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On the Roman domination in the lexicographic product of graphs

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ABSTRACT

A Roman dominating function of a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex with $f(v) = 0$ is adjacent to some vertex with $f(v) = 2$. The Roman domination number of G is the minimum of $w(f) = \sum_{v \in V} f(v)$ over all such functions. Using a new concept of the so-called dominating couple we establish the Roman domination number of the lexicographic product of graphs. We also characterize Roman graphs among the lexicographic product of graphs.

1. Introduction and preliminaries

The concept of domination in graphs has been studied extensively and many results are known as well as many different variations and generalizations [10,11]. Through our investigation of the Roman domination in the lexicographic product of graphs, the concept of the total domination naturally appears. In [13], several problems concerning the total domination were proposed. One of them is the characterization of graphs that attain natural bounds for this type of domination (it is known that the total domination number $\gamma_t(G)$ of a graph G lies between $\gamma(G)$ and $2\gamma(G)$).

Several authors have presented a historical problem of defending the Roman Empire [1,17,18] and in connection with it, Cockayne et al. defined the Roman domination [6]. They investigated properties of Roman dominating functions and Roman graphs, i.e. graphs that satisfy $\gamma_R(G) = 2\gamma(G)$. Two simple characterizations of these graphs were obtained, but the authors suggest finding some families of Roman graphs. A constructive characterization of Roman trees is given in [12], for further classes of Roman graphs we refer to [8]. The concept of Roman domination and related concepts still present an active area of research, as recent papers show [9,19].

As many other graph invariants, domination has been studied on different graph products. In [16], authors gave bounds for different graph invariants, including domination and total domination, for all four standard graph products (Cartesian, strong, direct, lexicographic). They observed that the domination number in the lexicographic product of graphs is multiplicative, i.e. for nontrivial graphs G and H

$$\gamma(G \circ H) \le \gamma(G)\gamma(H).$$

This bound can be improved if G has no isolated vertex:

$$\gamma(G \circ H) \le \gamma_t(G).$$

Recently, the results of [16] have been improved for direct product of graphs [3,15].

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One of the most important and widely studied problems on the domination concerns the Cartesian product. That is the Vizing's conjecture, which states that the domination number of the Cartesian product of two graphs is at least the product of the domination numbers of the factors [2]. Since this conjecture remains unsolved, the authors focused on proving Vizing's-like results for other types of domination. Concerning the Roman domination, Wu shows in [20] that $\gamma_R(G\Box H) \geq \gamma(G)\gamma(H)$. On the Roman domination in the Cartesian products, some results are known when the factors are paths [7], and in the case of products $C_{5k}\Box C_{5m}$ [8]. The last ones were proven to be Roman. However, no results on other graph products (lexicographic, strong, direct) have been published.

The rest of the paper is organized as follows: first, definitions and known results needed in the sequel are given. We characterize graphs with Roman domination number 2 and 3. In Section 2, the Roman domination number of the lexicographic product of connected graphs is presented. In the last section, graphs attaining natural bounds for the Roman domination number of the lexicographic product of graphs are characterized. Also, we show which lexicographic product of graphs are Roman.

All graphs considered in this paper are nontrivial, finite and simple. Let G be a graph and $v \in V(G)$. The *open neighborhood of* v is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$.

For a graph $G = (V(G), E(G))$, a set $D \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) \setminus D$ is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a *minimum dominating set*, or a γ-set for short.

If each vertex of a dominating set D has a neighbor in D, then D is called a *total dominating set*. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set [5].

A Roman dominating function (RDF) of a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u such that $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of an RDF is the value $w(f) = \sum_{v\in V(G)} f(v)$. The minimum weight of an RDF on a graph G is called Roman domination number, denoted by $\gamma_R(G)$. Observe that an RDF $f : V \to \{0, 1, 2\}$ can be presented by an ordered partition $f = (V_0, V_1, V_2)$ of V, where $V_i = \{v \in V | f(v) = i\}$. We will also write V_i^f when the function f is not clear from the context. Note that vertices with weight 1 in an RDF serve only to dominate themselves. We say that a function f is a *γ_R-function* if it is an RDF and $w(f) = \gamma_R(G)$.

