Bismut–Elworthy’s formula and random walk representation for SDEs with reflection

Jean-Dominique Deuschel\textsuperscript{a}, Lorenzo Zambotti\textsuperscript{b,*}

\textsuperscript{a}TU Berlin, Inst. Mathematik, Strasse des 17. Juni 136, 10623 Berlin, Germany
\textsuperscript{b}Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

Received 18 November 2003; received in revised form 29 December 2004; accepted 14 January 2005
Available online 8 March 2005

Abstract

We study the existence of first derivatives with respect to the initial condition of the solution of a finite system of SDEs with reflection. We prove that such derivatives evolve according to a linear differential equation when the process is away from the boundary and that they are projected to the tangent space when the process hits the boundary. This evolution, rather complicated due to the structure of the set at times when the process is at the boundary, admits a simple representation in terms of an auxiliary random walk. A probabilistic representation formula of Bismut–Elworthy’s type is given for the gradient of the transition semigroup of the reflected process.

\copyright 2005 Elsevier B.V. All rights reserved.

MSC: 60H10; 60J55; 60H07

Keywords: Stochastic differential equations with reflection; Malliavin calculus

\textsuperscript{*}Corresponding author. Tel.: +39 02 2399 4570; fax: +39 02 2399 4513.
E-mail addresses: deuschel@math.tu-be.de (J.-D. Deuschel), zambotti@mate.polimi.it (L. Zambotti).
1. Introduction

Let $I$ be a finite set of indexes. We consider a system of Stochastic differential equations of the Skorohod type

$$X_i^t(x) = x^i + \int_0^t b_i^i(X_r(x)) \, dr + \int_0^t l_i^i(x) \, dw_i^r, \quad t \geq 0, \ i \in I,$$

(1)

for all $x \in \mathbb{R}_I^+ = [0, \infty)^I$, where $b_i^j : \mathbb{R}_I^+ \to \mathbb{R}$ is continuously differentiable and $(w_i^r)_{i \in I}$ are independent Brownian motions on a probability space $(\Omega, \mathbb{P})$.

The aim of this paper is to provide probabilistic representations for the first derivatives of the process $X_i^t(x)$ w.r.t. the initial datum $x$ and of the transition semigroup of $X$

$$P_t f(x) := \mathbb{E}[f(X_t(x))], \quad t > 0, \ x \in \mathbb{R}_I^+$$

for all $f : \mathbb{R}_I^+ \to \mathbb{R}$ bounded and continuous.

Our approach is fully probabilistic and is based on a representation for the first partial derivatives of the process $X_i^t(x)$ w.r.t. to the initial datum $x$ in terms of an auxiliary random walk $\xi$ with values in the set $I$ of indexes. The existence of such derivatives, classical for smooth coefficients, is not trivial because of the presence of the non-smooth reflection term $l_i^i(x)$.

Notice that boundary of $\mathbb{R}_I^+$ is not $C^1$ because of the presence of corners. Nevertheless, we obtain $C^1$ regularity for the semigroup up to the boundary. Moreover, we obtain that the semigroup satisfies the Neumann condition at any point of the boundary, in particular along the corners where the normal vector field is multi-valued: see Corollary 1 below.

The random walk representation has been introduced in [3], as a probabilistic counterpart of an analytic framework given in [7]. Unlike [3], we do not assume positivity of $\partial b_i^j / \partial x^i, i \neq j$, and we generalize the random walk representation to path-integrals w.r.t. finite signed measures, following an idea of [6, pp. 8–9]. However, [6] remains at the level of transition semigroups, while we develop a kind of stochastic calculus w.r.t. signed measures: see Lemma 2 below.

We illustrate the result in two simple examples. First consider the trivial case $b_i^j(x) = b_i^j(x^i), \ i.e. \ the \ system \ is \ decoupled \ and \ we \ have \ a \ family \ of \ independent \ one-dimensional \ SDEs \ with \ reflection. \ In \ this \ case \ we \ have

$$\frac{\partial X_i^t(x)}{\partial x^i}(x) = \delta_{ij} \mathbf{1}_{(t_i > t)} \exp \left( \int_0^t \frac{\partial b_i^j}{\partial x^i}(X_{s}^i(x)) \, ds \right), \quad t \geq 0,$$

(2)

where $\tau_i : \inf \{ s > 0 : X_s^i(x) = 0 \}$. Clearly we have no interaction between the coordinates of $X$ and before $\tau_i$ the derivative is simply the solution of the
linearized equation

$$\eta^i_t := \frac{\partial X^i_t}{\partial x^i}(x) = 1 + \int_0^t \frac{\partial b^i(X^i_s(x))}{\partial x^i} \eta^i_s \, ds \quad t < \tau_i.$$

After $\tau_i$, by the strong Markov property $X^i$ forgets the initial condition $x^i$, so that $\partial X^i_t(x)/\partial x^i = 0$ for $t > \tau_i$.

We consider now a simple system with interaction, setting $b^i(x) = \sum_{k \in I} (x^k - x^i)$, $x \in \mathbb{R}^I$, $i \in I$.

We introduce the space of $I$-valued càdlàg functions $E := D([0, \infty); I)$ with coordinate process $(\xi_t)_{t \geq 0}$ and the probability measure $P_j$ on $E$ under which $\xi_0 = j$ and $\xi$ is the continuous time Markov chain with infinitesimal generator $Lf(i) := \sum_{k \in I} (f(k) - f(i))$, $i \in I$, $f \in \mathbb{R}^I$.

Consider the solution of the SDE in $\mathbb{R}^I$:

$$Y^i_t(x) = x^i + \int_0^t b^i(Y_r(x)) \, dr + w^i_t, \quad t \geq 0, \ i \in I.$$

Then, with our choice of $b$

$$\psi^i_t := \frac{\partial Y^i_t}{\partial x^i} = \delta^i + \int_0^t \sum_k (\psi^{kj}_s - \psi^i_s) \, ds, \quad t \geq 0,$$

and the solution of this equation admits the probabilistic representation

$$\psi^i_t = P_j(\xi_t = i) = E_j[1_{\{\xi_t = i\}}], \quad t \geq 0.$$

If we consider again the reflected SDE, our main result shows that the effect of the local time is the following: defining the random variables on $\Omega$, $\zeta_i := \inf\{s > 0 : X^i_s(x) = 0\}$ and $\sigma^i_t := \sup\{s \leq t : X^i_s(x) = 0\}$, then

$$\eta^i_t := \frac{\partial X^i_t}{\partial x^i} = \delta^i + \int_0^t \sum_k (\eta^{kj}_s - \eta^i_s) \, ds, \quad t < \zeta_i, \quad (3)$$

and

$$\eta^i_t = \int_{\sigma^i_t}^t \sum_k (\eta^{kj}_s - \eta^i_s) \, ds, \quad t \geq \zeta_i. \quad (4)$$

Up to the first hitting time of $X$ at the boundary, $\eta$ evolves like in the case without reflection. When $X^i_t(x) = 0$, $\eta^i_t$ has a discontinuity, becomes 0, and then $\eta$ evolves further according to the usual differential equation. Geometrically, $\eta^i_t$ is projected to the tangent space when $X^i_t(x)$ hits the boundary. Since $\partial b^i/\partial x^k \neq 0$ for $i \neq k$, $\eta^i_t$ does not remain identically zero after $\zeta_i$, and the complicated structure of the set $\{s : X^i_s(x) = 0\}$ makes the evolution of $\eta$ quite non-trivial. Nevertheless, a random walk representation for $\eta$ holds also in this case: defining the stopping
time on \( E \)

\[
\tau := \inf\{ t > 0 : X_t^\xi(x) = 0 \} = \inf_{k \in I} \inf\{ s > 0 : X_s^k(x) = 0, \xi_s = k \},
\]

we have the following probabilistic representation for \( \eta \):

\[
\eta_t^i = E_j[1_{\{\xi_t = i\}} 1_{\{\tau > 0\}}], \quad t \geq 0,
\]

i.e. the random walk \( \xi \) is killed at the first time \( \tau \) when \( \xi_t \) is in a site \( i \in I \) such that \( X_t^i = 0 \).

