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END INVARIANTS OF AMALGAMATED FREE PRODUCTS

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It is known that a closed aspherical manifold of dimension greater than or equal to five is covered by Euclidean space if and only if its fundamental group is 1-LC at infinity. In this paper theorems will be proved giving sufficient conditions for G, an amalgamated free product or HNN group, to be 1-LC at infinity and other conditions will be given which are sufficient for G to be not 1-LC at infinity. It is also proved that none of the groups which are shown to be not 1-LC at infinity can actually be the fundamental group of an aspherical manifold.

Introduction

This paper continues the study of the end invariant $e_1(G)$ for finitely presented groups G with one end. Earlier results are contained in [9, 10, 11, 15]. It was first shown in [11] that a closed aspherical manifold of dimension greater than or equal to five with fundamental group G is covered by Euclidean space if and only if $e_1(G)=1$. This is the most important application of the invariant $e_1(G)$.

All groups will be understood to be finitely presented unless stated otherwise. Mainly two kinds of groups will be considered. The amalgamated free product of A and B with amalgamated subgroup C will be denoted $A *_C B$. Also $[A: C, f]$ will denote the HNN group where C is a subgroup of A and f is a monomorphism from C into A. Then a presentation of $[A: C, f]$ can be obtained from a presentation of A by adding one additional generator t and the relations $f(g)=t^{-1}gt$ for all g in C. Also let K be any finite complex with $\pi_1(K)=G$ and let $\tilde{K}$ be the universal covering space of K. The group G is said to be 1-LC at infinity if for any compact set C of $\tilde{K}$ there is a larger compact set D in $\tilde{K}$ so that any loop in $\tilde{K}-D$ is contractible in $\tilde{K}-C$. A group is 1-LC at infinity if and only if $e_1(G)=1$. In this paper theorems will be proved giving sufficient conditions for G, an amalgamated free product or HNN group, to be 1-LC at infinity and other conditions will be given which are sufficient for G to be not 1-LC at infinity. I would like to thank the referee for his many helpful suggestions in revising this paper.

Let K and $\tilde{K}$ be as above. End invariants of the group G are defined as in [9, 11, 15]. One takes an increasing sequence of compact sets $U_1, U_2, \ldots$ whose union is all of $\tilde{K}$. The number of ends of $G, e_0(G)$, is defined to be

$$\operatorname{card}\left[\varprojlim\left\{\pi_0(\tilde{K}-U_1) \xleftarrow{i_1} \pi_0(\tilde{K}-U_2) \xleftarrow{i_2} \cdots\right\}\right].$$

For a group with one end the U_i can be chosen so that each $\tilde{K}-U_i$ is connected and thus one also has an inverse system of fundamental groups

$$\pi_1(\tilde{K}-U_1, u_1) \xleftarrow{f_1} \pi_1(\tilde{K}-U_2, u_2) \xleftarrow{f_2} \cdots$$

where the maps f_i are induced by paths between the respective basepoints u_{i+1} and u_i. A group G is said to be stable at infinity if for some subsequence of the U_i's

$$\pi_1(\tilde{K}-U_{i_1}, u_{i_1}) \xleftarrow{g_1} \pi_1(\tilde{K}-U_{i_2}, u_{i_2}) \xleftarrow{g_2} \cdots$$

with $g_1 = f_{i_2-1} \circ \cdots \circ f_2 \circ f_1$, etc., the induced maps

$$\operatorname{Im}[g_1] \xleftarrow{g_1} \operatorname{Im}[g_2] \xleftarrow{g_2} \operatorname{Im}[g_3] \xleftarrow{g_3} \cdots$$

are all isomorphisms. A group G is said to be semistable at infinity if for some subsequence of U_i's the induced maps are epimorphisms. The end invariant $e_1(G)$ can be defined, for any group with one end which is semistable at infinity, to be

$$\varprojlim\left\{\pi_1(\tilde{K}-U_1, u_1) \xleftarrow{f_1} \pi_1(\tilde{K}-U_2, u_2) \xleftarrow{f_2} \cdots\right\}.$$

A group G is 1-LC at infinity if and only if G is semistable and $e_1(G)=1$.

