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Fourier Analysis and Limit Theorems for Convolution Semigroups on a Locally Compact Group

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INTRODUCTION

Let G be a locally compact group. We consider a uniformly infinitesimal triangular system $\mathfrak{I} = (\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ of probability distributions on G and its sequence $(\mu_n)_{n\geqslant 1}$ of row products $\mu_n := \mu_{n1} * \cdots * \mu_{nk_n}$. For $G = \mathbb{R}$ the investigation of the possible limit points of (μ_n) has been one of the most important and stimulating problems of classical probability theory [8]. The starting point for the present paper was the question of the limiting behaviour of (μ_n) on an arbitrary locally compact group.

In order to restrict the number of new difficulties appearing in this general situation it is convenient to consider only systems $\mathfrak{I}$ that are (rowwise) commutative. There are also some results for non-commutative systems (cf. [15, 6.6]). But up to now they have not been very satisfying. So we do not go into this and deal always with a commutative system $\mathfrak{I}$. This has the immediate consequence that the row products ν_n of the accompanying system $\mathfrak{I}_a = (\nu_{nk})$ (where ν_{nk} is the Poisson measure with exponent μ_{nk}) are again Poisson measures. Since it is therefore much easier to study the system $\mathfrak{I}_a$ instead of $\mathfrak{I}$ one has to look for conditions which yield the equality of the limit points of (μ_n) and (ν_n). In the classical case or more generally for an

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Abelian group G this can always be achieved by applying appropriate shifts to the μ_{nk}. But for a general group G this does not work. So we will have to impose an additional condition on our system $\mathfrak{I}$.

Instead of considering the sequence (ν_n) it is more effective to study the sequence of the associated Poissson semigroups S_n. Yet the (infinitesimal) generator of S_n can be expressed with the aid of the row sum $\mu_{n1} + \cdots + \mu_{nk_n}$ of $\mathfrak{I}$. So our task is to investigate the convergence of the sequence (S_n) in terms of its generators. But at this point we can gain more information if we pose the problem more generally. A Poisson semigroup is the prototype of a convolution semigroup (of probability measures) on G. Thus we shall consider a sequence (or a net) of convolution semigroups S_n on G together with their generators A_n. And we shall connect the convergence (in an appropriate sense) of the sequences (S_n) and (A_n). This is the subject of the second part of the present paper.

In the classical situation of the real line, limit theorems for the system $\mathfrak{I}$ are proved by applying Fourier transformation. It will turn out that this is also an appropriate method for locally compact groups. So at first we have to study the convolution semigroups and their generators in terms of their Fourier transforms. In contrast to a maximally almost periodic group G, where this problem has been solved completely [15, 1.5, 4.3; 24], for a general group G infinite dimensional irreducible representations come into play. This demands a refinement of the usual Fourier analysis on groups. Differentiable vectors for representations have to be considered. These preparations make up the first part of this paper.

Of course the convergence behaviour of commutative and infinitesimal systems $\mathfrak{I}$ has attracted most mathematicians interested in probabilities on groups. But almost all of them have restricted their attention to special classes of groups. So Urbanik [27] proved a central limit theorem on some compact Abelian groups. Some years later Parthasarathy [20] derived limit theorems on locally compact Abelian groups in complete analogy with the classical situation. Parthasarathy [19] and Heyer [14] treated the central limit problem on compact Lie groups, and Carnal [6] worked in the context of compact groups. For systems of probability measures on Lie groups rather far reaching results were obtained by Wehn [29]. The first step to general locally compact groups apparently has been taken by Grenander [9]. In [21] the author studied triangular systems on locally compact groups. The results specialized to maximally almost periodic groups are presented in [15, 6.5]. Finally Hazod [10] obtained some interesting results on totally disconnected groups.

Now we give a summary of this paper. It is divided into two parts. The first deals with the Fourier analysis of convolution semigroups. Though preparatory in its character it nevertheless contains results which are of some interest in itself. In Section 1 the concept of a differentiable vector for a representation is introduced. In Section 2 we assemble some well-known facts on Fourier transformation and convolution operators needed in the sequel.

Section 3 is concerned with the Fourier transforms of a convolution semigroup S. The Lévy–Chintschin formula of S is given in terms of the coefficient functions of group representations (Proposition 3.2). In Section 4 we prove that a convolution semigroup is uniquely determined by its infinitesimal generator restricted to the differentiable vectors of the irreducible group representations.