Once this is proved, we can give probabilistic representations for the gradient of the transition semigroup of \( (X_t) \): for \( t > 0 \) and \( f \) bounded on \( \mathbb{R}_+^I \):

\[
\frac{\partial}{\partial x^j} \mathbb{E}[f(X_t(x))] = \frac{1}{t} \sum_i \mathbb{E} \otimes \mathbb{E}_j \left[ f(X_t(x)) \int_0^t 1_{\{\xi_t = i\}} 1_{\{\tau > s\}} \, dw_s^j \right]
\]

which is a Bismut–Elworthy formula (see e.g. [10]), and if \( f \) is differentiable with bounded continuous derivatives

\[
\frac{\partial}{\partial x^j} \mathbb{E}[f(X_t(x))] = \sum_i \mathbb{E} \otimes \mathbb{E}_j \left[ \frac{\partial f}{\partial x^j}(X_t(x)) 1_{\{\xi_t = i\}} 1_{\{\tau > 0\}} \right].
\]

The random walk representation has been applied to \( \nabla \phi \) interface models in [3], i.e. to the case where \( I \subset [-N, N]^d \cap \mathbb{Z}^d \), \( N, d \in \mathbb{N} \) and \( b'(x) = \sum_{|j| = 1} V'(x^l - x^j) \), where \( x^j = 0 \) for \( j \notin I \), \( V \in C^2(\mathbb{R}) \) uniformly convex and even and \( V' = dV/dx \). Formulas (5) and (6) find various applications to \( \nabla \phi \) interface models near a hard wall in [4] and [14]. In [4] formula (6) is used to prove monotonicity of the variance of \( X_t^i \) w.r.t. \( N \). In [14] formula (5) is used for \( d = 1 \) to prove the Lipschitz continuity, uniform in \( N \), for the transition semigroup of a \( \nabla \phi \) interface models near a wall under diffusive rescaling, a fundamental technical tool in the study of the equilibrium fluctuations.

Finally we remark that an SDE similar to (3)–(4) has been introduced in [1] in order to give a probabilistic representation formula for the solution of a system of linear PDEs with mixed Dirichlet–Neumann boundary condition on a regular domain in \( \mathbb{R}^n \). The approach of [1] has been later generalized to the manifold case, see [8, Section V.6]. The results of this paper treat a similar problem on a domain, whose boundary is piecewise flat but has corners: see the discussion at the end of Section 2 below. We recall that a general theory on Lipschitz continuity of solutions of Skorohod Problems in domains with corners has been given in [5]. However, it seems that this paper is the first which studies \( \mathbb{P} \)-almost sure differentiability properties w.r.t. the initial datum of a process with reflection.

The paper is organized as follows: after giving the precise definitions and statements in Section 2, we give the main technical lemma in Section 3; then we prove differentiability of \( X \) in Section 4 and the Bismut–Elworthy formula in Section 5; finally Section 6 contains the proof of the random walk representation.
2. Definitions and main results

We denote by \( E := D([0, \infty); I) \) the space of \( I \)-valued càdlàg functions, by \( \xi_t : E \mapsto I \), \( t \in [0, \infty) \), the coordinate process and by \( \mathcal{N}_t := \sigma(\xi_s, s \in [0, t]) \) the natural filtration of \( \xi \). If \( P \) is a probability measure on \( E \), by a \( P \)-martingale we always mean a \((E, P, (\mathcal{N}_t)_{t \geq 0})\)-martingale.

Let \( c : [0, \infty) \times I \times I \mapsto \mathbb{R} \) be continuous. For all \( i \in I \) and \( s \in [0, \infty) \) we denote by \( P^c_{s,i} \) the probability measure on \( E \), under which

- \( \xi_t = i \) for all \( t \in [0, s] \)
- \((\xi_t)_{t \in [s, \infty)} \) has the law of the time continuous Markov chain with values in \( I \) starting at \( t = s \) from \( i \) and with time-dependent generator \((L_t)_{t \geq 0}\):

\[
L_t : \mathbb{R}^I \mapsto \mathbb{R}^I, \quad L_tf(i) := \sum_k |c_i(i, k)| (f(k) - f(i)), \quad i \in I.
\]

We denote by \((\eta_l)_{l} \) the sequence of jump moments of \( \xi \in E \):

\[
\eta_l : E \mapsto [0, \infty), \quad \eta_0 := 0, \quad \eta_{l+1} := \inf\{s > \eta_l : \xi_s \neq \xi_{\eta_l}\}.
\]

We define the sign function \( \sigma : \mathbb{R} \mapsto \{-1, 0, 1\} \)

\[
\sigma(r) = \begin{cases} -1, & \forall r \in (-\infty, 0), \\ 0, & \sigma(0) = 0, \\ 1, & \forall r \in (0, \infty). \end{cases}
\]

For all \( s, t \in [0, \infty) \) such that \( s \leq t \) we define the real bounded measurable function \( \rho^c_{s,t} \) on \( E \)

\[
\rho^c_{s,t} := \exp \left( \int_s^t \sum_{k \neq \xi_r} |c_r(\xi_r, k)| \, dr + \int_s^t c_r(\xi_r, \xi_r) \, dr \right) \prod_{s < \eta_k \leq t} \sigma(c_{\eta_k}(\xi_{\eta_k-1}, \xi_{\eta_k})).
\]

Finally we define the stopping time on \( \Omega \times E \)

\[
\tau : \inf\{s > 0 : X^c_s(x) = 0\} = \inf\{s > 0 : X^c_s(x) = 0, \xi_x = k\}
\]

with \( \inf \emptyset := +\infty \). We say that \( f : \mathbb{R}^I_+ = [0, \infty)^I \mapsto \mathbb{R} \) is differentiable at \( x \in \mathbb{R}^I_+ \) if there exists a vector \((\partial_i f(x), i \in I) \in \mathbb{R}^I \) such that

\[
f(x + h) = f(x) + \sum_i \partial_i f(x) h^i + o(|h|), \quad \forall h : x + h \in \mathbb{R}^I_+,
\]

and we remark that if \( x \) lies in the boundary of \( \mathbb{R}^I_+ \), then the requirement \( x + h \in \mathbb{R}^I_+ \) becomes essential.

We consider continuously differentiable \( b_i : \mathbb{R}^I_+ \mapsto \mathbb{R}, \ i \in I \), with bounded derivatives \( \partial b_i / \partial x^i \) for all \( i,j \). We recall that pathwise existence and uniqueness of solutions of Eq. (1) have been proved in [13]. We can now state the main results of the paper.

**Theorem 1.** Let

\[
c_{i,j} := \frac{\partial b_i}{\partial x^j}(X_t(x)), \quad t \geq 0, \ i,j \in I.
\]
Then for all $T > 0$ and all $x \in \mathbb{R}^d_+$, a.s. the map $y \mapsto X_T(y)$ is continuously differentiable at $x$ and, setting $\eta_T^j = \partial X_T^j / \partial x^i$, we have a.s.

$$\eta_T^j = \mathbf{E}_{0,j}^c [1_{(\xi_{t,T}^j = i)} 1_{(t,T)} \rho_{0,T}^c], \quad i,j \in I.$$  

(10)

Moreover, the r.h.s. of (10) defines a right-continuous modification of $\eta$ such that, setting

$$C^i : = \{ s \geq 0 : X_s^i (x) = 0 \}, \quad \sigma^j_i : = \sup \{ s \leq t : X_s^i (x) = 0 \}, \quad i \in I,$$

we have a.s. for all $t \geq 0$

$$\eta_t^j = \delta_j + \sum_k \int_0^t \frac{\partial b^j}{\partial x^k} (X_s(x)) \eta_s^k \, ds, \quad t \in [0, \inf C^i),$$

(11)

$$\eta_t^j = \sum_k \int_{\sigma^j_i}^t \frac{\partial b^j}{\partial x^k} (X_s(x)) \eta_s^k \, ds, \quad t \in [\inf C^i, \infty).$$

(12)

**Theorem 2.** For all $f : \mathbb{R}^d_+ \mapsto \mathbb{R}$ bounded and continuous, $t > 0$ and $x \in \mathbb{R}^d_+$

$$\frac{\partial}{\partial x^i} P_t f (x) = \frac{1}{t} \sum_i \mathbb{E} \left[ f (X_t (x)) \int_0^t \eta_s^j \, dw_s^i \right]$$

$$= \frac{1}{t} \sum_i \mathbb{E} \left[ \mathbf{E}_{0,j}^c \left[ f (X_t (x)) \int_0^t 1_{(\xi_{s,T}^j = 0)} 1_{(t,T)} \rho_{0,s}^c \, dw_s^i \right] \right]$$

