic and unicyclic graphs

We are now in a position to characterize trees for which equality holds between minimum rank and diameter. Recall that a vertex v is said to be *appropriate* if there exist at least two pendant paths from v, while v is called a *peripheral leaf* if v has degree 1 and its only neighbor has degree no more than 2. Observe that it follows from [3, Propositions 4.1 and 4.3] that $r_v(G) = 2$ for an appropriate vertex, and $r_v(G) = 1$ for a peripheral leaf.

Theorem 6. *Let T be a tree. Then* $\operatorname{mr}(T) = \operatorname{diam}(T)$ *if and only if T is a regular centipede.*

Proof. If T is a regular centipede, then we have $\operatorname{mr}(T) = \operatorname{diam}(T)$ by Proposition 2. So assume that $\operatorname{mr}(T) = \operatorname{diam}(T)$, and let P be a diametrical path of T, and W the set of vertices of T that do not lie on P. By a similar argument to that given in [3, Lemma 5.1], it follows that if there is a vertex of W of degree greater than 2, then W contains an appropriate vertex. By Corollary 5, the vertices $v \in W$ satisfy $r_v(T) = 0$, so they can be neither appropriate vertices, nor peripheral leaves. In particular all vertices in W have degree at most 2. Moreover, W has no vertices of degree two, since this would imply the existence of a peripheral leaf. Thus, all vertices in W have degree one. Hence T is a centipede, and so by Proposition 2 it follows that T must be regular. This completes the proof. □

We now wish to apply the results for trees to unicyclic graphs (for which equality exists between minimum rank and diameter).

Theorem 7. *Let U be a connected unicyclic graph satisfying* $\operatorname{mr}(U) = \operatorname{diam}(U)$. *Then*

(i) *there is a vertex w in the unique cycle such that $U - w$ is a regular centipede, and*

(ii) *the cycle in* U *has length three or four and the vertices other than* w *that are on this cycle lie on a diametrical path in* $U - w$.

Proof. Let P be a diametrical path of U, and let Π be the corresponding diametrical path cover. Then there is at least one vertex w on the cycle that is not on P. Since U is unicyclic, $P(U) \geqslant M(U)$ (see [3]). Moreover, since $\mathrm{mr}(U) = \mathrm{diam}(U)$, it follows from Lemma 3 that $P(U) = M(U)$, and that Π is a minimal path cover. Therefore, w cannot be connected to any vertex in $U - P$, as this would lead to a path cover with fewer paths, and contradict the minimality of Π. Thus, if we define $T = U - w$, then T is connected and, by part (ii) of Lemma 4, we know that $\mathrm{mr}(T) = \mathrm{diam}(T)$. Thus, by Theorem 6, T is a regular centipede.

To demonstrate that (ii) holds, note that since w cannot be connected to a vertex in $U - P$, w is adjacent to two vertices u and v that lie on P. In P, if the distance between u and v were more than two, then we have a shorter path, namely u, w, v from u to v than that on P, contradicting the fact that P is a diametrical path and completing the proof. $\square$

Theorem 7 provides a necessary condition for a unicyclic graph to have minimum rank equal to the diameter. Note that Lemma 3 implies that the path cover number must equal the maximal multiplicity for these unicyclic graphs. Therefore, to study minimum rank, it is sufficient to study the path cover number. We will start by considering the case in which the cycle has length three in Lemma 8, while the case of length four will be studied in the following Lemma 9.

Lemma 8. *Let* C *be a regular centipede with associated diametrical path* P *and let* u *and* v *be adjacent vertices of* P. *Append a new vertex* w *to* u *and* v *to obtain a unicyclic graph* U. *Then* $P(U) = P(C) + 1$ *if and only if the degrees of both* u *and* v *are at most two in* C.