In [6], the authors noted that for a graph G of order n, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ (where the equality in the lower bound holds if and only if $G = \overline{K}_n$). In the same paper, the following properties of a γ_R-function are given.

Proposition 1.1 ([6]). *Let $f = (V_0, V_1, V_2)$ be any γ_R-function of G. Then:*

(i) *a vertex of V_1 is adjacent to at most one another vertex of V_1,*
(ii) *no edge of G joins V_1 and V_2.*

Another useful upper bound for the Roman domination number of a graph was established in [4].

Proposition 1.2 ([4]). *If G is a graph of order n, then $\gamma_R(G) \leq n - \Delta(G) + 1$.*

In [6], the authors show that if G is a graph of order n that contains a vertex of degree $n-1$ (i.e. a *universal vertex*), then $\gamma(G) = 1$ and $\gamma_R(G) = 2$. But even more can be said if the order of a connected graph is at least 2. Namely, if $\gamma(G) = 1$ and $u \in V(G)$ dominates G, then $f = (V(G) \setminus \{u\}, \emptyset, \{u\})$ is a γ_R-function of G since $n \geq 2$. Also, if $f = (V(G) \setminus \{u\}, \emptyset, \{u\})$ is a γ_R-function of G, then every vertex of $V(G) \setminus \{u\}$ is adjacent to u, which implies that u is of degree $n-1$. If $f = (V(G) \setminus \{u, v\}, \{u, v\}, \emptyset)$, which is the second possible γ_R-function of G of weight 2, then $G \cong K_2$ where both vertices are universal. Let us summarize the above:

Observation 1. Let G be a connected graph of order $n \geq 2$. Then the following are equivalent:

(i) $\gamma(G) = 1$,
(ii) $\gamma_R(G) = 2$,
(iii) G contains a universal vertex.

Now, consider a connected graph G of order $n \geq 2$ with $\gamma_R(G) = 3$ and let $f = (V_0, V_1, V_2)$ be a γ_R-function of G. If $V_2 = \emptyset$, then $|V_1| = 3$ and clearly, $n = 3$. Since G is connected, G is isomorphic to K_3 or P_3, but $\gamma_R(K_3) = \gamma_R(P_3) = 2$, a contradiction. Thus $|V_1| = |V_2| = 1$. Let u and v be vertices with $f(v) = 2$ and $f(u) = 1$ (note that $uv \notin E(G)$ by Proposition 1.1). Then all vertices from $V(G) \setminus \{u, v\}$ are adjacent to v, which implies $\Delta(G) = n - 2$.

If we assume that $\Delta(G) = n - 2$, then $\gamma_R(G) \leq n - (n-2) + 1 = 3$ (by Proposition 1.2), and Observation 1 implies $\gamma_R(G) \geq 3$. Thus $\gamma_R(G) = 3$. We established:

Observation 2. Let G be a connected graph of order $n \geq 2$. Then $\gamma_R(G) = 3$ if and only if $\Delta(G) = n - 2$.

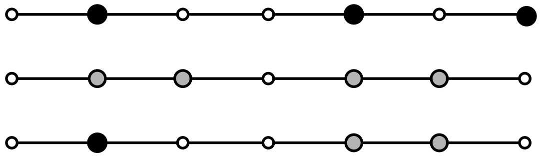

Fig. 1. Dominating couples of P_7.