(13)

and if $f$ is continuously differentiable with bounded derivatives on $\mathbb{R}^d_+$

$$\frac{\partial}{\partial x^i} P_t f (x) = \sum_i \mathbb{E} \left[ \frac{\partial}{\partial x^i} (X_t (x)) \eta_t^i \right]$$

$$= \sum_i \mathbb{E} \left[ \mathbf{E}_{0,j}^c \left[ \frac{\partial}{\partial x^i} (X_t (x)) 1_{(\xi_{t,T}^j = 0)} 1_{(t,T)} \rho_{0,t}^c \right] \right].$$

(14)

**Remark 1.** If $\partial b^j / \partial x^i \geq 0$ for $j \neq i$ then $\rho_{0,T}^c \geq 0$ and if furthermore $\sum_j \partial b^j / \partial x^i \leq 0$, then $0 \leq \rho_{0,T}^c \leq 1$. For instance, in the case of the $\nabla \phi$ interface model on the torus we have

$$I := (\mathbb{Z} / N \mathbb{Z})^d, \quad b^i (x) = \sum_{|i-j|=1} V'(x^i - x^j),$$

where $N, d \in \mathbb{N}$, $x^j \neq 0$ for $j \notin I$. Then in this case

$$c_0 (i,j) = V'' (X_t^i - X_t^j) \cdot 1_{(|i-j|=1)} \geq 0, \quad i \neq j, \quad c_r (i,i) = - \sum_{k \neq i} c_r (i,k),$$

so that $\rho_{0,t}^c \equiv 1$.

**Remark 2.** In the example of the introduction the random walk is independent of the process $X$, because the jump rates $\partial b^j / \partial x^i$ are constant in $x$. In the general case the law of $\xi$ depends on the trajectory of $X$ and $\xi$ is therefore a random walk in a random environment.
Remark 3. In [3], the set of indexes is decomposed into an interior and a boundary, say \( \overline{I} = I \cup \partial I \), and Dirichlet boundary conditions are imposed on \( \partial I \). This case and other boundary conditions are covered by our setting, interpreting the jump rate from \( i \in I \) to \( k \in \partial I \) as an additional diagonal term \( c_r(i, i) \).

Notice that \( \mathbb{R}_+^I = [0, \infty)^I \) is not a regular domain. In particular, the normal vector field can be multi-valued (see e.g. [5]): for all \( x \in \partial \mathbb{R}_+^I \), \( n(x) = \{e^j : x^j = 0, j \in I\} \), where \((e^j)_j\) is the canonical basis of \( \mathbb{R}^I \). Nevertheless, the Neumann condition holds in the strongest form, as the following Corollary states.

Corollary 1. For all bounded continuous \( f \) and \( t > 0 \), the transition semigroup \( P_t^f \) satisfies the Neumann condition at \( \partial \mathbb{R}_+^I := \{x \in \mathbb{R}_+^I : \min_i x_i = 0\} \):

\[
x \in \mathbb{R}_+^I, \ x_i = 0 \implies \frac{\partial}{\partial x_i} P_t^f(x) = 0.
\]

Proof. Since \( b \) is globally Lipschitz, by the Girsanov Theorem the law of \( X^i \) is absolutely continuous w.r.t. to the law of a reflected BM in \([0, \infty)\). Then if \( x^i = 0 \), a.s. \( t = 0 \) is an accumulation point of \( C^i \), and since \( \xi \) is piecewise constant and right-continuous, \( \mathbf{P}_{0, \tau}^f \)-almost surely \( \tau = 0 \). □

We end this section by comparing the results of this paper with those of [1]. In [1, Theorem 4.4], a probabilistic representation formula is given for any \( f : M \mapsto \mathbb{R}^q \), where \( M \) is regular bounded domain in \( \mathbb{R}^n \) and \( q, n \in \mathbb{N} \), such that \( f = (\pi_1, \ldots, \pi_q) \), \( \Delta \pi_i = 0 \) on \( M \) and

\[
P(x)f(x) - (I - P(x)) \frac{\partial f}{\partial n}(x) = \phi(x), \quad x \in \partial M,
\]

where \( P(x) \) is a symmetric matrix of order \( q \) of constant rank such that \( P(x)P(x) = P(x), \) \( \partial M \ni x \mapsto P(x) \) is smooth, \( \phi : \partial M \mapsto \mathbb{R}^q \) is continuous and \( \partial / \partial n \) denotes the normal derivative. The probabilistic formula for \( f \) contains a matrix-valued process, which solves a SDE similar to (11)--(12): see [1, Theorem 4.1].

If we set now \( u(t,x) := P_t^f \phi(x) \), then at least formally \( u \) solves

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \sum_k b^k \frac{\partial u}{\partial x^k}, \quad u(0, \cdot) = \phi,
\]

\[
x^i = 0 \implies \frac{\partial u}{\partial x^i}(x) = 0.
\]

Setting \( v^i := \frac{\partial u}{\partial x^i} \) and differentiating formally (15), we get

\[
\frac{\partial v^i}{\partial t} = \frac{1}{2} \Delta v^i + \sum_k b^k \frac{\partial v^i}{\partial x^k} + \sum_j \frac{\partial b^k}{\partial x^j} v^j, \quad v^i(0, \cdot) = \frac{\partial \phi}{\partial x^i},
\]

\[
x^i = 0 \implies v^i(t,x) = 0, \quad x^i = 0 \quad \text{and} \quad x^i \neq 0 \implies \frac{\partial v^i}{\partial x^i}(t,x) = 0.
\]

Then (16) and (17) are a system of linear PDEs on \( M := \mathbb{R}_+^I \) with mixed Dirichlet and Neumann conditions at \( \partial M := \{x : \min_j x^j = 0\} \). Notice that \( \partial M \) is not a regular
boundary, since it contains corners. Then the normal vector field at $\partial M$ is multi-valued
\[ n(x) = \{ e^j : j \in I, x^j = 0 \}, \quad x \in \partial M, \]
where $(e^j)_{j \in I}$ is the canonical basis of $\mathbb{R}^I$. Then it is natural to define the normal derivative of a smooth $f$ on $M$
\[ \frac{\partial f}{\partial n}(x) = \left\{ \frac{\partial f}{\partial x^j} : e^j \in n(x) \right\}, \quad x \in \partial M. \]
Moreover, for all $x \in \partial M$ we denote by $P(x)$ the linear projector onto the span of $n(x)$. Then, setting $\mathbb{R}^I \ni v(t, x) = (v^i(t, x) : i \in I)$, the boundary condition (17) is equivalent to
\[ P(x)v(t, x) - (I - P(x)) \frac{\partial v}{\partial n}(t, x) = [0], \quad x \in \partial M \]
which shows that (14) is analogous to Theorem 4.4 in [1], although in our case $\partial M$ and $x \mapsto P(x)$ are not smooth. Finally, we set $O := \mathbb{R}^I_+ \times I$
\[ v : [0, \infty) \times O \mapsto \mathbb{R}, \quad v(t, z) = v^i(t, x) \quad \text{for} \quad z = (x, i), \]
\[ \partial O := (\mathbb{R}^I_+) \times I = \left\{ (x, i) : \min_{j \in I} x^j = 0 \right\}, \]
\[ \partial^1 O := \{ (x, i) \in \partial O : x^i = 0 \}, \quad \partial^2 O := \partial O \setminus \partial^1 O. \]
Then we can write (17) in the form
\[ v = 0 \text{ on } \partial^1 O, \quad \frac{\partial v}{\partial n} = [0] \text{ on } \partial^2 O \]
and (16)–(17) can be interpreted as a PDE on $O$ with Dirichlet boundary condition on $\partial^1 O$, which explains the definition (8) of $\tau$, and Neumann boundary condition on $\partial^2 O$, which shows why $X_t$ is reflected for $t < \tau$.

3. Minima of perturbed BM

The following Lemma is crucial in the proof of Theorem 1 below.