Related invariants can be defined using cohomology. As in Epstein [4] one has an exact sequence

$$0 \to C_f^*(\tilde{K}) \to C^*(\tilde{K}) \to C_e^*(\tilde{K}) \to 0$$

where $C_f^*(\tilde{K})$ is the finite cochains on $\tilde{K}, C^*(\tilde{K})$ is the regular cochains on $\tilde{K}$, and $C_e^*(\tilde{K})$ is defined by the exact sequence. This induces a long exact sequence in cohomology

$$0 \to H_f^0(\tilde{K}) \to H^0(\tilde{K}) \to H_e^0(\tilde{K}) \to H_f^1(\tilde{K}) \to H^1(\tilde{K}) \to H_e^1(\tilde{K}) \to \cdots.$$

It is well known that $\dim_{\mathbb{Z}_2}\{H_e^0(\tilde{K}; \mathbb{Z}_2)\} = e_0(G) = \dim_{\mathbb{Z}_2}\{H^1(G; \mathbb{Z}_2 G)\} + 1$ and that an infinite group G has either one, two, or an infinite number of ends. Also an important result of Stallings [16] shows that any finitely generated group G with more than one end can be nontrivially written as either $A *_C B$ or $[A: C, f]$ where C is a finite subgroup in either case.

Less information is known about the other end invariants, stability, semistability, and $e_1(G)$, for finitely presented infinite groups with one end and their relation to cohomology. Results of Farrell in [5] show that if G is a finitely presented group with at least one element of infinite order then $H^2(G;\mathbb{Z}G)$ is either 0, infinite cyclic, or not finitely generated. He also shows that $H^2(G;\mathbb{Z}_2G)$ and $H_e^1(\tilde{K};\mathbb{Z}_2)$ are isomorphic for any finite complex K with $\pi_1(K)=G$. Houghton in [8] has shown that if G is semistable at infinity and $e_1(G)=M$ then $H^2(G,\mathbb{Z}G)=\mathrm{Hom}(M/M',\mathbb{Z})$ [also $H^2(G,\mathbb{Z}_2G)=\mathrm{Hom}(M/M',\mathbb{Z}_2)$] where M' is the commutator subgroup of M. A special case of this is a consequence of the following proposition.

Proposition 1. *If $e_1(G)=1$ (G is 1-LC at infinity) then $H_e^1(\tilde{K};\mathbb{Z})=0$ for any finite simplicial complex K with $\pi_1(K)=G$.*

Proof. An element of $H_e^1(\tilde{K};\mathbb{Z})$ is represented by a simplicial 1-cochain f, whose coboundary is a finite 2-cochain. So δf is zero outside some finite complex C. Since $\tilde{K}$ is a locally finite simplicial complex which is 1-LC at infinity then there exists a subcomplex D in $\tilde{K}-C$ whose complement is a finite number of cells and so that any edge loop α in D is collapsible to a vertex in the complement of C. Suppose α starts at a_0, follows a 1-simplex to a_1, then follows a 1-simplex to $a_2,\dots,$then follows a 1-simplex to a_n, and finally follows another 1-simplex back to a_0. Then $f(\alpha)=f(a_0a_1)+f(a_1a_2)+\cdots+f(a_na_0)=0$ since α is collapsible in the complement of C and thus is a boundary in the simplicial homology. Now we extend f restricted to D to a simplicial cocycle g on all of $\tilde{K}$. The cocycle g will be the same as f on D. Let $b_1b_2, b_3b_4,\dots,b_{2n-1}b_{2n}$ be the 1-simplices of $\tilde{K}$ not in D. If b_1 and b_2 are joined by a path p in D and $f(p)=k$ then for any other path q in D joining b_1 and b_2, $f(q)=k$, and so we can define $g(b_1b_2)=k$. Otherwise if b_1 and b_2 are not joined by a path by a path in D one defines $g(b_1b_2)=0$. Now let D_1 be D plus the vertices b_1 and b_2 and the 1-simplex joining them. The new map g restricted to D_1 will again satisfy the property that $g(\alpha)=0$ for any loop α in D_1 since if p_1 and p_2 are any two paths in D_1 with the same endpoints then $g(p_1)=g(p_2)$. Define g for b_3b_4 in a similar fashion and continue in this way until g is defined on all of $\tilde{K}$. Thus g is a 1-cocyle on K, since $g(\alpha)=0$ for any loop α in $\tilde{K}$. Since $H_1(\tilde{K};\mathbb{Z})=0$ then by the universal coefficient theorem $H^1(\tilde{K};\mathbb{Z})=0$ and g is also a coboundary. But g differs from f on at most a finite number of simplices so f is a coboundary in $C_e^*(\tilde{K};\mathbb{Z})$. Therefore $H_e^1(\tilde{K};\mathbb{Z})=0$.