Finally Section 5 is very technical but at the same time very important for our further analysis. Lemmata 5.2 and 5.3 compare the coordinate functions an a Lie group with the differentiable coefficient functions of irreducible representations. An application of these lemmata yields Lemma 5.4, where we return to measures. This lemma will enable us to extend results from Lie groups to Lie projective groups.

Part II of this paper is concerned with our central subject: the convergence of convolution semigroups and its applications to triangular systems of probability measures. For this G will be assumed to be a Lie projective group (in most cases). In Section 6 we introduce first an appropriate convergence concept for convolution semigroups, namely, the uniform convergence on compact intervals of the parameter set (= real numbers $\geqslant 0$). Our main result here is Proposition 6.4: If a net $(S_\alpha)_{\alpha \in I}$ of convolution semigroups with generators A_α converges to a convolution semigroup S with generator A then the net $(A_\alpha)_{\alpha \in I}$ converges to A in a very precise manner.

In Section 7 we are concerned with the converse situation. Corollary 1 of Proposition 7.1 assures that $\lim A_\alpha = A$ is also sufficient for $\lim S_\alpha = S$. As a consequence we obtain compactness criteria for the space of convolution semigroups. These criteria are so sharp that they yield the compactness conditions of Parthasarathy for infinitely divisible probability measures on Abelian groups [20].

In Section 8 we study a commutative and infinitesimal system $\mathfrak{J}$ of probability distributions on G. Our first result is Proposition 8.1: If $\mathfrak{J}$ satisfies a certain boundedness condition (B) then the sequence (μ_n) of row products of $\mathfrak{J}$ converges if and only if the sequence (ν_n) of row products of its accompanying system $\mathfrak{J}_a$ converges. In the affirmative case these two limits coincide. By a counterexample it is shown that without condition (B) this result becomes incorrect even on the real line. Proposition 8.2 gives sufficient conditions for $\mathfrak{J}$ that the sequence (μ_n) converges to a probability measure embeddable into a convolution semigroup.

The final Section, Section 9, is devoted to the important central limit problem, i.e., the convergene of the sequence (μ_n) of row products of $\mathfrak{J}$ to a Gaussian measure. Our version of a central limit theorem is Proposition 9.3. Condition (G) figuring in it is classical (cf. [8, p. 126, Theorem 1]). Here we need one more condition (WB). But we obtain also a stronger result than in the classical case, namely, the existence of limit points for (μ_n). A special case of this appears as a law of large numbers (Proposition 9.4 and its corollary) which is concerned with the convergence of (μ_n) to degenerate measures.

The problem of the convergence to Poisson measures has been excluded here since in this case other methods become involved. We have studied it in [25].

Preliminaries

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are the sets of positive integers, integers, rational numbers, real numbers and complex numbers, resp. We define $\mathbb{R}_+ := \{r \in \mathbb{R} \mid r \geqslant 0\}$, $\mathbb{Q}_+ := \mathbb{R}_+ \cap \mathbb{Q}$, $\mathbb{R}_+^* := \{r \in \mathbb{R} \mid r > 0\}$, $\mathbb{Q}_+^* := \mathbb{R}_+^* \cap \mathbb{Q}$. For any ordered vector space $(V, \geqslant)$ let V_+ be the cone of its elements $\geqslant 0$.

Let E be a locally compact space. A^- denotes the closure of a subset A of E. If T is a mapping with domain E its restriction to A is denoted by $T \mid A$. By $\mathscr{C}^b(E)$ we denote the space of bounded continuous complex-valued functions on E equipped with the supremum norm $\|\cdot\|_\infty$. $\mathscr{K}(E)$ and $\mathscr{C}^0(E)$ are the subspaces of functions with compact support and of functions vanishing at infinity. The support of a function f on E is denoted by $\operatorname{supp}(f)$, and $\bar{f}$ is the complex conjugate function of f.