**Lemma 1.** Let $(w_t)_{t \geq 0}$ be a Brownian motion on $(\Omega, \mathbb{P})$. For all $T > 0$, let $\theta : \Omega \mapsto [0, T]$ be a random variable such that a.s. $w$ attains its minimum over $[0, T]$ only at $\theta$, i.e.
\[ w_0 = \min_{[0, T]} w, \quad w_\theta < w_s, \quad \forall s \in [0, T] \setminus \{\theta\}. \]

There exists a random variable $\gamma > 0$, such that for any continuous process $f : [0, T] \times \Omega \mapsto \mathbb{R}$ with
\[ f(0) = 0, \quad |f(s) - f(t)| \leq \gamma |t - s|, \quad t, s \in [0, T] \]
(18)
a.s. $\theta$ is the only time when $w + f$ attains its minimum over $[0, T]$

$$w_0 + f_\theta = \min_{[0, T]} (w + f), \quad w_0 + f_\theta < w_s + f_s, \quad \forall s \in [0, T] \setminus \{\theta\}.$$

**Proof.** It is enough to consider the case $T = 1$.

**Step 1:** We recall the following path decomposition of a Brownian path around its minimum, proven in [2]: let $(U, M, \hat{M})$ be an independent triple, such that $U : \Omega \to [0, 1]$ has the arcsine law, $M$ and $\hat{M}$ are two standard Brownian meanders. Then $V(U, M, \hat{M}) \overset{d}{=} w$, where

$$E := \{ u \in C([0, 1]) : u(0) = 0 \}, \quad V : [0, 1] \times E \times E \mapsto E,$$

$$V(r, f, g)(t) := -\sqrt{r} f(1) + \sqrt{r} 1_{[0, 1]}(t) f \left(1 - \frac{t}{r}\right) + \sqrt{1 - r} 1_{[0, 1]}(t) g \left(\frac{t}{1 - r}\right), \quad t \in [0, 1].$$

This formula has the following meaning: $U$ is the unique time in $[0, 1]$ where the path attains its minimum and $-\sqrt{U} M(1)$ is the value of the minimum; over the intervals $[0, U]$ and $[U, 1]$ the path has the shape of two rescaled meanders, the first one being run backward from 0 to $U$. Moreover, in [9] it was proved that the law of the Brownian meander is absolutely continuous w.r.t. the law of the Bessel process $(\hat{\lambda}_t)_{t \geq 0}$ of dimension 3 starting from 0 over the interval $[0, 1]$. This proves in particular that $\theta$ is well-defined, since a.s. $\lambda > 0$ on $[0, 1]$.

**Step 2:** We want to prove now that

$$\lim_{s \to \theta} \frac{w(s) - w(\theta)}{|s - \theta|} = +\infty, \quad \text{almost surely.} \quad (19)$$

By the path decomposition of the previous step, it is enough to prove that

$$\lim_{t \to 0^+} \frac{\hat{\lambda}_t}{t} = +\infty. \quad (20)$$

Recall that, by time-inversion, $\hat{\lambda}_s := s \lambda_1/s, \ s > 0, \ \hat{\lambda}_0 = 0$ is again a Bessel process of dimension 3. Then (20) is equivalent to $\hat{\lambda}_s \to \infty$ as $s \to \infty$, which follows by the transience of the Bessel process of dimension 3. Arguing by contradiction, by (19) it is now easy to conclude the proof. □

4. The derivative of $X$

This section is devoted to the proof of Theorem 1. First we give in Proposition 1 below a representation formula for the value at a fixed time $T > 0$ for solution of certain linear differential equations in $\mathbb{R}^I$.

We consider for each $i \in I$ a compact set $C^i \subset [0, T]$ with zero Lebesgue measure. We set $C := \bigcup_i C^i$ and

$$\sigma^i_t := \sup \left(C^i \cap [0, t] \right), \quad t \in [0, T], \ i \in I, \quad (21)$$
with the convention \( \sup \emptyset := 0 \). Notice that \( t \mapsto \sigma^i_t \) is locally constant on \([0, T] \setminus C^i \). We also introduce the stopping time on \( E \)

\[
\tau := \inf_{i \in I} \inf\{s > 0 : \xi_s = i, \ s \in C^i\}, \quad \inf \emptyset := + \infty.
\]

**Proposition 1.** Let \( y : [0, T] \to \mathbb{R}^I \) a bounded measurable function such that

\[
y^i_t = y^i_0 + \int_0^t \sum_k c^i_r(k, i) y^k_r \, dr, \quad t \in [0, \inf C^i),
\]

\[
y^i_t = \int_{\sigma^i_t} \sum_k c^i_r(k, i) y^k_r \, dr, \quad t \in (\inf C^i, T] \setminus C^i.
\]

for all \( i \in I \). Then for all \( i \in I \)

\[
y^i_T = \sum_{j \in I} y^i_0 \mathbb{E}_0^{c_i} \left[ 1_{(\xi_T = i)} 1_{(\tau > T)} \rho_0^{(i)} \right].
\]

The Proof of Proposition 1 is postponed to Section 6.

**Remark 4.** If \( C^i = \emptyset \) for all \( i \in I \), then \( y \) satisfies (22) for all \( t \in [0, T] \) and (24) gives a probabilistic representation for the solution of a linear differential equation in \( \mathbb{R}^I \) with continuous coefficients. If \( t \in \bigcap_{i \in I} C^i \) then \( \tau \leq t \) and, by (24), \( y^i_T = 0 \) for all \( i \in I \). Formula (24) becomes more interesting when \( \bigcup_{i} C^i \neq \emptyset \) and \( \bigcap_{i} C^i = \emptyset \). In general \( C^i \) can be a very complicated set, as it is our case in Section 3, where we consider

\( C := \{s \in [0, T] : X^i_s(x) = 0\} \), which is known to be a.s. a set with zero Lebesgue measure without isolated points.

**Proof of Theorem 1.** We recall that \((e^i)_{i \in I}\) is the canonical basis of \( \mathbb{R}^I \), i.e. \( e^i(i) = \delta_{ij} \). We divide the proof into several steps:

*Step 1:* Since for all \( t \geq 0 \): \( X^i_t dI^i_t = 0 \), then we have

\[
(X^i_t(x) - X^i_t(y))[dI^i_t(x) - dI^i_t(y)] = -X^i_t(x) dI^i_t(y) - X^i_t(y) dI^i_t(y) \leq 0.
\]

By the Cauchy–Schwarz inequality for the Euclidean norm \( \| \cdot \| \) in \( \mathbb{R}^I \)

\[
d\|X_t(x) - X_t(y)\|^2 = 2 \sum_i (X^i_t(x) - X^i_t(y))[b^i(X_t(x)) - b^i(X_t(y))] \, dt
\]

\[
+ 2 \sum_i (X^i_t(x) - X^i_t(y))dI^i_t(x) - dI^i_t(y) \| \leq 2K\|X_t(x) - X_t(y)\|^2 \, dt,
\]

where \( K \) denotes the Lipschitz norm of \( b \), so that

\[
\|X_t(x) - X_t(y)\|^2 \leq \|x - y\|^2 \exp(2Kt), \quad t \geq 0.
\]

*Step 2:* For every \( i \in I \) we set \( C^i := \{s \in [0, T] : X^i_s = 0\} \). We claim that almost surely the sets \((C^i)_{i \in I}\) satisfy the properties required in the hypothesis of Proposition 1. Indeed, since \( X^i \) is a continuous semimartingale with \( \langle X^i, X^j \rangle_t = t \) for all \( t \geq 0 \), by
the occupation time formula the Lebesgue measure of $C^i$ is equal to

$$|C^i| = \int_0^1 1_{[0]}(X^i_s) \, ds = \int_{[0]} L^a_s \, da = 0,$$

where $L^a$ is the local time process of $X^i$ at $a$. Moreover a.s. $T \not\in \bigcup_i C^i$, since a.s. $X^{i,\tau}(x) > 0$ for all $i \in I$, and the claim is proved.

Let now $(A_n)_n$ be the countable collection of the connected components of the set $[0, T] \setminus C$, open in $[0, T]$. We set $a_n := \inf A_n$. Since $A_n$ is open in $[0, T]$ there exists a rational $q_n \in A_n$. For $i \in I$ we denote by $A^i_n$ the connected component of $[0, T] \setminus C^i$ which contains $q_n$. Then $A_n \subseteq A^i_n$.