The *lexicographic product* of graphs G and H is the graph $G \circ H$ with the vertex set $V(G) \times V(H)$ and the edge set:

$$E(G \circ H) = \{(a, x)(b, y) \mid ab \in E(G),\ \text{or } a = b \text{ and } xy \in E(H)\}.$$

For $g \in V(G)$, the H-layer ${}^gH$ is defined as ${}^gH = \{(g, h) \in V(G \circ H) \mid h \in V(H)\}$. Similarly, the G-layer through $h \in V(H)$ is defined, and denoted ${}^hG$. For a set $A \subseteq V(G \circ H)$ let $\mathrm{proj}_G(A) = \{g \in V(G) \mid (g, h) \in A \text{ for some } h \in V(H)\}$. The distance between two vertices in the lexicographic product depends on whether they lie in the same copy of H:

$$d_{G\circ H}((g, h), (g', h')) = \begin{cases} d_G(g, g'); & \text{if } g \neq g', \\ 1; & \text{if } g = g' \text{ and } hh' \in E(H), \\ 2; & \text{if } g = g' \text{ and } hh' \notin E(H). \end{cases}$$

The aim of this paper is to explore the Roman domination number in the lexicographic product of graphs. All of our graphs will be connected since, according to the definition, the Roman domination number of a disconnected graph is the sum of the Roman domination numbers of its connected components. Also, $G \circ H$ is disconnected if and only if G is disconnected [14].

2. Roman domination number of the lexicographic product of graphs

If a graph G has no isolated vertex and H is an arbitrary graph, the upper bound for the domination number of $G \circ H$, already observed in [16], is $\gamma_t(G)$. Thus $\gamma_R(G \circ H) \leq 2\gamma(G \circ H) \leq 2\gamma_t(G)$.

To observe that $2\gamma(G)$ is the lower bound for $\gamma_R(G \circ H)$ we need the following proposition.

Proposition 2.1. *Let G and H be nontrivial connected graphs and $f = (V_0, V_1, V_2)$ a γ_R-function of $G \circ H$ with the minimum cardinality of V_1. Then $\mathrm{proj}_G(V_2)$ is a dominating set of G.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-function of $G \circ H$ with the minimum cardinality of V_1. Suppose that $D = \mathrm{proj}_G(V_2)$ is not a dominating set of G. Then there is a vertex $g \in V(G) \setminus D$ such that none of its neighbors is in D. Therefore $N_G[g] \times V(H) \subseteq V_0 \cup V_1$. Moreover, all vertices in ${}^gH$-layer belong to V_1. Since H is connected it follows by Proposition 1.1 that H is isomorphic to K_2. Let $V(H) = \{h_1, h_2\}$. Now, define $\widehat{f}$ on $V(G\circ H)$ by letting $\widehat{f}(a, b) = f(a, b)$ for any $(a, b) \in V(G\circ H)$ except for $\widehat{f}(g, h_1) = 0$ and $\widehat{f}(g, h_2) = 2$. Obviously $\widehat{f}$ is an RDF, the weight of which is the same as the weight of f. Hence $\widehat{f}$ is a γ_R-function with $|V_1^{\widehat{f}}| < |V_1^f|$, a contradiction. $\square$

Corollary 2.2. *Let G and H be nontrivial connected graphs. Then $2\gamma(G) \leq \gamma_R(G \circ H)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-function of $G \circ H$ such that the cardinality of V_1 is minimum. Since $\mathrm{proj}_G(V_2)$ is a dominating set of G by Proposition 2.1, we have $\gamma(G) \leq |\mathrm{proj}_G(V_2)|$. Suppose that $\gamma_R(G \circ H) < 2\gamma(G)$. Then $2|V_2| + |V_1| < 2\gamma(G) \leq 2|\mathrm{proj}_G(V_2)| \leq 2|V_2|$, a contradiction. Hence $2\gamma(G) \leq \gamma_R(G \circ H)$. $\square$

We observed that $2\gamma(G) \leq \gamma_R(G\circ H) \leq 2\gamma_t(G)$, and we will show in what follows that both bounds are sharp. Moreover, it turns out that the lower bound is actually the exact value for the Roman domination number of $G \circ H$ in the case when $\gamma_R(H) = 2$, and the upper bound is in fact the Roman domination number of $G \circ H$ for every H with $\gamma_R(H) \geq 4$. To be able to examine the case when $\gamma_R(H) = 3$ we introduce the following concept.