$\mathscr{M}(E)$ is the space of all real Radon measures on E, $\mathscr{M}^b(E)$ the subspace of bounded measures (equipped with the norm $\|\cdot\|$) and $\mathscr{M}^1(E)$ the subset of probability measures or distributions (i.e., positive measures μ such that $\mu(E) = 1$). If F is a locally compact subspace of E and $\mu \in \mathscr{M}(E)$ then $\mu \mid F$ denotes the restriction of μ to F. It is $\mu \mid F \in \mathscr{M}(F)$. The Dirac measure in $x \in E$ is denoted by ε_x. $\mu \in \mathscr{M}(E)$ is said to be degenerate if $\mu = \varepsilon_x$ for some $x \in E$. For $\mu \in \mathscr{M}(E)$ and appropriate functions f (on E) $f \cdot \mu$ denotes the measure with μ-density f. The image of $\mu \in \mathscr{M}^b(E)$ under a continuous mapping T with domain E is denoted by $T(\mu)$. The vague topology $\mathscr{T}_v$ in $\mathscr{M}(E)$ and the weak topology $\mathscr{T}_w$ in $\mathscr{M}^b(E)$ are defined as the topologies of simple convergence on $\mathscr{K}(E)$ and $\mathscr{C}^b(E)$, resp. A set M in $\mathscr{M}_+^b(E)$ is said to be uniformly tight if $\sup\{\|\mu\| \mid \mu \in M\} < \infty$ and if for every $\varepsilon > 0$ there exists a compact set K_ε in E such that $\mu(\complement K_\varepsilon) < \varepsilon$ for all $\mu \in M$. By Prohoroff's theorem M is uniformly tight iff M is relatively $\mathscr{T}_w$-compact.

By G we always denote a locally compact group. $\mathfrak{U}(G)$ is the system of all neighborhoods of the identity e in G that are Borel sets. Let $G^\times := G \backslash \{e\}$. If f is a function on G and $y \in G$ let ${}_yf, f_y, f^*, f^\sim$ be the functions defined by ${}_yf(x) := f(yx)$, $f_y(x) := f(xy)$, $f^*(x) := f(x^{-1})$, $f^\sim := \bar{f}^*$ resp. (all $x \in G$). $\mathscr{C}_u(G)$ is the subspace of $\mathscr{C}^b(G)$ of uniformly continuous functions with respect to the left uniform structure on G. If ψ is a homomorphism of G into some other group then $\ker(\psi)$ denotes its kernel. $\mathscr{M}_+^b(G)$ is a topological semigroup with respect to convolution $*$ and topology $\mathscr{E}_w$. If $\kappa \in \mathscr{M}^b_+(G)$ the *Poisson measure* $\exp(\kappa - \kappa(G)\varepsilon_e) \in \mathscr{M}^1(G)$ with exponent κ is defined by $\exp(\kappa - \kappa(G)\varepsilon_e) := e^{-\kappa(G)}(\varepsilon_e + \kappa + (1/2!)\kappa * \kappa + \cdots)$. For $\mu \in \mathscr{M}(G)$ (resp. $\mu \in \mathscr{M}(G^x)$) the adjoint measure $\tilde{\mu}$ is defined by $\tilde{\mu}(f) := \mu(f^*)$ ($f \in \mathscr{K}(G)$ resp. $f \in \mathscr{K}(G^*)$). μ is said to be symmetric if $\mu = \tilde{\mu}$. By ω or dx we denote a left Haar measure on G. $L^1(G)$ (resp. $L^2(G)$) is the Banach space (resp. Hilbert space) of integrable (resp. square integrable) complex-valued functions on G with respect to ω.

G is said to be a *Lie projective group* if there exists a family $\mathfrak{H}$ of compact normal subgroups in G descending to $\{e\}$ such that G/H is a Lie group for any $H \in \mathfrak{H}$. It is well known that any locally compact group G admits an open Lie projective subgroup.

If $\mathscr{H}$ is a complex Hilbert space and T a densely defined linear operator on $\mathscr{H}$ the adjoint operator of T exists and is denoted by T^*.

Finally some remarks on nets: A net in a set X is said to be *universal* iff for each subset A of X the net is eventually in A or eventually in $X \backslash A$. There is a universal subnet of each net in X. The image of a universal net is again a universal net [16, p. 81]. Let I be a non-void set directed by $>$. If ψ denotes the identity mapping on I then $(\psi, >)$ is a net (in I) with domain I. In this case we will also say that I is a net. Thus it should be also clear what we mean by a subnet of I.

Let $(x_\alpha)_{\alpha \in I}$ be a net in a topological Hausdorff space X. $(x_\alpha)_{\alpha \in I}$ is said to be a *compact net* if any of its universal subnets converges. Let X be a completely regular space and A a subset of X. Then A is relatively compact iff any net in A is a compact net or equivalently, iff any universal net in A converges (in X). In particular, a sequence $(x_n)_{n \geq 1}$ in X is a compact net iff the subset $\{x_n \mid n \in \mathbb{N}\}$ of X is relatively compact.