**Step 3:** We set

$$W^i_t(x) := \int_0^t b^i(X^i_r(x)) \, dr + w^i_t, \quad t \in [0, T], \; i \in I.$$

Since $b$ is globally Lipschitz, by the Girsanov Theorem the law of $W^i(x)$ over $[0, T]$ is absolutely continuous w.r.t. the law of a BM. Therefore, by Lemma 1 a.s. for every rational $q \in [0, T] \cap \mathbb{Q}$ we can find a random variable $\theta^i_q \in [0, q]$ such that $W^i(x)$ attains its minimum over $[0, q]$ only at $\theta^i_q$ and moreover $\sqrt{\gamma^i} + W^i(\theta^i_q(x)) \neq 0$.

For all $t \in [0, T]$, we have by Skorohod’s Lemma (see e.g. [11, VI.2.1])

$$l^i_t(x) = \sup_{s \leq t} \left[ x^i + \int_0^s b^i(X^i_r(x)) \, dr + w^i_s \right] = \left[ -x^i + \inf_{s \leq t} (W^i_s(x)) \right]^+,$$

where $2(r)^- := |r| - r, (r)^+ := (-r)^+$. Using this representation, it is easy to see that a.s. for all $n$: $\theta^i_{q_n} = \sigma^i_{q_n}$, where $\sigma^i$ is defined in (21). In particular, for all $t \in A^i_n$:

$$l^i_t(x) = l^i_{\sigma^i_{q_n}}(x) = [-x^i - W^i_{\sigma^i_{q_n}}(x)]^+ \tag{26}.$$

**Step 4:** Fix $i, j \in I$ and for all $\varepsilon > 0$ let $x^i_\varepsilon := x + \varepsilon e^i$. The key point of the proof is the following: for $\varepsilon > 0$ small enough, we can represent $l^i_t(x^i_\varepsilon)$ by (26) as follows:

$$l^i_t(x^i_\varepsilon) = [-x^i_\varepsilon - W^i_{\sigma^i_{q_n}}(x^i_\varepsilon)]^+$$

with $\sigma^i_{q_n}$ independent of $\varepsilon$. This will allow to differentiate $l^i_t(x^i_\varepsilon)$ in $\varepsilon$.

Since the law of $W^i(x)$ over $[0, T]$ is absolutely continuous w.r.t. the law of a BM, by Lemma 1 a.s. for every rational $q \in [0, T] \cap \mathbb{Q}$ we can find a random variable $\gamma^i_q > 0$ such that every $\gamma^i_q$-Lipschitz perturbation of $W^i(x)$ attains its minimum over $[0, q]$ only at $\sigma^i_q$. Fix now $n \in \mathbb{N}$. By (25), there exists a random $\Delta^i_n > 0$ such that

$$\sup_{r \in [0, q_n]} |b^i(X^i_r(x^i_\varepsilon)) - b^i(X^i_r(x))| \leq \gamma^i_{q_n}, \quad \forall \varepsilon \in (0, \Delta^i_n),$$

which implies, setting $f(s) := W^i_s(x^i_\varepsilon) - W^i_s(x)$

$$|f(t) - f(s)| \leq \gamma^i_{q_n} |t - s|, \quad t, s \in [0, q_n].$$

By Lemma 1, since $W^i(x)$ attains its minimum over $[0, q_n]$ only at $\sigma^i_{q_n}$ we have for $\varepsilon \in (0, \Delta^i_n)$ that also $W^i(x^i_\varepsilon)$ attains its minimum over $[0, q_n]$ only at $\sigma^i_{q_n}$. 

---

and in particular

\[ l^i_{q_n}(x_\varepsilon) = \left[ -X^{i}_\varepsilon - W^{i}_{\sigma^{i}_{q_n}}(x_\varepsilon) \right]^+ \]

\[ = \left[ -X^{i}_\varepsilon - W^{i}_{\sigma^{i}_{q_n}}(x) - \int_0^{\sigma^{i}_{q_n}} [b^{i}(X^{i}_r(x_\varepsilon)) - b^{i}(X^{i}_r(x))] dr \right]^+. \tag{27} \]

Recall that \( X^{i}_n + W^{i}_{\sigma^{i}_{q_n}}(x_\varepsilon) \neq 0 \). Then either \( X^{i}_n + W^{i}_{\sigma^{i}_{q_n}}(x) > 0 \) or \( X^{i}_n + W^{i}_{\sigma^{i}_{q_n}}(x) < 0 \), according to whether \( q_n < \inf C^i \) or \( q_n > \inf C^i \). By (27), possibly after choosing a smaller \( \Delta'_n > 0 \), for \( \varepsilon \in (0, \Delta'_n) \), in the first case

\[ l^i_{q_n}(x_\varepsilon) = l^i_{q_n}(x) = 0 \]

and in the second case

\[ l^i_{q_n}(x_\varepsilon) - l^i_{q_n}(x) = -\varepsilon \delta_{ij} - \int_0^{\sigma^{i}_{q_n}} [b^{i}(X^{i}_r(x_\varepsilon)) - b^{i}(X^{i}_r(x))] dr. \]

Since for all \( t \in [0, T] \)

\[ X^{i}_r(x_\varepsilon) - X^{i}_r(x) = \varepsilon \delta_{ij} + \int_0^t [b^{i}(X^{i}_r(x_\varepsilon)) - b^{i}(X^{i}_r(x))] dr + l^i_{q_n}(x_\varepsilon) - l^i_{q_n}(x), \]

then, setting for all \( i \in I, t \in [0, T] \) and \( \varepsilon > 0 \)

\[ \Delta_n := \min\{\Delta'_n\}, \quad \eta^{i}_{n}(\varepsilon) := \frac{X^{i}_r(x_\varepsilon) - X^{i}_r(x)}{\varepsilon}, \]

\[ X^{2,i}_r := \alpha X^{i}_r(x_\varepsilon) + (1 - \alpha) X^{i}_r(x), \quad \alpha \in [0, 1], \]

we obtain a.s. for all \( n \in \mathbb{N}, \varepsilon \in (0, \Delta'_n) \) and \( t \in \Delta^{i}_n \)

\[ \eta^{i}_{n}(\varepsilon) = \delta_{ij} + \int_0^t \sum_k \left[ \int_0^{1} \frac{\partial b^{i}}{\partial X^k}(X^{2,i}_r) dz \right] \eta^{k}_{n}(\varepsilon) dr, \text{ if } t \in [0, \inf C^i), \tag{28} \]

\[ \eta^{i}_{n}(\varepsilon) = \int_0^t \sum_k \left[ \int_0^{1} \frac{\partial b^{i}}{\partial X^k}(X^{2,i}_r) dz \right] \eta^{k}_{n}(\varepsilon) dr, \text{ if } t \in (\inf C^i, T], \tag{29} \]

\[ \eta^{i}_{n}(\varepsilon) = \eta^{i}_{a_n}(\varepsilon) + \int_0^t \sum_k \left[ \int_0^{1} \frac{\partial b^{i}}{\partial X^k}(X^{2,i}_r) dz \right] \eta^{k}_{n}(\varepsilon) dr. \tag{30} \]

**Step 5:** By (25), \( \|\eta_{n}(\varepsilon)\| \leq \exp(Kt) \) for all \( \varepsilon > 0, t \geq 0 \). Let \( (\tilde{c}_{m})_m \) be any monotone non-increasing sequence converging to 0. By a diagonal procedure, we can extract a subsequence \((m_l)_l\) such that \( \eta_{n_l}(\tilde{c}_{m_l}) \) has a limit \( \eta_{a_n} \in \mathbb{R}^l \) as \( l \to \infty \) for all \( n \in \mathbb{N} \). Let \( \eta : [0, T] \times \mathbb{C} \to \mathbb{R}^l \) be the unique solution of

\[ \eta^{i}_{n}(\varepsilon) := \eta^{i}_{a_n} + \int_0^t \sum_k \frac{\partial b^{i}}{\partial X^k}(X^{i}_r(x)) \eta^{k}_{n}(\varepsilon) dr, \quad t \in A_n. \]

By a standard application of Gronwall’s Lemma we obtain that \( \eta_{n}(\tilde{c}_{m}) \to \eta_{t} \) uniformly in \( t \in A_n \). Since this is true for every \( n \) and \( C \) has zero Lebesgue measure,
derivatives are continuous. To this aim, we use formula (31).