Proof. If $\deg_C(u) \leqslant 2$ and $\deg_C(v) \leqslant 2$, then, in each minimal path cover of C, u and v are contained in the same path (else, we could achieve a path cover for C with fewer paths by joining the path containing u and the path containing v with the edge $\{u, v\}$). Let Π be a minimal path cover for U, and let Q be the path in Π that contains w. Since u, v, w form a triangle, u and v cannot both be in Q. Thus, w is an endpoint of Q or $Q = \{w\}$. Let Π' be the path cover of C obtained from Π by removing w from Q. Note that Π' cannot be a minimal path cover for C, as u and v now fall in distinct paths. Therefore

$$P(C) < |\Pi'| \leqslant |\Pi| = P(U),$$

that is, $P(U) = P(C) + 1$.

Conversely, suppose, without loss of generality, that $\deg_C(u) \geqslant 3$. Consider a LTR minimal path cover Π for C. We easily see (Fig. 5) that v is an endpoint of a path in Π. Let Π' be the path cover of U obtained from Π by joining w and v. We then have

$$P(C) = |\Pi| = |\Pi'| \geqslant P(U),$$

that is, $P(U) \neq P(C) + 1$. $\square$

Lemma 9. *Let* C *be a regular centipede with diametrical path* P *and let* u, x, v *be a subpath of* P, *and assume* $\deg_C(u) \geqslant \deg_C(v)$. *Append the vertex* w *to both* u *and* v *to obtain the unicyclic graph* U (*of girth* 4). *Then* $P(U) = P(C) + 1$ *if and only if*

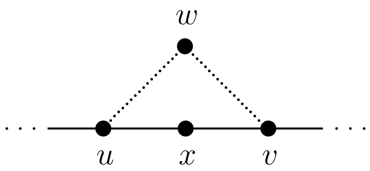

Fig. 7. Case I: $\deg_C(u), \deg_C(x), \deg_C(v) \leqslant 2$.

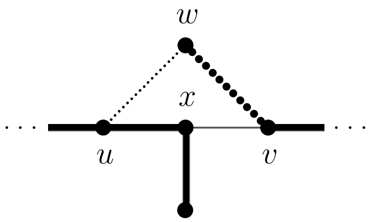

Fig. 8. Case II: $\deg_C(u), \deg_C(v) \leqslant 2$ and $\deg_C(x) > 2$.

(i) $\deg_C(u), \deg_C(x), \deg_C(v) \leqslant 2$, *or*
(ii) $\deg_C(u) \geqslant \deg_C(v) \geqslant 3$; *or*
(iii) $\deg_C(u) \geqslant 3$, $\deg_C(v) = 2$, *and* $\deg_C(y) \leqslant 2$, *where y is the other vertex adjacent to v.*

Proof. We prove this lemma by considering the following exhaustive cases. For each case a figure is provided for clarity:

I. $\deg_C(u), \deg_C(x), \deg_C(v) \leqslant 2$,
II. $\deg_C(u), \deg_C(v) \leqslant 2$ and $\deg_C(x) > 2$,
III. $\deg_C(u) \geqslant 3$ and
   A. $\deg_C(v) = 1$,
   B. $\deg_C(v) = 2$ and $\deg_C(y) \geqslant 3$, where y is the other neighbor of v on P,
   C. $\deg_C(v) = 2$ and $\deg_C(y) \leqslant 2$,
   D. $\deg_C(v) \geqslant 3$.

Case I. Assume that $\deg_C(u), \deg_C(x), \deg_C(v) \leqslant 2$ (see Fig. 7).

Clearly, u, x, v lie on the same path in any minimal path cover of C. Moreover, we also claim that there is a minimal path cover Π for U in which w is an isolated vertex. To verify this, suppose to the contrary that w belongs to a path of positive length in every minimal path cover in U. Then in any such path cover, x must be an endpoint of a path in that path cover. Assume, without loss of generality, that w and v lie on the same path in this path cover. If, in addition, x lies on the same path in this path cover, then delete the edge $\{w, v\}$ and append the edge $\{u, x\}$ to achieve a minimal path cover with w isolated. Otherwise, delete the edge $\{w, v\}$ and append the edge $\{x, v\}$ to form a minimal path cover with w as an isolated vertex. Hence, it follows that $P(U) = P(C) + 1$.