Now let $X := (\mathscr{M}^b_+(E), \mathscr{E}_w)$, where E is a locally compact space. A net $(\mu_\alpha)_{\alpha \in I}$ in X is said to be tight if $\overline{\lim}_\alpha \|\mu_\alpha\| < \infty$ and if for every $\varepsilon > 0$ there exists a compact set C_ε in E such that $\overline{\lim}_\alpha \mu_\alpha(\complement C_\varepsilon) < \varepsilon$. Then we have the following characterization: A net $(\mu_\alpha)_{\alpha \in I}$ in $(\mathscr{M}^b_+(E), \mathscr{E}_w)$ is a compact net if and only if it is a tight net [23, Lemmas 1.1 and 1.2].

I. Infinite Dimensional Fourier Transformation

1. *Unitary Representations and Differentiable Vectors*

Let G be a locally compact group. A continuous *unitary representation* of G is a homomorphism D of G into the group of unitary operators on a complex Hilbert space $\mathscr{H}$ such that the mapping $x \to D(x)u$ of G into $\mathscr{H}$ is continuous for all $u \in \mathscr{H}$. The space $\mathscr{H}$ is called the representation space of D and is denoted by $\mathscr{H}(D)$. The inner product and the norm in $\mathscr{H}(D)$ are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. If $u, v \in \mathscr{H}(D)$ then $\langle Du, v\rangle$ denotes the coefficient function $x \to \langle D(x)u, v\rangle$ of G into $\mathbb{C}$. Obviously we have $\langle Du, v\rangle \in \mathscr{C}_u(G)$.

The class of all continuous unitary representations of G is denoted by $\operatorname{Rep}(G)$. The direct sum and the tensor product of two representations $D_1, D_2 \in \operatorname{Rep}(G)$ are denoted by $D_1 \oplus D_2$ and $D_1 \otimes D_2$, resp. A representation $D \in \operatorname{Rep}(G)$ is said to be *irreducible* if the only closed subspaces of $\mathscr{H}(D)$ invariant under D are $\{0\}$ and $\mathscr{H}(D)$. By $\operatorname{Irr}(G)$ we denote the class of all irreducible representations in $\operatorname{Rep}(G)$.

Let $\mathscr{D}(G)$ be the space of infinitely differentiable complex-valued functions with compact support on G in the sense of Bruhat [5]. The space $\mathscr{E}(G)$ of bounded *regular functions* on G is defined by

$$\mathscr{E}(G) := \{f \in \mathscr{C}^b(G) \mid f \cdot g \in \mathscr{D}(G) \text{ for all } g \in \mathscr{D}(G)\}.$$

DEFINITION. Let $D \in \operatorname{Rep}(G)$. The vector $u \in \mathscr{H}(D)$ is said to be *differentiable* (for D) if the function $\langle Du, v\rangle$ is in $\mathscr{E}(G)$ for any $v \in \mathscr{H}(D)$.

By $\mathscr{H}_0(D)$ we denote the space of all vectors in $\mathscr{H}(D)$ differentiable for D. $\mathscr{H}_0(D)$ is invariant under D. (We have $\langle DD(x)u, v\rangle = (\langle Du, v\rangle)_x$ for all $x \in G$, and $\mathscr{E}(G)$ is invariant under right translations [5]). We want to show that $\mathscr{H}_0(D)$ is dense in $\mathscr{H}(D)$. We need a preparation:

For $D \in \operatorname{Rep}(G)$ and $f \in L^1(G)$ there is a bounded linear operator $D(f)$ on $\mathscr{H}(D)$ defined by

$$\langle D(f)u, v\rangle := \int f(x)\langle D(x)u, v\rangle\, dx \qquad (\text{all } v \in \mathscr{H}(D)).$$

Then $f \to D(f)$ is the (continuous) representation of the group algebra $L^1(G)$ associated with D [18, p. 381]. The linear space $\mathscr{H}_1(D)$ generated by $\{D(f)u \mid f \in \mathscr{D}(G),\ u \in \mathscr{H}(D)\}$ is called the *Garding space* of D.