Let $A, B \subseteq V(G)$. We say that an ordered couple (A, B) of disjoint sets A and B is a *dominating couple* of G if for every vertex $x \in V(G) \setminus B$ there exists a vertex $w \in A \cup B$ such that $x \in N_G(w)$. Note that $(\emptyset, B)$ is a dominating couple if and only if B is a dominating set, and $(A, \emptyset)$ is a dominating couple if and only if A is a total dominating set. Fig. 1 shows examples of dominating couples for the graph P_7. Grey and black circles represent vertices of A and B, respectively. The first dominating couple is formed by an ordinary dominating set, the second by a total dominating set, while the last is a mixture of both dominations.

Theorem 2.3. *Let G and H be nontrivial connected graphs. Then*

$$\gamma_R(G \circ H) = \begin{cases} 2\gamma(G); & \gamma_R(H) = 2, \\ \zeta(G); & \gamma_R(H) = 3, \\ 2\gamma_t(G); & \gamma_R(H) \geq 4, \end{cases}$$

where $\zeta(G) = \min\{2|A| + 3|B|;\ (A, B) \text{ is a dominating couple of } G\}$.

We will prove Theorem 2.3 in three separate propositions.

Proposition 2.4. *Let G be a nontrivial connected graph and H a connected graph with $\gamma_R(H) = 2$. Then $\gamma_R(G \circ H) = 2\gamma(G)$.*

Proof. Let G and H be connected graphs and $\gamma_R(H) = 2$. Let $\{h\}$ be a γ-set of H and $D_G = \left\{g_1, \dots, g_{\gamma(G)}\right\}$ a γ-set of G. Then $f = (V(G \circ H) \setminus \{(g_i, h) | g \in D_G\}, \emptyset, \{(g_i, h) | g_i \in D_G\})$ is an RDF of $G \circ H$ and $w(f) = 2\gamma(G)$. Thus $\gamma_R(G \circ H) \le 2\gamma(G)$ and the equality $\gamma_R(G \circ H) = 2\gamma(G)$ follows from Corollary 2.2. □

Proposition 2.5. *Let G be a nontrivial connected graph and H a connected graph with $\gamma_R(H) \ge 4$. Then $\gamma_R(G \circ H) = 2\gamma_t(G)$.*

Proof. Let G and H be nontrivial connected graphs. We already know that $\gamma_R(G \circ H) \le 2\gamma_t(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_R-function of $G \circ H$ (with the minimum cardinality of V_1).

First, suppose $\gamma_R(H) > 4$. By Proposition 2.1, $\mathrm{proj}_G(V_2)$ is a dominating set of G. Moreover, we claim that in this case $\mathrm{proj}_G(V_2)$ is a total dominating set of G. Suppose to the contrary that there exists $g' \in \mathrm{proj}_G(V_2)$ that is not adjacent to any other vertex in $\mathrm{proj}_G(V_2)$. This implies that every vertex in the ${}^{g'}H$ –layer is dominated within the layer (note that the ${}^{g'}H$-layer contributes at least 5 to the weight of f). Let $(g', h') \in V(G \circ H)$ be such that $f(g', h') = 2$ and define $\widehat{f}$ on $V(G \circ H)$ by $\widehat{f}(g', h') = f(g', h') = 2$, $\widehat{f}(g', h) = 0$ for every $h \in V(H) \setminus \{h'\}$, $\widehat{f}(g'', h') = 2$ for some $g'' \in V(G)$ such that $g'g'' \in E(G)$, and $\widehat{f}(g, h) = f(g, h)$ for every other vertex in $G \circ H$. It is straightforward to check that $\widehat{f}$ is an RDF, but its weight is less than the weight of f, a contradiction. Thus in the case when $\gamma_R(H) > 4$, $\mathrm{proj}_G(V_2)$ is a total dominating set of G. Hence $\gamma_R(G \circ H) \ge 2\gamma_t(G)$.