Case II. Assume that $\deg_C(u), \deg_C(v) \leqslant 2$ and $\deg_C(x) > 2$ (see Fig. 8).

Consider the LTR minimal path cover Π for C. By appending w to v in the path cover Π, we obtain a path cover for U of size $P(C)$, and hence $P(U) \neq P(C) + 1$.

Case III. Assume that $\deg_C(u) \geqslant 3$. Thus, since C is regular, $\deg_C(x) = 2$.

Subcase A. ($\deg_C(v) = 1$) (see Fig. 9).

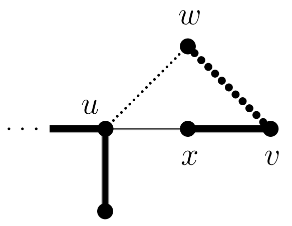

Fig. 9. Case III-A: $\deg_C(u) \geqslant 3$, $\deg_C(v) = 1$.

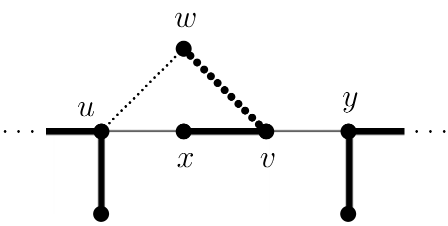

Fig. 10. Case III-B: $\deg_C(u) \geqslant 3$, $\deg_C(v) = 2$, and $\deg_C(y) \geqslant 3$.

Consider the LTR minimal path cover Π for C. Since the path x, v is in Π, we can append w to v to obtain a path cover for U of size $P(C)$. Again $P(U) \neq P(C) + 1$.

Subcase B. ($\deg_C(v) = 2$ and $\deg_C(y) \geqslant 3$, where y is the other neighbor of v on P) (see Fig. 10).

We need to consider here the following path cover: cover all the vertices to the left of x with the paths of a LTR path cover, and all the vertices to the right of v with the paths of a RTL path cover. Finally, cover x and v by an edge. It is easy to see that such a path cover is again minimal for C. Now, at v, append w to the path containing v to obtain a path cover for U of size $P(C)$, and hence $P(U) \neq P(C) + 1$.

Subcase C. ($\deg_C(v) = 2$ and $\deg_C(y) \leqslant 2$) (see Fig. 11).

Observe that any minimal path cover Π for U must satisfy:

(a) w is an isolated vertex in Π, or
(b) exactly one of the edges $\{u, w\}$ or $\{w, v\}$ is contained in some path in Π, or
(c) both of the edges $\{u, w\}$ and $\{w, v\}$ are contained in some path in Π.

If (a) holds, then obviously $P(U) = P(C) + 1$. If (b) holds, then w belongs to a path of positive length in Π and one can follow the argument in the proof of Case I to obtain $P(U) = P(C) + 1$. If (c) holds, then x is an isolated vertex and we can add the edges $\{u, x\}$ and $\{x, v\}$ while removing the edges $\{u, w\}$ and $\{w, v\}$ to obtain a minimal path cover for U for which w is an isolated vertex. Thus, $P(U) = P(C) + 1$.

Subcase D. ($\deg_C(v) \geqslant 3$) (see Fig. 12).

As in Subcase C, one can easily prove that there exists a minimal path cover for U that has w as an isolated path. Hence, $P(U) = P(C) + 1$.

This completes the proof. □

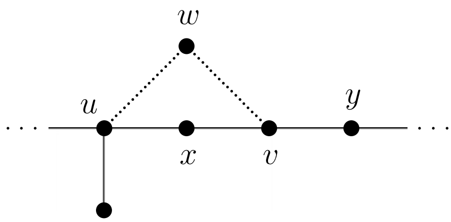

Fig. 11. Case III-C: $\deg_C(u) \geqslant 3$, $\deg_C(v) = 2$, and $\deg_C(y) \leqslant 2$.