Remark. Also it is true that $H_e^1(\tilde{K};\mathbb{Z}_2)=H^2(G;\mathbb{Z}_2G)=0$ if $e_1(G)=1$. In addition another consequence of Houghton's result is that if $H^2(G;\mathbb{Z}G)$ [or $H^2(G;\mathbb{Z}_2G)$] is nonzero then G is not 1-LC at infinity. For if G is semistable at infinity and $H^2(G;\mathbb{Z}G)=\mathrm{Hom}(M/M';\mathbb{Z})$ is nontrivial then $M=e_1(G)$ is nontrivial and G is not 1-LC at infinity. In addition if G is not semistable at infinity then trivially G is not 1-LC at infinity.

1. Amalgamated free products

Earlier results of Houghton [8] and Jackson [10] show that if $1\to H\to G\to K\to 1$ is an exact sequence of finitely presented infinite groups where at least one of H and K has one end then $e_1(G)=1$. Stallings' classification of groups with more than one end indicates that amalgamated free products and HNN groups also play an important role. The following theorems can be used to compute $e_1(G)$ and $H^2(G;\mathbb{Z}_2G)$ for certain amalgamated free products. Thus consider the standard amalgamated free product $G=G_1 *_H G_2$. Suppose H has generators $a_1,a_2,\ldots,a_n$ with relations $r_1=1,\ldots,r_j=1$, G_1 has generators $w_1,\ldots,w_k,a_1,\ldots,a_n$ with relations $q_1=1,\ldots,q_t=1$, $r_1=1,\ldots,r_j=1$, and G_2 has generators $m_1,m_2,\ldots,m_d,a_1,\ldots,a_n$ with relations $p_1=1,\ldots,p_s=1$, $r_1=1,\ldots,r_j=1$. Then G can be presented with generators $m_1,\ldots,m_d,w_1,\ldots,w_k,a_1,\ldots,a_n$ and relations $p_1=1,\ldots,p_s=1$, $q_1=1,\ldots,q_t=1$, $r_1=1,\ldots,r_j=1$. So G_1,G_2, and H will be thought of as subgroups of G. Let $\mathscr{H}$, $\mathscr{G}$, $\mathscr{G}_1$, and $\mathscr{G}_2$ be the 2-dimensional CW complexes corresponding to these presentations, constructed as in [12, p. 117] with one vertex, a loop for each generator, and a 2-cell for each relation, with $\pi_1(\mathscr{H})=H$, $\pi_1(\mathscr{G})=G$, $\pi_1(\mathscr{G}_1)=G_1$, and $\pi_1(\mathscr{G}_2)=G_2$ respectively. Then $\mathscr{G}$ is constructed from attaching a copy of $\mathscr{G}_1$ and $\mathscr{G}_2$ at a copy of $\mathscr{H}$. Also since G_1,G_2, and H inject into G then the universal covering space of G, $\tilde{\mathscr{G}}$, is constructed from coset copies of the universal covering spaces $\tilde{\mathscr{G}}_1$ and $\tilde{\mathscr{G}}_2$ which are attached at coset copies of $\tilde{\mathscr{H}}$. As usual $e_1(G)=e_1(\tilde{\mathscr{G}})$.

Theorem 1. *If H is a subgroup of G_1 and G_2 where G_1 and G_2 both have one end and H has more than one end, then the amalagamated free product $G=G_1 *_H G_2$ has one end but is not stable at infinity.*

Proof. One has a Mayer–Vietoris sequence for the amalgamated free product of groups [18]:

$$\cdots\leftarrow H^1(G_1;\mathbb{Z}_2G)\oplus H^1(G_2;\mathbb{Z}_2G)\leftarrow H^1(G;\mathbb{Z}_2G)\leftarrow H^0(H;\mathbb{Z}_2G)\leftarrow\cdots.$$

The groups H,G_1, and G_2 are subgroups of G of infinite index so $\mathbb{Z}_2G$ is a free $\mathbb{Z}_2H,\mathbb{Z}_2G_1$, and $\mathbb{Z}_2G_2$ module respectively of infinite rank. Since H is infinite then $H^0(H;\mathbb{Z}_2H)=H^0(H;\mathbb{Z}_2G)=0$ and since G_1 and G_2 both have one end then $H^1(G_1;\mathbb{Z}_2G)=H^1(G_1;\mathbb{Z}_2G_1)=0=H^1(G_2;\mathbb{Z}_2G_2)=H^1(G_2;\mathbb{Z}_2G)$. Thus it is also true that $H^1(G;\mathbb{Z}_2G)=0$ and hence G has one end.