ð

which does not depend on the chosen subsequence 

we obtain

\[ \eta^i_T = E_{0,j}^c \left[ 1_{\{\xi_T=i\}} 1_{\{t>T\}} \rho^c_{0,T} \right] \]

that does not depend on the chosen subsequence \((e_m)_n\), so that

\[ \lim_{\varepsilon \to 0} \frac{X^i_T(x + \varepsilon e^j) - X^i_T(x)}{\varepsilon} = \eta^i_T = E_{0,j}^c \left[ 1_{\{\xi_T=i\}} 1_{\{t>T\}} \rho^c_{0,T} \right]. \] (31)

\textbf{Step 6:} We have proved so far that right partial derivatives exist a.s. at every 

\(x \in \mathbb{R}^d_I\). To prove differentiability, it is enough to show that such right partial 

dervatives are continuous. To this aim, we use formula (31).

We denote by \(Q_j\) the law on \(E_T\) of the random walk with values in \(I\) starting from \(j\) 

at time 0 and with generator \(M\)

\[ M : \mathbb{R}^I \to \mathbb{R}^I, \quad Mf(i) := \sum_k m(f(k) - f(i)), \quad i \in I \]

with \(m := \sup_{[0,T] \times I} |c|\). By Theorem IV.22.4 in [12], \(P_{0,j}^c\) is absolutely continuous 

w.r.t. to \(Q_j\) on \(E_T\) with bounded density

\[ \gamma_T := \exp \left( - \int_0^T \left[ \sum_{k \neq \xi} (|c_{ij}(\xi, k)| - m) \right] dr \right) \prod_{0 < \eta_k \leq T} \frac{|c_{ij}(\xi_{\eta_k-1}, \xi_{\eta_k})|}{m}. \]

Notice that \(Q_j\) does not depend on \(x\), while \(\rho^c_{0,T} \gamma_T\) depends continuously on \(c = c(X, x)\) and in particular on \(x\) and is uniformly bounded. We claim that also the indicator function \(1_{\{t>T\}}\) depends continuously on \(x\), \(Q_j\)-a.s. Indeed, by Lemma 3, 

\(P_{0,j}^c\)-a.s. no jump moment \(\eta_k\) of \(\xi\) is in \(C\), so that \(\xi \in \{t>T\}\) iff

\[ \forall \eta_k \in (0, T), \quad \xi_{\eta_k} = i \implies X^i_j(x) > 0, \quad \forall t \in [\eta_{k-1}, \eta_k]. \]

Since \(P_{0,j}^c\)-a.s. the number of jumps of \(\xi\) before time \(T\) is finite, by the compactness of 

\([\eta_{k-1}, \eta_k]\) and the continuity of \(x \mapsto (X^i_j(x))_{t \in [\eta_{k-1}, \eta_k]}\) in the sup-norm topology given 

by (25), we obtain the claim. By the Dominated Convergence Theorem we can 

conclude the proof of the theorem. \(\square\)

In fact, the same proof yields the following more general result:

\textbf{Corollary 2.} Let \(H : [0, \infty) \to \mathbb{R}^d_I\) be adapted to the filtration of \(w\) and such that for all 

\(i \in I\), \(dH^i_s = h^i_s \, ds \) with \(h^i \in L^\infty(\Omega \times [0, T])\) for all \(T > 0\). Let \(Z_t = Z_t(H) \to \mathbb{R}^d_I\) be the solution of the system of SDEs with reflection:

\[ Z_t^i = Z_t^i(H) = x^i + \int_0^t b^i(Z_s) \, ds + l^i_t + w^i_t + H^i_t, \quad t \geq 0, \quad i \in I. \]
Notice that $Z(0) = X(x)$. Then for all $T > 0$ a.s. $Z'_T(\varepsilon H)$ is differentiable in $\varepsilon$ and:

$$
\frac{d}{d\varepsilon} Z'_T(\varepsilon H) \bigg|_{\varepsilon = 0} = \sum_k \eta_{ik}^T H_0^k + \sum_k \int_0^T \eta_{ik,T}^T h_s^k ds,
$$

(32)

where we set $\tau_s := \inf \{ r > s : X^{\varepsilon}(x) = 0 \}$ and

$$
\eta_{ij,s,T}^T := E_{s,j}^c \left[ 1_{(t_\varepsilon_s = 0)} 1_{(t_\tau_s > t)} \rho_{s,t}^c \right], \quad i,j \in I, \quad t \geq s \geq 0.
$$

**Proof.** Arguing like in the proof of Theorem 1, we obtain that a.s. the derivative of $Z'_T(\varepsilon H)$ at $\varepsilon = 0$ exists and is equal to $\eta_T(H)$, where $\eta(H)$ satisfies $\eta_0(H) = H_0$ and for $t \in A_{\eta}$:

$$
\eta_t^j(H) := \eta_{ia_n}^j(H) + \int_{a_n}^t \sum_k \frac{\partial b_i^j}{\partial x_k}(X_T(x)) \eta_t^k(H) dr + \int_0^t h_s^j ds.
$$

(33)

By Proposition 1 the only solution of this equation is given by the right-hand side of (32). \qed

5. **Bismut–Elworthy’s formula**

This section is devoted to the proof of Theorem 2. The proof of (14) is straightforward by Theorem 1 and the chain rule. We now prove (13).

First let $f$ be continuously differentiable with bounded derivatives. In the notation of Corollary 2, let $H_0 = 0$ and $h_s^k := \eta_{ij}^k / T$, $s \in [0, T]$. Then, by (11)–(12)–(33) and the uniqueness of solutions of linear equations

$$
\sum_k k \int_0^T \eta_{ik,T}^k h_s^k ds = \frac{1}{T} \sum_k \int_0^T \eta_{ik,T}^k \eta_s^{kj} ds = \frac{1}{T} \int_0^T \eta_s^{ij} ds = \eta_T^{ij}.
$$

Therefore by (32)

$$
\frac{d}{d\varepsilon} E[f(Z_T(\varepsilon H))] \bigg|_{\varepsilon = 0} = E \left[ \sum_i \frac{\partial f}{\partial x_i}(X_T(x)) \eta_T^{ij} \right] = \frac{\partial}{\partial y^j} E[f(X_T(y))] \bigg|_{y=x}.
$$

On the other hand, since $H_0 = 0$ by the Girsanov Theorem we have

$$
E \left[ f(Z_T(\varepsilon H)) \exp \left\{ -\sum_k \varepsilon \int_0^T h_s^k dw_s^k - \frac{\varepsilon^2}{2} \int_0^T |h_s^k|^2 ds \right\} \right] = E[f(X_T(x))]
$$

and differentiating at $\varepsilon = 0$ we obtain (13). By a density argument, we obtain (13) for $f$ bounded and continuous. \qed

**Remark 5.** Arguing analogously, we obtain an integration by parts formula on the law of $X(x)$. Let $h : [0, \infty) \rightarrow \mathbb{R}^I$ adapted to the filtration of $w$ and such that $h^i \in
\( L^\infty(\Omega \times [0, T]) \) for all \( T > 0 \), and set

\[
\mathcal{F} h := K, \quad K^i_t := \sum_k \int_0^t \eta_{s,t}^{ik} h^k_s \, ds, \quad t \in [0, T].
\]

Then the law of \((X_t(x))_{t \in [0,T]}\) admits an integration by parts formula along all process of the form \( \mathcal{F} h \), i.e. for all continuous \( F : C([0,T]; \mathbb{R}^I) \rightarrow \mathbb{R} \)

\[
\frac{d}{d\varepsilon} \mathbb{E}[\Phi(X(x) + \varepsilon \mathcal{F} h)] \bigg|_{\varepsilon=0} = \mathbb{E} \left[ \Phi(X(x)) \int_0^T \sum_k h^k_s \, dw^k_s \right].
\]


### 6. Random walk representation

This section is devoted to the proof of Proposition 1. First we give three preliminary lemmas. Notice that we can write

\[
\rho_{s,t}^c := R_{s,t} r_{s,t},
\]

where

\[
R_{s,t} := \exp \left( \int_s^t \sum_k c_r(\zeta_r, k) \, dr \right),
\]

\[
r_{s,t} := \exp \left( 2 \int_s^t \sum_{k \neq \zeta_r} (c_r(\zeta_r, k))^- \, dr \right) \prod_{s < \eta_k \leq t} \sigma(c_{\eta_k}(\zeta_{\eta_{k-1}}, \zeta_{\eta_k})),
\]

where \( 2(r)^- := |r| - r \). We set for all \( j \in I \) and \( t \in [s, \infty) \)

\[
X_{s,t}^c(j) := 1_{\{\zeta_r=j\}} \rho_{s,t}^{c_r} - 1_{\{\xi_r=j\}} - \int_s^t c_r(\zeta_r, j) \rho_{s,t}^{c_r} \, dr.
\]

**Lemma 2.** \((r_{s,t})_{t \in [s,S]}\) and \((X_{s,t}^c(j))_{t \in [s,S]}\) are bounded \( P_{s,t}^c \)-martingales for all \( S \geq s \).