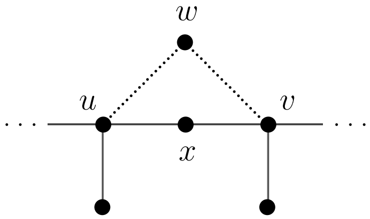

Fig. 12. Case III-D: $\deg_C(u) \geqslant \deg_C(v) \geqslant 3$.

Our main result for unicyclic graphs, namely a characterization of the unicyclic graphs whose minimum rank equals the diameter, follows.

Theorem 10. *Let U be a connected unicyclic graph. Then*, $\mathrm{mr}(U) = \mathrm{diam}(U)$ *if and only if*

(i) *the cycle in U has length* 3 *or* 4;
(ii) *there is a vertex w on the cycle such that when this vertex is deleted from U we obtain a regular centipede C, and the remaining vertices from the cycle lie on a diametrical path P of C*;
(iii) *if the cycle has length* 3, *then* $\deg_C(u) \leqslant 2$ *and* $\deg_C(v) \leqslant 2$, *where u and v are the other vertices from the cycle lying on P*;
(iv) *if the cycle has length* 4 *and u, x, v is the subpath on P formed from the remaining vertices on the cycle, where we can assume* $\deg_C(u) \geqslant \deg_C(v)$, *then*:
  (i) $\deg_C(u), \deg_C(x), \deg_C(v) \leqslant 2$, *or*
  (ii) $\deg_C(u) \geqslant \deg_C(v) \geqslant 3$; *or*
  (iii) $\deg_C(u) \geqslant 3, \deg_C(v) = 2$, *and* $\deg_C(y) \leqslant 2$, *where y is the other vertex adjacent to v.*

Proof. Let U be a unicyclic graph. If (i)–(iv) hold, then it follows from Lemmas 8 and 9 that $P(U) = P(C) + 1$. Also, since the cycle of U has length 3 or 4, $M(U) = P(U)$ [3, Corollary 5.3]. Hence

$$\begin{aligned}
\mathrm{mr}(U) &= |U| - M(U) \\
&= |U| - P(U) \\
&= (|C| + 1) - (P(C) + 1) \\
&= |C| - P(C) \\
&= \mathrm{diam}(C) \\
&= \mathrm{diam}(U).
\end{aligned}$$

Conversely, assume that $\mathrm{mr}(U) = \mathrm{diam}(U)$. From Lemma 3 we know that $P(U) = M(U)$. From the latter part of the argument above, it follows that $P(U) = P(C) + 1$. It then follows from Lemmas 8 and 9 that (i)–(iv) hold. □

5. Future exploration

We close this paper with a brief summary of this work and a discussion of possible future analysis. The main purpose of this note was to initiate the study of the graphs with certain extremal values of minimum rank. Such probing seems justified, as a general description of $\mathrm{mr}(G)$ (in terms of G) is probably a long way off.

Specifically, we have begun the study of the (connected) graphs G for which equality holds in the general inequality $\mathrm{mr}(G) \geqslant \mathrm{diam}(G)$. To this end, we described all of the acyclic and unicyclic graphs G that satisfy $\mathrm{mr}(G) = \mathrm{diam}(G)$.

A natural next step in this line of inquiry is to characterize more general graphs for which this equality holds, or to restrict the minimum rank. For example, in the latter case, if $\mathrm{mr}(G)$ is 1 or 2, then, indeed, it follows that $\mathrm{mr}(G) = \mathrm{diam}(G)$. On the other hand, restricting the diameter does not seem to be worthy of consideration, as families of graphs with diameter 2 and arbitrary minimum rank can be obtained easily.

In addition, another avenue for future exploration is to consider alternative graph parameters in conjunction with minimum rank (e.g. girth, tree-width, minimum degree, etc.). We have started to delve into these issues and related challenges in hopes of comprehending more about the minimum rank problem in general.

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