Now let C_i be an increasing sequence of finite subcomplexes in $\tilde{\mathscr{G}}$ whose union is all of $\tilde{\mathscr{G}}$ and consider the inverse sequence of fundamental groups. For convenience basepoints will be omitted.

$$\pi_1(\tilde{\mathscr{G}}-C_1)\leftarrow\pi_1(\tilde{\mathscr{G}}-C_2)\leftarrow\pi_1(\tilde{\mathscr{G}}-C_3)\leftarrow\cdots.$$

For any compact set C_i there is always a coset copy of $\tilde{\mathscr{G}}_1$ intersecting a coset copy of $\tilde{\mathscr{G}}_2$ so that C_i doesn't intersect either one. Thus any loop contained in this $\tilde{\mathscr{G}}_1$ and $\tilde{\mathscr{G}}_2$ is trivial in $\pi_1(\tilde{\mathscr{G}}-C_j)$ for $j\le i$. But some larger C_r intersects this copy of $\tilde{\mathscr{G}}_1$ and $\tilde{\mathscr{G}}_2$ and divides the copy of $\tilde{\mathscr{H}}$ in which they intersect into at least two infinite components. Using the fact that $\tilde{\mathscr{G}}_1$ and $\tilde{\mathscr{G}}_2$ both have one end one can construct a loop α in $\tilde{\mathscr{G}}-C_r$ which consists of a path in the copy of $\tilde{\mathscr{G}}_1$ connecting two infinite components of $\mathscr{H}$ followed by a path in $\mathscr{G}_2$ connecting the endpoints of the first path. In fact in the complement of any compact set larger than C_r there is also such a loop connecting two points in the remaining parts of the original infinite components of $\mathscr{H}$. Let L be the union of all remaining parts of copies of $\mathscr{G}_1$ in $\mathscr{G}-C_r$ and let M be the union of all remaining parts of copies of $\mathscr{G}_2$ in $\mathscr{G}-C_r$. Subdivide $\mathscr{G}$ by putting a new vertex at the center of each 1-cell and 2-cell and creating new edges that join the vertex at the center of a 2-cell to each vertex on its boundary. Thus each 2-cell is subdivided into triangles. Then further subdivide each of these triangles and edges. In this second subdivision of $\mathscr{G}$ let $L'=L$ together with the interiors of all triangles and edges whose boundaries intersect L and let $M'=M$ together with the interiors of all triangles and edges whose boundaries intersect M. Then L', M', and $L'\cap M'$ are all open and L, M, and $L\cap M$ are deformation retracts of $L', M', L'\cap M'$ respectively. Thus one has the following Mayer–Vietoris sequence.

$$\cdots\to H_1(L)\oplus H_1(M)\to H_1(L\cup M)\xrightarrow{d} H_0(L\cap M)\to\cdots.$$

Since $d(\alpha)$ is nontrivial in $H_0(L\cap M)$ the loop α is nontrivial in $H_1(\mathscr{G}-C_r)$ and also nontrivial in $\pi_1(\mathscr{G}-C_r)$. So far arbitrarily large $k>r$ one can construct a loop which is nontrivial in $\pi_1(\mathscr{G}-C_j)$ for $r\le j\le k$ but trivial in $\pi_1(\mathscr{G}-C_j)$ for $j\le i$. Since i and k are arbitrarily large then this shows that the inverse sequence of fundamental groups is not stable and hence $\mathscr{G}$ is not stable at infinity. It is however possible that $\mathscr{G}$ is semistable at infinity but in that case $e_1(G)$ is infinitely generated [9].