Now, assume that $\gamma_R(H) = 4$. If $\mathrm{proj}_G(V_2)$ is a total dominating set of G, then again the result follows. So, suppose that $\mathrm{proj}_G(V_2)$ is not a total dominating set. But in this case we can construct a γ_R-function $\widehat{f}$ such that $\mathrm{proj}_G(V_2^{\widehat{f}})$ is a total dominating set in the following way. Let g' be a vertex from $\mathrm{proj}_G(V_2)$ that is not adjacent to any other vertex in $\mathrm{proj}_G(V_2)$. Then every vertex in the ${}^{g'}H$-layer is dominated within this layer and, since f is a γ_R-function of $G \circ H$ with the minimum cardinality of V_1, we infer $|{}^{g'}H \cap V_2^f| = 2$ and $|{}^{g'}H \cap V_1^f| = 0$. Let (g', h') and (g', h'') be the vertices with $f(g', h') = f(g', h'') = 2$. We assign $\widehat{f}(g', h') = 2$, $\widehat{f}(g', h'') = 0$, $\widehat{f}(g'', h') = 2$ where $g'g'' \in E(G)$, and $\widehat{f}(g, h) = f(g, h)$ for every other vertex in $G \circ H$. It is obvious that $\widehat{f}$ is an RDF and $w(f) = w(\widehat{f})$. Thus, also in the case when $\gamma_R(H) = 4$, we can find a γ_R-function $\widehat{f}$ such that by projecting vertices (g, h) with $\widehat{f}(g, h) = 2$ on G, the total dominating set of G is obtained. Hence, as in the previous case we conclude $\gamma_R(G \circ H) \ge 2\gamma_t(G)$. □

Proposition 2.6. *Let G be a nontrivial connected graph and H a connected graph with $\gamma_R(H) = 3$. Then*

$$\gamma_R(G \circ H) = \min\{2|A| + 3|B|;\ (A, B) \text{ is a dominating couple of } G\}.$$

Proof. Let (A, B) be a dominating couple of a nontrivial connected graph G. Let $\widehat{f} = (V_0^{\widehat{f}}, V_1^{\widehat{f}}, V_2^{\widehat{f}})$ be a γ_R-function of H with $w(\widehat{f}) = 3$. If $|V_1^{\widehat{f}}| = 3$, then $\widehat{f}(h) = 1$ for every $h \in V(H)$ and since H is connected it is isomorphic to P_3 or K_3. But in both cases this is a contradiction by Proposition 1.1(i). Hence $|V_1^{\widehat{f}}| = 1$. Let $V_1^{\widehat{f}} = \{h_1\}$ and $V_2^{\widehat{f}} = \{h_2\}$. We claim that $f = (V_0^f, V_1^f, V_2^f)$, where $V_1^f = B \times \{h_1\}$ and $V_2^f = (A \times \{h_2\}) \cup (B \times \{h_2\})$, is an RDF of $G \circ H$.

If $g \in B$, then it is obvious that all vertices in ${}^gH \cap V_0^f$ are adjacent to some vertex from V_2^f. Now, let $g \in V(G) \setminus B$. Since (A, B) is a dominating couple of G there exists $g' \in A \cup B$ such that $gg' \in E(G)$. We infer (by the structure of $G \circ H$) that (g', h_2) dominates every vertex in the ${}^gH$-layer. It follows that f is an RDF and $\gamma_R(G \circ H) \le \min\{2|A| + 3|B|;\ (A, B)$ is a dominating couple of $G\}$.

For the reversed inequality we need to prove that given a γ_R-function $f = (V_0^f, V_1^f, V_2^f)$ of $G \circ H$ there exists a dominating couple (A, B) of G such that $\gamma_R(G \circ H) \ge 2|A| + 3|B|$.

First, we claim that if f is a γ_R-function of $G \circ H$, then for every $g \in G$ such that ${}^gH \cap \left(V_1^f \cup V_2^f\right) \ne \emptyset$ only two options exist: either $|{}^gH \cap V_1^f| = 1$ and $|{}^gH \cap V_2^f| = 1$ or $|{}^gH \cap V_1^f| = 0$ and $|{}^gH \cap V_2^f| = 1$.