**Remark 6.** In the proof of Lemma 2, \( R_{s,t} \) is shown to be a Feynman–Kac-type term, and \( r_{s,t} \) a Girsanov-type term. Indeed, like in Girsanov’s Theorem, the density \( r_{s,t} \) is proved to be a martingale in \( t \), and more precisely an exponential martingale, which in the framework of discontinuous martingales need not be non-negative.

**Proof of Lemma 2.** The boundedness follows easily since \( c \) is continuous and \( I \) is finite. We divide the rest of the proof into two steps.

**Step 1:** Let \( S \geq s \). For all bounded continuous \( g : [s, S] \times I \times I \rightarrow \mathbb{R} \) we define for all \( t \in [s, S] \)

\[
N_{s,t}^g := \sum_{s < \eta_k \leq t} g_{\eta_k}(\zeta_{\eta_{k-1}}, \zeta_{\eta_k}) - \int_s^t \sum_{k \neq \zeta_r} |c_r(\zeta_r, k)| g_r(\zeta_r, k) \, dr.
\]
By Lemma IV.21.13 in [12], $N^\vartheta$ is a local $P^c_{s,t}$-martingale over $[s, S]$. We denote now by $\mathcal{E}(N^\vartheta)$ the exponential martingale of $N^\vartheta$, unique solution of

$$y_t = 1 + \int_{(s,t]} y_{r-} \, dN^\vartheta_r, \quad t \in [s, S].$$

Then $\mathcal{E}(N^\vartheta)$ is a local $P^c_{s,t}$-martingale and has the explicit form

$$\mathcal{E}(N^\vartheta)_t = \exp \left( - \int_s^t \sum_{k \neq \xi_r} |c_r(\xi_r, k)| g_r(\xi_r, k) \, dr \right) \prod_{s < \eta_k \leq t} \left( 1 + g_{\eta_k}(\xi_{\eta_k-1}, \xi_{\eta_k}) \right)$$

see [12, Section IV.19]. Setting $\overline{g}_i(i,j):=\sigma(c_i(i,j)) - 1$, we have $r_{s,t} = \mathcal{E}(N^\vartheta)$, and therefore $(r_{s,t})_{t \in [s, S]}$ is a bounded $P^c_{s,t}$-martingale.

Step 2: We set for all $t \in [s, \infty)$ and $j \in I$

$$M^j_{t} := 1(\xi_r = j) - 1(\xi_r = j) - \int_s^t \sum_k |c_r(\xi_r, k)|(1(k=j) - 1(\xi_r = j)) \, dr,$$

$$\tilde{M}^j_{t} := 1(\xi_r = j) - 1(\xi_r = j) - \int_s^t c_r(\xi_r, k)(1(k=j) - 1(\xi_r = j)) \, dr.$$

We recall that $(M^j_{t})_{t \in [s, S]}$ is a $P^c_{s,t}$-martingale. Now, arguing as in the proof of [12, Theorem IV.22.4], we find

$$d(\tilde{M}^j_{t} r_{s,t}) = d(M^j_{t} \mathcal{E}(N^\vartheta)_t)$$

$$= \mathcal{E}(N^\vartheta)_{t-} \, dM^j_{t} + \tilde{M}^j_{t-} \, d\mathcal{E}(N^\vartheta)_t + \mathcal{E}(N^\vartheta)_{t-} \, dN^G_t,$$

where $G_i(i,k) := \overline{g}_i(i,k)(1(k=j) - 1(i=j))$. Then $(M^j_{t} r_{s,t})_{t \in [s, S]}$ is a $P^c_{s,t}$-martingale that we call $m^j$. In particular

$$d(1(\xi_r = j) r_{s,t}) = \sum_k c_r(\xi_r, k)(1(k=j) - 1(\xi_r = j)) \, r_{s,t} \, dt + dm^j_t.$$

Then, since $t \to R_{s,t}$ is absolutely continuous:

$$d \left( 1(\xi_r = j) \rho^c_{s,t} \right) = R_{s,t} \, d(1(\xi_r = j) r_{s,t}) + 1(\xi_r = j) \, r_{s,t} \, dR_{s,t}$$

$$= c_r(\xi_r, j) \rho^c_{s,t} \, dt + R_{s,t} \, dm^j_t,$$

so that for all $t \geq s$: $X^c_{s,t}(j) = \int_s^t R_{s,t} \, dm^j_t$, and the thesis is proved. \qed

Lemma 3. $P^c_{0,t}$-a.s. there exists no $k \geq 1$ such that $\eta_k \in C$.

Proof. In the notation of the proof of Lemma 2, setting $g_i(i,j) = 1_c(t)$ and

$$N^\vartheta = \sum_{s < \eta_k \leq t} 1_c(\eta_k) - \int_s^t \sum_{k \neq \xi_r} |c_r(\xi_r, k)| \, 1_c(r) \, dr,$$

then $N^\vartheta$ is a $P^c_{s,t}$-martingale. Since $C$ has zero Lebesgue measure, then $N^\vartheta_t = \sum_{s < \eta_k \leq t} 1_c(\eta_k) = 0$ $P^c_{0,t}$-a.s. for all $t \geq 0$. \qed
Let now $T>0$. In the proof of Proposition 1 a time-reversal argument is needed. For this reason, we set for all $0 \leq s \leq t$ and $i,j \in I$

\[ \hat{c}_s(i,j) = c_{T-s}(j,i) \quad \text{if} \quad t \leq T, \quad \hat{c}_s(i,j) = c_0(j,i) \quad \text{if} \quad t > T, \]

\[ \hat{P}_s^{c_{i,j}} = \mathbf{P}_s^{c_{i,j}}, \quad \hat{X}_s^{c_{i,j}} = X_s^{c_{i,j}}. \]

Let $E_T := D([0,T]; I)$. If $e : [0,T] \mapsto I$ has right limit at any $t \in [0,T]$, we set $e^*(t) := \lim_{s \uparrow t} e(s)$. Then for all $e \in E_T$, $[e_{T-}]^* \in E_T$.

**Lemma 4.** For all bounded Borel $\Phi : E_T \mapsto \mathbb{R}$ and $i,j \in I$

\[ E_{0,i}^c[\Phi(\xi) 1_{(\xi_T = j)} \rho_{0,T}^c] = E_{0,i}^c[\Phi(\xi_{T-}^c)] 1_{(\xi_T = j)} \rho_{0,T}^c]. \]

**Proof.** We divide the proof into several steps.

**Step 1:** Notice first that for all $s, i,j \in I$

\[ E_{0,i}^c[\Phi(\xi) 1_{(\xi_T = j)} \rho_{s,T}^c] = E_{0,i}^c[\Phi(\xi_{T-}^c)] 1_{(\xi_T = j)} \rho_{s,T}^c]. \]

Then the following analogue of the Markov Property holds for $\rho_{s,T}^c \cdot \mathbf{P}_{s,i}^{c_{i,j}}$ for all bounded Borel $\psi : E_T \mapsto \mathbb{R}$ and $A \in \mathcal{N}_i$, $t \in [s,T]$, we have

\[ E_{s,i}^c[1_A(\xi) \psi(\xi_{t+}^c) \rho_{s,T}^c] = E_{s,i}^c[1_A(\xi) \Psi(t, \xi) \rho_{s,T}^c], \]

where $\Psi : [0,T] \times I \mapsto \mathbb{R}$ is

\[ \Psi(t, k) := E_{i,k}^c[\psi(\xi_{t+}^c) \rho_{s,T}^c]. \]

and an analogous formula holds for $\hat{P}_{s,T}^c \cdot \mathbf{P}_{s,i}^{c_{i,j}}$.