Theorem 1 allows one to construct many examples of groups which are not 1-LC at infinity. In fact if $n=\max\{\mathrm{cd}(G_1),\mathrm{cd}(G_2)\}$ and H is a subgroup of both G_1 and G_2 with $\mathrm{cd}(H)<n$ then the amalgamated free product $G=G_1 *_H G_2$ has $\mathrm{cd}(G)=n$ [14]. Thus one can construct groups which are not 1-LC at infinity of any cohomological dimension. However none of these groups can be a Poincaré duality group which is a condition that is satisfied by the fundamental group of a closed aspherical manifold. One has the following Mayer–Vietoris sequence

$$\cdots\leftarrow H^2(G;\mathbb{Z}_2G)\leftarrow H^1(H;\mathbb{Z}_2G)\leftarrow H^1(G_1;\mathbb{Z}_2G)\oplus H^1(G_2;\mathbb{Z}_2G)\leftarrow\cdots.$$

Since $e_0(H)>1$ and $e_0(G_1)=e_0(G_2)=1$ then $H^1(H;\mathbb{Z}_2H)\neq 0$ and $H^1(G_1;\mathbb{Z}_2G_1)=H^1(G_2;\mathbb{Z}_2G_2)=0$. Also H, G_1, and G_2 are subgroups of G of infinite index so $\mathbb{Z}_2G$ is a free $\mathbb{Z}_2H, \mathbb{Z}_2G_1, \mathbb{Z}_2G_2$ module respectively of infinite rank. Therefore $\dim_{\mathbb{Z}_2}[H^1(H;\mathbb{Z}_2G)]$ is infinite and $\dim_{\mathbb{Z}_2}[H^1(G_1;\mathbb{Z}_2G)]=\dim_{\mathbb{Z}_2}[H^1(G_2;\mathbb{Z}_2G)]=0$ and so $\dim_{\mathbb{Z}_2}[H^2(G;\mathbb{Z}_2G)]$ is infinite. For a Poincaré duality group of dimension n, $\dim_{\mathbb{Z}_2}[H^n(G;\mathbb{Z}_2G)]=1$ and $\dim_{\mathbb{Z}_2}[H^i(G;\mathbb{Z}_2G)]=0$ for $i\neq n$.

Theorem 2. *If both G_1 and G_2 are* 1-LC *at infinity and H has one end then $G=G_1 *_H G_2$ is also* 1-LC *at infinity.*

Proof. As in Theorem 1 it is easy to see that as long as G_1 and G_2 both have one end and H is infinite then G has one end. Suppose C is some compact set in $\mathscr{G}$. Then C is restricted to any copy of $\mathscr{G}_1$ or $\mathscr{G}_2$ is compact. Since each copy of $\mathscr{G}_1$ and $\mathscr{G}_1$ and $\mathscr{G}_2$ in $\mathscr{G}$ is 1-LC at infinity one can find a larger compact set F in $\mathscr{G}$ so that any loop in one copy of a $\mathscr{G}_i$ in the complement of F can be contracted in this copy of $\mathscr{G}_i$ in the complement of C. Also F restricted to each copy of $\mathscr{H}$ is compact so F can be enlarged to a compact set E so that for every copy of $\mathscr{H}$ in $\mathscr{G}$ the 1-skeleton of $\mathscr{H}$ in the complement of E is connected. Let D be a compact set containing E for which any loop in the complement of D is homotopic to an edge loop in $\mathscr{G}-E$. I now prove by induction that any loop in the complement of D is contractible in the complement of C.

A loop in the complement of D is homotopic to an edge loop in the complement of E. A loop in the complement of E contained in one coset copy of $\mathscr{G}_i$ can be contracted in the complement of C. Suppose an edge loop α in the complement of E starts in some copy of $\mathscr{H}$, follows a path α_1 in some $\mathscr{G}_1$ to another copy of $\mathscr{H}$, where it follows a path α_2 in some $\mathscr{G}_2$ and so on until α_n comes back to the starting point of α_1. Also suppose the path α_1 starts at a vertex corresponding to some g in G. Then α_1 ends at a vertex corresponding to $g\alpha_1$ where α_1 is an element of G_1, α_2 ends at ga_1b_2 where b_2 is in G_2 until α_n ends at $ga_1b_2\cdots b_n$. Since $ga_1b_2\cdots b_n=g$ then $a_1b_2\cdots b_n=1$ in G. This can only happen if at least one of the a_k's or b_k's is in H [13]. This means at least one of the paths α_j starts at a vertex c_1 and this path α_j ends at a vertex c_2 in the same copy of $\mathscr{H}$. By the construction of E, c_1 and c_2 are connected by an edge path p entirely in this copy of $\mathscr{H}$ and in the complement of E. The loop consisting of the path α_j followed by the path p^{-1} is contractible in the complement of C since it is contained in one coset copy of a $\mathscr{G}_1$ or $\mathscr{G}_2$. This means α_j is homotopic to p in the complement of C. However α_{j-1} followed by p followed by α_{j+1} is entirely in a copy of some $\mathscr{G}_1$ or $\mathscr{G}_2$ so α is homotopic in the complement of C to a new edge path in the complement of E which is contained in $n-2$ copies of $\mathscr{G}_1$ and $\mathscr{G}_2$ or 1 copy if $n=2$. By induction α is contractible in the complement of C so $\mathscr{G}$ is 1-LC at infinity.