Suppose that a ${}^gH$-layer contains only vertices from $V_0^f \cup V_1^f$. Then, using Proposition 1.1(ii), we infer that ${}^gH \subset V_1^f$. Since H is a connected graph with $\gamma_R(H) = 3$ we conclude again that H can only be isomorphic to P_3 or K_3, which is impossible by Proposition 1.1(i). Thus, observe the case when the ${}^gH$-layer contains at least one vertex from V_2^f. One can notice that as soon as this layer contains another vertex from V_2^f or another two vertices from V_1^f, one can construct a new RDF $\widehat{f}$ of smaller weight by changing only function values of vertices from the ${}^gH$-layer in such a way that this layer contributes only 3 to the weight of $\widehat{f}$. Thus the claim is proved.

Now, let $A' = \{g \in V(G);\ |{}^gH \cap V_1^f| = 0 \text{ and } |{}^gH \cap V_2^f| = 1\}$ and $B' = \{g \in V(G);\ |{}^gH \cap V_1^f| = 1 \text{ and } |{}^gH \cap V_2^f| = 1\}$. We claim that (A', B') is a dominating couple of G. Suppose to the contrary that there exists $g' \in V(G) \setminus B'$ that is not adjacent to any vertex from $A' \cup B'$. We infer that then all vertices from the ${}^{g'}H$-layer would have to be dominated within this layer, which is impossible. Hence (A', B') is a dominating couple of G such that $2|A'| + 3|B'| \le w(f) = \gamma_R(G \circ H)$. □

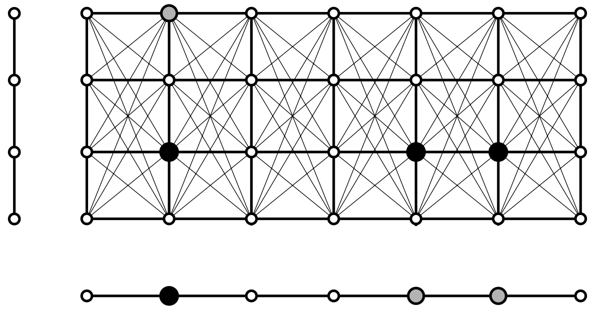

Fig. 2. Roman domination of a graph $P_7 \circ P_4$.

A dominating couple (A, B) for which the minimum in the above proposition is attained is called a *minimum dominating couple*. For example, the last dominating couple of Fig. 1 yields a γ_R-function of $P_7 \circ P_4$, which is depicted in Fig. 2. Here, in the graph P_7, grey and black circles represent vertices of A and B, respectively, while in the product $P_7 \circ P_4$, grey and black circles represent vertices of V_1^f and V_2^f, respectively.

3. Characterization of graphs that attain the natural bounds

Theorem 2.3 gives the exact Roman domination number of the lexicographic product of graphs. However, this number is not always easy to compute, yet the bounds for it can easily be estimated. We already observed that $\gamma_R(G \circ H) \le 2\gamma_t(G)$. Since $\gamma_t(G) \le 2\gamma(G)$, we derive that for arbitrary nontrivial connected graphs G and H,

$$\gamma_R(G \circ H) \le 4\gamma(G).$$

This bound is sharp and it is easy to characterize graphs for which it is attained.

Proposition 3.1. *Let G and H be nontrivial connected graphs and let $\gamma_R(H) \ge 4$. Then $\gamma_R(G \circ H) = 4\gamma(G)$ if and only if $\gamma_t(G) = 2\gamma(G)$.*

Proof. Assume that G and H are connected graphs and $\gamma_R(H) \ge 4$. If $\gamma_t(G) = 2\gamma(G)$, then the result $\gamma_R(G \circ H) = 4\gamma(G)$ is a direct corollary of Theorem 2.3. For the converse, let $\gamma_R(G \circ H) = 4\gamma(G)$. By Theorem 2.3, $\gamma_R(G \circ H) = 2\gamma_t(G)$, therefore $\gamma_t(G) = 2\gamma(G)$. □

Observe that in the case when $\gamma_R(H) = 3$, the upper bound $4\gamma(G)$ can be improved. Let $D = \{g_1, g_2, \ldots, g_{\gamma(G)}\}$ be a dominating set of G and let $(V(H) \setminus \{h_1, h_2\}, \{h_1\}, \{h_2\})$ be a γ_R-function of H. Then $(V_0, D \times \{h_1\}, D \times \{h_2\})$ is an RDF of $G \circ H$. Thus for $\gamma_R(H) = 3$ we obtain $\gamma_R(G \circ H) \le 3\gamma(G)$.