LEMMA 1.1. (i) *$\mathscr{H}_1(D)$ is a subspace of $\mathscr{H}_0(D)$.*

(ii) *$\mathscr{H}_1(D)$ (and thus also $\mathscr{H}_0(D)$) is dense in $\mathscr{H}(D)$.*

(iii) *$D(x)D(f^*) = D((f_x)^*)$ for all $x \in G$ and $f \in \mathscr{D}(G)$.*

In particular, $\mathscr{H}_1(D)$ is invariant under D.

Proof. (i) By [3] and Lemma 3.1 we have $D(f)u \in \mathscr{H}_0(D)$ for all $f \in \mathscr{D}(G)$ and $u \in \mathscr{H}(D)$.

(ii) For any $U \in \mathfrak{U}(G)$ there exists a function $f_U \in \mathscr{D}_+(G)$ such that $\operatorname{supp}(f_U) \subset U$ and $\int f_U\, d\omega = 1$. Hence $\lim_U D(f_U)u = u$.

(iii) A simple calculation proves the formula. Since $\mathscr{D}(G)$ is invariant under right translations and inversion [5] the second assertion follows from it. ∎

EXAMPLES. (a) For any $f \in L^2(G)$ let $L(y^{-1})f := {}_yf$ (all $y \in G$). Each $L(y)$ is a unitary operator on $L^2(G)$, and L is a continuous unitary representation of G with representation space $L^2(G)$. It is called the *left regular representation* of G. We collect the following *properties of* L:

(1) $\mathscr{D}(G) \subset \mathscr{H}_0(L)$. (This is proved in [28, p. 252] for Lie groups. In the general case it follows along the same lines taking into account the definitions of $\mathscr{D}(G)$ and $\mathscr{E}(G)$.)

(2) We have $L(f)g = f * g$ for all $f \in L^1(G)$, $g \in L^2(G)$. Hence $f * g \in \mathscr{H}_0(L)$ for all $f \in \mathscr{D}(G)$, $g \in L^2(G)$ (Lemma 1.1).

(3) $\langle L(x)f, g\rangle = (f * g^{\sim})(x^{-1})$ for all $f, g \in L^2(G)$ and $x \in G$. Moreover $\operatorname{supp}(\langle Lf, g\rangle) \subset \operatorname{supp}(g)(\operatorname{supp}(f))^{-1}$.

(4) $\langle Lf, f\rangle = (f * f^{\sim})^*$ is a positive definite function in $\mathscr{C}^0(G)$ (all $f \in L^2(G)$) [12, Vol. II, (32.43 e)].

(b) For any $D \in \operatorname{Rep}(G)$ such that $\mathscr{H}(D)$ has finite dimension we have $\mathscr{H}_1(D) = \mathscr{H}_0(D) = \mathscr{H}(D)$. (This is an immediate consequence of Lemma 1.1.)

2. *Fourier Transforms and Convolution Operators*

Let G be a locally compact group. For a measure $\mu \in \mathscr{M}^b(G)$ we define its *Fourier transform* $\hat{\mu}$ by

$$\langle \hat{\mu}(D)u, v\rangle := \int \langle D(x)u, v\rangle \mu(dx)$$

for all $D \in \operatorname{Rep}(G)$ $(u, v \in \mathscr{H}(D))$. This Fourier transformation has the following *properties*:

1. $\hat{\mu}(D)$ is a bounded linear operator on $\mathscr{H}(D)$ such that $\|\hat{\mu}(D)\| \leqslant \|\mu\|$.

2. $\hat{\tilde{\mu}}(D) = \hat{\mu}(D)^*$.

3. (*Uniqueness theorem*) The correspondence $\mu \to \hat{\mu}$ is injective; i.e., $\hat{\mu}_1(D) = \hat{\mu}_2(D)$ for all *irreducible* representations, $D \in \operatorname{Irr}(G)$, implies $\mu_1 = \mu_2$.

4. $(a\mu_1 + b\mu_2)^{\wedge}(D) = a\hat{\mu}_1(D) + b\hat{\mu}_2(D)$ for $\mu_1, \mu_2 \in \mathscr{M}^b(G)$ and $a, b \in \mathbb{R}$ $(D \in \operatorname{Rep}(G))$.

5. $(\mu_1 * \mu_2)^{\wedge}(D) = \hat{\mu}_1(D)\hat{\mu}_2(D)$ $(D \in \operatorname{Rep}(G))$.

6. (*Continuity theorem*) The correspondence $\mu \to \hat{\mu}$ is *sequentially* continuous in the following sense. For any sequence $(\mu_n)_{n \geqslant 0}$ in $\mathscr{M}^b_+(G)$ the following assertions are equivalent:

(i) $\mathscr{T}_w\text{-}\lim \mu_n = \mu_0$.