Remark. If in Theorem 2 one just assumes that $H^2(G_1;\mathbb{Z}_2G_1)$ and $H^2(G_2;\mathbb{Z}_2G_2)$ are both zero in addition to the fact that H has one end then using the Mayer–Vietoris sequence

$$\cdots\leftarrow H^2(G_1;\mathbb{Z}_2G)\oplus H^2(G_2;\mathbb{Z}_2G)\leftarrow H^2(G;\mathbb{Z}_2G)\leftarrow H^1(H;\mathbb{Z}_2G)\leftarrow\cdots$$

it is easy to see that $H^2(G,\mathbb{Z}_2G)=0$. Also in Theorems 1 and 2 it is only necessary that the amalgamated subgroup H be finitely generated since a presentation of G can be constructed using the presentation of G_1 and the presentation of G_2 (each of which includes the generators of H) but it is unnecessary to include the relations of H.

2. HNN Groups

Now consider the groups $G=[G_1\colon H,f]$. Call the subgroup $f(H)$ of G_1, K. Suppose that H has generators $a_1, \ldots, a_n$, similarly K has generators $a_1^*, \ldots, a_n^*$, where $f(a_i) = a_i^*$, and G_1 has generators $b_1, \ldots, b_m, a_1, \ldots, a_n, a_1^*, \ldots, a_n^*$ with relations $p_1 = 1, \ldots, p_k = 1$. Then G can be presented with generators $b_1, \ldots, b_m, a_1, \ldots, a_n, a_1^*, \ldots, a_n^*, t$ and relations $p_1 = 1, \ldots, p_k = 1$, $a_1^* = t^{-1}a_1t, \ldots, a_n^* = t^{-1}a_nt$. Let $\mathscr{G}$ and $\mathscr{G}_1$ be the finite 2-complexes corresponding to the presentations of G and G_1 respectively and let $\mathscr{H}$ and $\mathscr{K}$ be the finite subcomplexes of G_1 corresponding to the generators of H and K. Then $\mathscr{G}$ is homeomorphic to the space constructed from a copy of $\mathscr{G}_1$ and a copy of $\mathscr{H} \times [0,1]$ where $\mathscr{H} \times 0$ is identified with the copy of $\mathscr{H}$ in $\mathscr{G}_1$ and $\mathscr{H} \times 1$ is identified with the copy of $\mathscr{K}$ in $\mathscr{G}_1$. Since G_1, K, and H inject into G then $\tilde{\mathscr{G}}$ can be constructed from copies of $\tilde{\mathscr{G}}_1$ and $\tilde{\mathscr{H}} \times [0,1]$. The complexes $\tilde{\mathscr{H}}$ and $\tilde{\mathscr{K}}$ will contain the edges of $\tilde{\mathscr{G}}_1$ corresponding to the generators of H and K. As before there will be one copy of $\tilde{\mathscr{H}}$ and $\tilde{\mathscr{K}}$ for each coset of H and K respectively. In $\tilde{\mathscr{G}}_1$ the copies of $\tilde{\mathscr{H}} \times [0,1]$ are attached in one copy of $\tilde{\mathscr{G}}_1$ to a copy of $\tilde{\mathscr{H}}$ and in another copy of $\tilde{\mathscr{G}}_1$ to a copy of $\tilde{\mathscr{K}}$.