By Corollary 2.2, $\gamma_R(G \circ H) \ge 2\gamma(G)$. We also know that this bound is sharp; it is attained in the case when $\gamma_R(H) = 2$. But these are not the only graphs with such property.

Proposition 3.2. *Let G and H be nontrivial connected graphs. Then $\gamma_R(G \circ H) = 2\gamma(G)$ if and only if one of the following holds:*

1. *$\gamma_R(H) = 2$;*
2. *$\gamma_t(G) = \gamma(G)$ and $\gamma_R(H) \ge 3$.*

Proof. Let G and H be connected graphs. If $\gamma_R(H) = 2$ or $\gamma_t(G) = \gamma(G)$ and $\gamma_R(H) \ge 4$, the result follows directly from Theorem 2.3. So let $\gamma_R(H) = 3$, $\gamma_t(G) = \gamma(G)$ and suppose D is a minimum dominating set of G (which is also minimum total dominating set of G). Then $(D, \emptyset)$ is a dominating couple of G. According to the Theorem 2.3, $\gamma_R(G \circ H) = \min\{2|A|+3|B|;\ (A, B)$ is a dominating couple of $G\}$. Therefore, $\gamma_R(G \circ H) \le 2|D| = 2\gamma_t(G) = 2\gamma(G)$. Since $\gamma_R(G \circ H) \ge 2\gamma(G)$ by Corollary 2.2, we have $\gamma_R(G \circ H) = 2\gamma(G)$.

To prove the converse suppose $\gamma_R(G \circ H) = 2\gamma(G)$. If $\gamma_R(H) \ge 4$, then, by Theorem 2.3, $\gamma_R(G \circ H) = 2\gamma_t(G)$, therefore $\gamma_t(G) = \gamma(G)$. To conclude the proof, let $\gamma_R(H) = 3$. By Theorem 2.3, $\gamma_R(G \circ H) = \min\{2|A| + 3|B|;\ (A, B)$ is a dominating couple of $G\}$. Let (A, B) be a minimum dominating couple. If $A = \emptyset$, then B is a minimum dominating set of G. Thus $\gamma_R(G \circ H) = 3|B| = 3\gamma(G)$, a contradiction. If $B = \emptyset$, then A is a minimal total dominating set, so $\gamma_R(G \circ H) = 2|A| = 2\gamma_t(G)$ and hence $\gamma_t(G) = \gamma(G)$. Finally, suppose (A, B) is an arbitrary minimum dominating couple where both A and B are nonempty. Since A and B are disjoint and $A \cup B$ is a dominating set, we have $\gamma_R(G \circ H) = 2|A| + 3|B| = 2|A| + 2|B| + |B| = 2|A \cup B| + |B| \ge 2\gamma(G) + |B| > 2\gamma(G)$, a contradiction. □

Proposition 3.2 enables the construction of new infinite families of Roman graphs. Before that, we need the following observation about dominating sets in $G \circ H$.

Lemma 3.3. *Let G be a nontrivial connected graph and H a connected graph with $\gamma(H) \geq 2$. Then*