**Step 2:** We set for all $s, t \in [0,T]$, $t \geq s$, and $i,j \in I$

\[ P_{s,i}(i,j) := E_{s,i}^c[1_{(\xi_T = j)} \rho_{s,T}^c], \]

\[ \hat{P}_{s,i}(i,j) := E_{s,i}^c[1_{(\xi_T = j)} \hat{\rho}_{s,T}^c]. \]

By Lemma 2, $E_{s,i}^c[X_{s,t}^c(j)] = 0$ for all $t \in [s,T]$ and $j \in I$. We obtain

\[ P_{s,i}(i,j) = \delta_{ij} + \int_s^t \sum_k P_{s,r}(i,k) c_r(k,j) \, dr, \quad t \in [s,T], i,j \in I. \]

By this formula and the Markov property proved in Step 1, we obtain

\[ P_{s,i}(i,j) = \delta_{ij} + \int_s^t \sum_k c_r(k,j) P_{s,r}(i,k) \, dr, \quad t \in [s,T]. \]

Analogously, we obtain for all $i,j \in I$:

\[ \hat{P}_{s,i}(i,j) = \delta_{ij} + \int_s^t \sum_k \hat{P}_{s,r}(i,k) \hat{c}_r(k,j) \, dr, \quad t \in [s,T]. \]

By the uniqueness of solutions of linear differential equations

\[ \hat{P}_{sij} = P_{T-s,T-s}(j,i), \quad t \in [s,T], i,j \in I. \]

**Step 3:** We prove now the thesis. It is enough to consider

\[ \Phi(\xi) = 1_{(\xi_{i_1} = i_1)} \cdots 1_{(\xi_{i_n} = i_n)} \]
for some $n \in \mathbb{N}$, $s := t_0 \leq t_1 < \cdots < t_n \leq t_{n+1} = T$, and $i_1, \ldots, i_n \in I$, $l_0 = i$, $l_{n+1} = j$. Since $\hat{\mathbf{P}}_{0, j}$-a.s. $\eta_k \in \{l_l, l = 0, \ldots, n + 1\}$, then $\Phi([-\xi_{T-}, \xi_T]) = \Phi([-\xi_{T-}, \xi_{T-}])$. Then by the previous

$$
\mathbf{E}^c_{0, j}[\Phi([-\xi_{T-}, \xi_T])1_{(\xi_{T} = j)}\hat{\rho}_{0, T}^c] = \prod_{k=1}^{n+1} \hat{P}_{T-t_k, T-t_{k-1}}(i_k, i_{k-1})
= \prod_{k=1}^{n+1} P_{t_k, t_{k-1}}(i_{k-1}, i_k) = \mathbf{E}^c_{0, j}[\Phi(\xi)1_{(\xi_T = j)}\hat{\rho}_{0, T}^c].
$$

**Proof of Proposition 1.** Since every $C^i$ has zero Lebesgue measure, we can suppose from now on that $y^i = 0$ on $C^i$ and that $y^i$ satisfies (22) for all $t \in [0, \inf C^i)$ and (23) for all $t \in [\inf C^i, T]$, $i \in I$. Moreover $t \mapsto y_t^i$ is right-continuous on $[0, T]$ and the right-derivative of $y^i$:

$$
\lim_{h \downarrow 0} \frac{y^i_{t+h} - y^i_t}{h} = \sum_k c_i(k, i) y^k_t
$$

is well-defined and right-continuous for $t \in [0, T] \setminus C^i$, $i \in I$. We set now

$$
\hat{\tau}_i := \inf\{s > 0 : \xi_s = i, T - s \in C^i\}, \quad \hat{\tau} := \inf_{i \in I} \hat{\tau}_i, \quad \inf \emptyset := +\infty,
$$

$$
F_i := \sum_j y^j_{(T-\tau) \wedge 0} 1_{(\xi_{T} = j)} \hat{\rho}_{0, T}^c, \quad t \geq 0.
$$

For all $t \in [0, \hat{\tau})$, if $\hat{\tau}_i = j$ then $T - t \notin C^j$. Then $F$ is continuous and left-differentiable on $[0, \hat{\tau}) \setminus \{\eta_k\}$ and has left limits at any $\eta_k \in [0, \hat{\tau})$. Moreover, by Lemma 3, $\hat{\mathbf{P}}_{0, j}$-a.s. no $\eta_k \in C$, so that $y^j_{(T-\tau) \wedge 0}$ and $1_{(\xi_T = j)} \hat{\rho}_{0, T}^c$ have no common jump moment. Then, by the integration by parts formula for BV functions (see e.g. [12, Section IV.18]),

$$
\hat{\mathbf{P}}_{0, j}$-a.s.
$$
$$
F_{\hat{\tau} \wedge T} - F_0
= \int_{(0, \hat{\tau} \wedge T]} \sum_j y^j_{T-\tau} d(1_{(\xi_T = j)} \hat{\rho}_{0, T}^c) - \sum_{j,k} c_{T-\tau}(k, j) y^k_{T-\tau} 1_{(\xi_T = j)} \hat{\rho}_{0, T}^c d\tau
= \sum_j \int_{(0, \hat{\tau} \wedge T]} y^j_{T-\tau} d(1_{(\xi_T = j)} \hat{\rho}_{0, T}^c) - c_{T-\tau}(j, \xi_T) \hat{\rho}_{0, T}^c d\tau
= \sum_j \int_{(0, \hat{\tau} \wedge T]} y^j_{T-\tau} d\hat{\chi}^c(j),
$$

since $c_{T-\tau}(k, j) = \hat{c}_r(j, k)$ and by the right-continuity of $y$

$$
y^j_{T-\tau} := \lim_{s \uparrow \tau} y^j_{T-s} = \lim_{s \downarrow \tau} y^j_{T-s} = y^j_{T-\tau}.
$$

Since by Lemma 2, $\hat{\chi}^c(j)$ is a bounded $\hat{\mathbf{P}}_{0, j}$-martingale for all $j \in I$, then

$$
0 = \mathbf{E}^c_{0, j}[F_{\hat{\tau} \wedge T} - F_0] = \sum_j \mathbf{E}^c_{0, j}[y^j_{T-\hat{\tau} \wedge T} 1_{(\xi_{T} = j)} \hat{\rho}_{0, \hat{\tau} \wedge T}^c] - y^j_{T}.
$$
By definition, if $\hat{t}\leq T$ and $\xi_{\hat{t}} = j$, then $T - \hat{t} \in C^j$ and therefore $y^j_{T-\hat{t}} = 0$. Then we obtain

$$y^j_T = \sum_j y^j_0 \hat{E}^c_{0,j}[1_{(\hat{t} > T)} 1_{(\xi_{\hat{t}} = j)} \hat{\rho}_{0,T}^c].$$

Notice now that $\hat{t}(\xi) > T$ if and only if $T - t \notin C^{\xi_t}$ for all $t \in [0, T]$, i.e. $t \notin C^{\xi_{\hat{t}}-t}$ for all $t \in [0, T]$. By the result of Lemma 3, $\hat{P}^c_{0,j}$-a.s. $t \notin C^{\xi_{\hat{t}}-t}$ for all $t \in [0, T]$ and only $t \notin C^{\xi_{\hat{t}}-t}(t)$ for all $t \in [0, T]$. Therefore,

$$1_{(\xi \in (\hat{t} > T))} = 1_{(\xi_{\hat{t}} \in (\hat{t} > T))}, \quad \hat{P}^c_{0,j} = \text{a.s.}$$

Therefore, by Lemma 4, for all $i, j \in I$

$$\hat{E}^c_{0,i}[1_{(\hat{t} > T)} 1_{(\xi_{\hat{t}} = j)} \hat{\rho}_{0,T}^c] = E^c_{0,j}[1_{(\hat{t} > T)} 1_{(\xi_{\hat{t}} = i)} \rho_{0,T}^c],$$

and (24) is proved. $\square$

Acknowledgements

L. Zambotti acknowledges the financial support provided through the European Community’s Human Potential Programme under the Contracts HPMF-CT-2002-01568 and HPRN-CT-2002-00281.

References