Theorem 1*. *If H is a subgroup of G_1 where G_1 has one end and H has more than one end, then the* HNN *group $G = [G_1 : H, f]$ has one end but is not stable at infinity.*

Proof. The proof will be essentially the same as in Theorem 1. As before G can easily be seen to have one end using the appropriate Mayer–Vietoris sequence. Let C_i be an increasing sequence of compact sets in $\tilde{\mathscr{G}}$ whose union is all of $\tilde{\mathscr{G}}$ and consider the inverse sequence of fundamental groups.

$$\pi(\tilde{\mathscr{G}} - C_1) \leftarrow \pi_1(\tilde{\mathscr{G}} - C_2) \leftarrow \pi_1(\tilde{\mathscr{G}} - C_3) \leftarrow \cdots.$$

For any compact set C_i there will always be two coset of $\tilde{\mathscr{G}}_1$ joined by a coset copy of $\tilde{\mathscr{H}} \times [0,1]$ disjoint from C_i. However some larger C_r intersects these copies of $\tilde{\mathscr{G}}_1$ and divides the copy of $\tilde{\mathscr{H}} \times [0,1]$ which joins them into at least two infinite components. Then in the complement of any compact set C_k larger than C_r there exists a loop α which first consists of a path entirely in the first copy of $\tilde{\mathscr{G}}_1$ joining points $p_1 \times 0$ and $p_2 \times 0$ in $\tilde{\mathscr{H}} \times 0$ which are in different infinite components of $\tilde{\mathscr{H}} \times [0,1]$, which is possible since $\tilde{\mathscr{G}}_1$ has one end. Since C_k is compact p_1 and p_2 can be chosen so that $p_1 \times [0,1]$ and $p_2 \times [0,1]$ are in the complement of C_k. Then α folows $p_2 \times [0,1]$, then follows a path in the second copy of $\tilde{\mathscr{G}}_1$ joining $p_2 \times 1$ and $p_1 \times 1$, and finally follows $p_1 \times [0,1]$ back to the starting point. Let L be the union of all parts of $\tilde{\mathscr{G}}_1$, and $\tilde{\mathscr{H}} \times [0, \frac{1}{4})$, and $\tilde{\mathscr{H}} \times (\frac{3}{4}, 1]$ in $\tilde{\mathscr{G}} - C_k$ and let M be the union of all parts of $\tilde{\mathscr{H}} \times (\frac{1}{8}, \frac{7}{8})$ in $\tilde{\mathscr{G}} - C_k$. As before the Mayer–Vietoris sequence

$$\cdots \to H_1(L) \oplus H_1(M) \to H_1(L \cup M) \xrightarrow{d} H_0(L \cap M) \to \cdots$$

shows that the loop α is nontrivial in $H_1(\tilde{\mathscr{G}} - C_k)$ and $\pi_1(\tilde{\mathscr{G}} - C_k)$. Once again we can construct a loop α so that α is trivial in $\pi_1(\tilde{\mathscr{G}} - C_j)$ for $j \leq i$ but nontrivial in $\pi_1(\tilde{\mathscr{G}} - C_j)$ for $r \leq j \leq k$. Since i can be chosen to be arbitrarily large and k can be chosen arbitrarily larger than r we have that $\tilde{\mathscr{G}}$ is not stable at infinity and thus G is not stable at infinity.

Remark. None of these groups can be Poincaré duality groups either. One has the following Mayer–Vietoris sequence for HNN groups [1].

$$\cdots \leftarrow H^n(H;\mathbb{Z}_2G) \leftarrow H^n(G_1;\mathbb{Z}_2G) \leftarrow H^n(G;\mathbb{Z}_2G) \leftarrow H^{n-1}(H;\mathbb{Z}_2G) \leftarrow \cdots.$$

If G is as in Theorem 1* then $\dim_{\mathbb{Z}_2}[H^1(H;\mathbb{Z}_2G)]$ is infinite and also

$$\dim_{\mathbb{Z}_2}[H^1(G_1;\mathbb{Z}_2G)] = 0.$$

Thus for $n=2$ one has

$$\cdots \leftarrow H^2(G;\mathbb{Z}_2G) \leftarrow H^1(H;\mathbb{Z}_2G) \leftarrow 0 \leftarrow \cdots.$$

Since $\dim_{\mathbb{Z}_2}[H^1(H;\mathbb{Z}_2G)]$ is infinite then one also has that $\dim_{\mathbb{Z}_2}[H^2(G;\mathbb{Z}_2G)]$ is infinite. Therefore G is not a Poincaré duality group.