(ii) $\lim \hat{\mu}_n(D)u = \hat{\mu}_0(D)u$ for all $D \in \operatorname{Irr}(G)$ and $u \in \mathscr{H}(D)$.

(iii) $\lim \langle \hat{\mu}_n(D)u, v\rangle = \langle \hat{\mu}_0(D)u, v\rangle$ for all $D \in \operatorname{Irr}(G)$ and $u, v \in \mathscr{H}(D)$.

(For the proofs of (1)–(5) see [13]. For the proof of (6) see [1, Proposition 6] or [26].)

For our purposes we need the following slightly stronger version of the continuity theorem:

LEMMA 2.1. *Let* $(\mu_n)_{n\geqslant 0}$ *be a sequence in* $\mathscr{M}_+^b(G)$ *such that* $\lim\langle\hat{\mu}_n(D)u, v\rangle = \langle\hat{\mu}_0(D)u, v\rangle$ *for all* $D\in \operatorname{Irr}(G)$ *and* $u, v\in\mathscr{H}_0(D)$. *Then we have* $\mathscr{E}_w\text{-}\lim\mu_n = \mu_0$.

Proof. We only have to verify condition (iii) of property (6). For any $x\in G$ let $D_0(x)$ be the identity operator on $\mathbb{C}$. Obviously $D_0\in\operatorname{Irr}(G)$ and $\|\mu_n\| = \mu_n(G) = \langle\hat{\mu}_n(D_0)1, 1\rangle$. Thus we have $\lim\|\mu_n\| = \|\mu_0\|$ by our assumption. By Lemma 1.1, $\mathscr{H}_0(D)$ is dense in $\mathscr{H}(D)$. Hence a simple estimation yields condition (iii). ∎

Let us say that the net $(\mu_\alpha)_{\alpha\in I}$ in $\mathscr{M}_+^b(G)$ $\mathscr{E}_f$*-converges* to $\mu\in\mathscr{M}_+^b(G)$ if we have $\lim_\alpha\langle\hat{\mu}_\alpha(D)u, v\rangle = \langle\hat{\mu}(D)u, v\rangle$ for all $D\in\operatorname{Irr}(G)$ and $u, v\in\mathscr{H}_0(D)$. In this case we will write $\mathscr{E}_f\text{-}\lim_\alpha\mu_\alpha = \mu$. Clearly $\mathscr{E}_w$-convergence implies $\mathscr{E}_f$-convergence whereas the converse implication holds, in general, only for sequences.

Sometimes we need a second type of transformation for measures. For $\mu\in\mathscr{M}_+^b(G)$ we define

$$T_\mu f(x) := \int f(xy)\,\mu(dy) \qquad (\text{all } f\in\mathscr{C}^b(G), x\in G).$$

We have $T_\mu f\in\mathscr{C}^b(G)$. T_μ is called the *convolution operator* of μ. It has *properties* similar to those of the Fourier transformation:

1. T_μ is a bounded linear operator on $\mathscr{C}^b(G)$ such that $\|T_\mu\| = \|\mu\|$.

2. $T_{\mu*\nu} = T_\mu T_\nu$ and $T_{a\mu+b\nu} = aT_\mu + bT_\nu$ for all $\mu, \nu\in\mathscr{M}_+^b(G)$ and $a, b\in\mathbb{R}_+$.

3. The correspondence $\mu\to T_\mu$ is continuous on $\mathscr{M}^1(G)$, i.e., for each net $(\mu_\alpha)_{\alpha\in I}$ in $\mathscr{M}^1(G)$ and each $\mu\in\mathscr{M}^1(G)$ the following assertions are equivalent:

(i) $\mathscr{E}_w\text{-}\lim_\alpha\mu_\alpha = \mu$.

(ii) $\lim_\alpha\|T_{\mu_\alpha}f - T_\mu f\|_\infty = 0$ for all $f\in\mathscr{C}^0(G)$.

(For the proofs see [15, 1.5.5] or [23, p. 440].)

Finally we have the following connections between the convolution operator and the Fourier transform: $\langle D\hat{\mu}(D)u, v\rangle = T_\mu(\langle Du, v\rangle)$ for all $D\in\operatorname{Rep}(G)(u, v\in\mathscr{H}(D))$, and $(T_\mu f)^* = \hat{\mu}(L)f^*$ for all $f\in\mathscr{K}(G)$.