$$\gamma(G \circ H) = \gamma_t(G).$$

Proof. Let G be a nontrivial connected graph and H a connected graph with $\gamma(H) \geq 2$. We already know that $\gamma(G \circ H) \leq \gamma_t(G)$. To prove the reversed inequality, we claim that there exists a minimum dominating set D' of $G \circ H$ such that its projection on G is a total dominating set of G. Let D be a minimum dominating set of $G \circ H$. Suppose that $\mathrm{proj}_G(D)$ is not a total dominating set of G. Then there is a vertex $g \in \mathrm{proj}_G(D)$ that is not adjacent to any other vertex in $\mathrm{proj}_G(D)$ (and let A be the set of vertices in G with such property). Since $\gamma(H) \geq 2$, it follows $|{}^gH \cap D| \geq 2$. Let D' be a set of vertices obtained from the set D in such a way that for every vertex $g' \in A$ we replace one vertex from ${}^{g'}H \cap D$ by a vertex in a neighboring H-layer. One can easily observe that D' is a dominating set of $G \circ H$, $|D| = |D'|$ and $\mathrm{proj}_G(D')$ is a total dominating set of G. Then $\gamma(G \circ H) = |D'| \geq |\mathrm{proj}_G(D')| \geq \gamma_t(G)$. □

Using Lemma 2.5 from [16], which states that $\gamma_t(G \circ H) \leq \gamma_t(G)$, we derive the following simple observation.

Corollary 3.4. *Let G be a nontrivial connected graph and H a connected graph with $\gamma(H) \geq 2$. Then $\gamma(G \circ H) = \gamma_t(G \circ H)$.*

With this corollary we obtain a large family of graphs for which the natural lower bound for the total domination number is attained. In the end, we characterize Roman graphs among the lexicographic products of graphs.

Theorem 3.5. *Let G and H be nontrivial connected graphs. Then $G \circ H$ is a Roman graph if and only if one of the following holds:*

1. $\gamma_R(H) = 2$ *or* $\gamma_R(H) \geq 4$,
2. $\gamma_R(H) = 3$ *and there exists a minimum dominating couple* (A, B)*, such that* $B = \emptyset$.

Proof. First, let $G \circ H$ be a Roman graph, i.e. $\gamma_R(G \circ H) = 2\gamma(G \circ H)$, and let $\gamma_R(H) = 3$. By Lemma 3.3, $\gamma_R(G \circ H) = 2\gamma_t(G)$ and by Theorem 2.3,

$$\min\{2|A| + 3|B|;\ (A, B) \text{ is a dominating couple of } G\} = 2\gamma_t(G).$$

Observe that for a minimum total dominating set D of G, $(D, \emptyset)$ is a minimum dominating couple.

To prove the converse, let G and H be nontrivial connected graphs and suppose D is a dominating set of $G \circ H$. It is straightforward to check that $\mathrm{proj}_G(D)$ is a dominating set of G (this fact also follows from Lemma 2.3 in [16]). Thus $\gamma(G) \leq \gamma(G \circ H)$.

Let $\gamma_R(H) = 2$ (i.e. $\gamma(H) = 1$ by Observation 1). Note that in this case $\gamma(G) \geq \gamma(G \circ H)$, hence $\gamma(G) = \gamma(G \circ H)$. By Theorem 2.3, we obtain $\gamma_R(G \circ H) = 2\gamma(G) = 2\gamma(G \circ H)$, thus $G \circ H$ is a Roman graph.

Now, let $\gamma_R(H) = 3$ and $(A, \emptyset)$ a minimum dominating couple. By Theorem 2.3, $\gamma_R(G \circ H) = 2|A|$. Since A is a minimum total dominating set of G, $\gamma_R(G \circ H) = 2\gamma_t(G)$. According to Lemma 3.3, $\gamma_t(G) = \gamma(G \circ H)$ and therefore $\gamma_R(G \circ H) = 2\gamma_t(G) = 2\gamma(G \circ H)$.

Finally, let $\gamma_R(H) \geq 4$. By Lemma 3.3, $\gamma(G \circ H) = \gamma_t(G)$ and by Theorem 2.3, $\gamma_R(G \circ H) = 2\gamma_t(G)$. Therefore, also in this case $G \circ H$ is a Roman graph. □

An example of a lexicographic product of graphs that is not a Roman graph is depicted in Fig. 2.

In this paper we established the formula that gives the Roman domination number of the lexicographic product of graphs. We believe that it is not easy to obtain such a formula in the case of other standard products of graphs, however, it would be reasonable to investigate Roman dominating sets in these products in order to obtain (improved) bounds for this graph parameter.

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