Theorem 2*. *If G_1 is* 1-LC *at infinity and H has one end then the* HNN *group $[G_1\colon H, f]$ is also* 1-LC *at infinity.*

Proof. By Britton's lemma if one has $g_0t^{i_1}g_1t^{i}\cdots t^{i_n}g_n=1$ where the g_j's are in G_1, then for some $j, i_j>0, i_{j+1}<0$, and g_j is in K or for some $j, i_j<0, i_{j+1}>0$, and g_j is in H. The proof now proceeds as in Theorem 2. Suppose C is some compact set in $\mathscr{G}$. Since each copy of $\mathscr{G}_1$ is 1-LC at infinity one can construct a larger compact set F in $\mathscr{G}$ so that any loop in one copy of $\mathscr{G}_1$ in the complement of F can be contracted in the complement of C. Since F is compact then it intersects a finite number of copies of $\mathscr{H}\times[0,1]$. Also since F restricted to each copy of $\mathscr{H}\times[0,1]$ is compact and $\mathscr{H}$ has one end it is possible to choose a compact set D in $\mathscr{H}$ so that F restricted to this copy of $\mathscr{H}\times[0,1]$ is contained in $D\times[0,1]$ and $\mathscr{H}$-D is connected. Add $D\times[0,1]$ to F and continue this for each $\mathscr{H}\times[0,1]$ which F intersects to get a new compact set E. Finally let T be a compact set containing E for which any loop in the complement of T is homotopic to an edge loop in the complement of E. Now any loop in the complement of T can first be contracted to an edge loop in α in the complement of E. Therefore α can be represented by a word $g_0t^{i_1}\cdots t^{i_n}g_n$ which is equal to 1 in G. The proof now proceeds by induction on $|i_1|+|i_2|+\cdots+|i_n|$. If $|i_1|+|i_2|+\cdots+|i_n|=0$ then α is contained in one copy of G_1 and hence is contractible in the complement of C. If $|i_1|+|i_2|+\cdots+|i_n|>0$ then α can be contracted in the complement of C to a new loop in the complement of E with $|i_1|+|i_2|+\cdots+|i_n|$ two less by using the combinatorial fact about HNN groups above. By induction α is then contractible in the complement of C and hence G is 1-LC at infinity.

Remark. If in Theorem 2* one just assumes that $H^2(G_1;\mathbb{Z}_2G_1)$ is zero in addition to the fact that H has one end then using the appropriate Mayer–Vietoris sequence one can show that $H^2(G;\mathbb{Z}_2G)$ is also zero.

Thus Theorems 1 and 1* show that certain amalgamated free products and HNN groups, $G_1 *_S G_2$, $[G_1\colon S, f]$ where G_1 and G_2 have one end and S has more than one end, are not stable at infinity. If G_1 and G_2 are not assumed to have one end, then $G_1 *_S G_2$ may be stable at infinity even if S has an infinite number of ends. For example let G_1 be the free group generated by a and b and G_2 the free group generated by c and d. Let the amalgamated subgroup S be the free group generated by a^2b and b which is identified with the free group generated by $d^{-2}c^{-1}$ and c so that $a^2b=d^{-2}c^{-1}$ and $b=c$. Then $G_1 *_S G_2$ is isomorphic to $K=\langle e,f,g \mid e^2f^2g^2=1\rangle$. Since K is isomorphic to the fundamental group of a closed aspherical 2-manifold covered by $\mathbb{R}^2$, then K is stable and $e_1(K)=\mathbb{Z}$.

Bieri in [2] shows that any extension G of H by K, where H and K both have more than one end, has $H^2(G,\mathbb{Z}_2G)$ nontrivial. So these groups have one end but are not 1-LC at infinity. Applying Stallings' classification to the group K with more than end shows that G can be written as $G_1 *_S G_2$ or $[G_1: S, f]$ where S is a group having more than one end which contains H as a subgroup of finite index. So all examples of groups that I know of which are not 1-LC at infinity can be written as $G_1 *_S G_2$ or $[G_1: S, f]$ where $e_0(S)>1$.

Theorems 2 and 2* show that certain amalgamated free products and HNN groups are 1-LC at infinity. Also it is well known that if a group G contains a subgroup H of finite index which is 1-LC at infinity then G is also 1-LC at infinity and thus undoubtedly one has many more groups which are 1-LC at infinity. Certainly some groups satisfying the conditions of these two theorems will be the fundamental groups of closed aspherical manifolds. It then follows that these manifolds have universal covering spaces homeomorphic to Euclidean space if they are of dimension greater than four. However in general it is not known to what extent the fundamental groups of aspherical manifolds can be written as amalgamated free products or HNN groups or when they contain such groups as subgroups of finite index, even when their dimension is three.

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