3. *Fourier Transformation of Convolution Semigroups*

Let G be a locally compact group. A family $(\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ is said to be a *convolution semigroup* if we have $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geqslant 0$, and $\mathscr{T}_w$-$\lim_{t\downarrow 0} \mu_t = \mu_0 := \varepsilon_e$. Its *generating functional* $(A, \mathscr{A})$ is defined by

$$\mathscr{A} := \left\{ f \in \mathscr{C}^b(G) \mid A(f) := \lim_{t\downarrow 0} \frac{1}{t} (\mu_t(f) - f(e)) \text{ exists} \right\}.$$

We have $\mathscr{E}(G) \subset \mathscr{A}$, and on $\mathscr{E}(G)$ the functional A admits the *canonical decomposition* (Lévy–Chintschin formula)

$$A(f) = A_1(f) + A_2(f) + \int_{G^\times} [f - f(e) - \Gamma(f)]\, d\eta. \tag{LC}$$

Here A_1 is a primitive form, A_2 a quadratic form, Γ a Lévy mapping for G and η a Lévy measure for G [15, 4.5.9; 24]. We will also say that the generating functional A admits the canonical decomposition (A_1, A_2, η). The Lévy measure η is uniquely determined by the semigroup $(\mu_t)_{t>0}$; in fact we have [24, Lemma 1]

$$\int_{G^\times} d\eta = \lim_{t\downarrow 0} \frac{1}{t} \int_G f\, d\mu_t \quad \text{for all } f \in \mathscr{C}^b(G) \text{ with } e \notin \operatorname{supp}(f).$$

If G is a Lie group with a system $\{x_1, ..., x_p\}$ of canonical coordinates in $\mathscr{D}(G)$ adapted to the basis $\{X_1, ..., X_p\}$ of its Lie algebra $\mathscr{L}(G)$ then (LC) takes the more explicit form

$$\begin{aligned} A(f) = {} & \sum_{i=1}^{p} a_i(X_i f)(e) + \sum_{i,j=1}^{p} a_{ij}(X_i X_j f)(e) \\ & + \int_{G^\times} \left[f(x) - f(e) - \sum_{i=1}^{p} x_i(x)(X_i f)(e) \right] d\eta(x), \end{aligned} \tag{LLC}$$

where $a_1, ..., a_p$ are real numbers and $(a_{ij})_{1<i,j<p}$ is a real symmetric positive semidefinite matrix [15, 4.2.4; 24].

Sometimes we have to consider the adjoint convolution semigroup $(\tilde{\mu}_t)_{t>0}$. We denote its generating functional by $\tilde{A}$. Obviously $\tilde{A}$ admits the canonical decomposition $(-A_1, A_2, \tilde{\eta})$ (or $(-a_i, a_{ij}, \tilde{\eta})_{1<i,j<p}$ in the Lie group case).

EXAMPLE. For $\kappa \in \mathscr{M}^b_+(G)$ let $\mu_t := \exp(t(\kappa - \kappa(G)\varepsilon_e))$ be the Poisson measure with exponent $t\kappa$ (all $t \geqslant 0$). Then it can be easily seen that $(\mu_t)_{t>0}$ is a convolution semigroup with generating functional $(\kappa - \kappa(G)\varepsilon_e, \mathscr{C}^b(G))$; it is called a *Poisson semigroup*.

If $(\mu_t)_{t>0}$ is a convolution semigroup in $\mathscr{M}^1(G)$ the family $(T_{\mu_t})_{t>0}$ of convolution operators defines a strongly continuous semigroup of contractions on the Banach space $\mathscr{C}_u(G)$ [23, p. 441, proof of Proposition 5.1]) whose infinitesimal generator is denoted by $(N, \mathscr{N})$. We have $\mathscr{D}(G) \subset \mathscr{N}$ [15, 4.5.8] and $(Nf)(x) = A({}_xf)$ for all $x \in G$ and $f \in \mathscr{N}$.

Let $D \in \mathrm{Rep}(G)$. By properties 1, 5 and 6 of the Fourier transformation $(\hat{\mu}_t(D))_{t>0}$ is a strongly continuous semigroup of contractions on $\mathscr{H}(D)$. We denote its infinitesimal generator by $(A(D), \mathscr{A}(D))$.

PROPOSITION 3.1. *$\mathscr{A}(D) = \{u \in \mathscr{H}(D) \mid \langle Du, v\rangle \in \mathscr{A}$ for all $v \in \mathscr{H}(D)\}$ and $\langle A(D)u, v\rangle = A(\langle Du, v\rangle)$ for all $u \in \mathscr{A}(D)$ and $v \in \mathscr{H}(D)$.*

Proof. Let $u \in \mathscr{H}(D)$ such that $\langle Du, v\rangle \in \mathscr{A}$ for any $v \in \mathscr{H}(D)$. Let $h(v) := A(\langle Du, v\rangle)$ and $h_n(v) := n[\langle \hat{\mu}_{1/n}(D)u, v\rangle - \langle u, v\rangle]$ $(n \in \mathbb{N})$. Then we have $\lim h_n(v) = h(v)$ by the definitions of A and the Fourier transforms. Any $\bar{h}_n$ is a continuous linear functional on $\mathscr{H}(D)$. Since $\mathscr{H}(D)$ is barreled also $\bar{h}$ is a continuous linear functional on $\mathscr{H}(D)$ by the Banach–Steinhaus theorem. Hence there exists a vector $B(D)u \in \mathscr{H}(D)$ such that

$$\langle B(D)u, v\rangle = h(v) = \lim_{t \downarrow 0} (1/t)[\langle \hat{\mu}_t(D)u, v\rangle - \langle u, v\rangle]$$

(all $v \in \mathscr{H}(D)$). Moreover,

$$\begin{aligned}
&\lim_{s \downarrow 0} (1/s)[\langle \hat{\mu}_{s+t}(D)u, v\rangle - \langle \hat{\mu}_t(D)u, v\rangle] \\
&\quad = \lim_{s \downarrow 0} (1/s)[\langle \hat{\mu}_s(D)u, \hat{\mu}_t(D)^* v\rangle - \langle u, \hat{\mu}_t(D)^* v\rangle] \\
&\quad = \langle B(D)u, \hat{\mu}_t(D)^* v\rangle = \langle \hat{\mu}_t(D)\, B(D)u, v\rangle.
\end{aligned}$$

Therefore the function $t \to \langle \hat{\mu}_t(D)u, v\rangle$ of $\mathbb{R}_+$ into $\mathbb{C}$ has a continuous right derivative. This yields

$$\hat{\mu}_t(D)u - u = \int_0^t \hat{\mu}_s(D)\, B(D)u\, ds$$

[30, IX, proof of Theorem 3.2]. Thus $u \in \mathscr{A}(D)$, $B(D)u = A(D)u$ and $\langle A(D)u, v\rangle = h(v) = A(\langle Du, v\rangle)$.

Conversely for $u \in \mathscr{A}(D)$ we have

$$\begin{aligned}
\langle A(D)u, v\rangle &= \lim_{t \downarrow 0} (1/t)[\langle \hat{\mu}_t(D)u, v\rangle - \langle u, v\rangle] \\
&= \lim_{t \downarrow 0} (1/t)[\mu_t(\langle Du, v\rangle) - \langle u, v\rangle] \\
&= A(\langle Du, v\rangle),
\end{aligned}$$

i.e., $\langle Du, v\rangle \in \mathscr{A}$ (all $v \in \mathscr{H}(D)$). ∎

COROLLARY 1. *For $D \in \operatorname{Rep}(G)$ and $u \in \mathscr{A}(D)$, $v \in \mathscr{H}(D)$ we have*

$$\hat{\mu}_t(D)u - u = \int_0^t \hat{\mu}_s(D)\, A(D)u\, ds$$

and

$$\langle \hat{\mu}_t(D)u, v\rangle - \langle u, v\rangle = \int_0^t \int_G \langle D(y)\, A(D)u, v\rangle \mu_s(dy)\, ds.$$

COROLLARY 2. *For all $u \in \mathscr{A}(D)$ and $v \in \mathscr{H}(D)$ we have $\langle Du, v\rangle \in \mathscr{N}$, $N(\langle Du, v\rangle) = \langle DA(D)u, v\rangle$ and $\|N(\langle Du, v\rangle)\|_\infty \leqslant \|A(D)\, u\|\, \|v\|$.*

Proof.

$$\begin{aligned}
&\left| \langle D(x)\, A(D)u, v\rangle - \frac{1}{t}\, [(T_{\mu_t}\langle Du, v\rangle)(x) - (\langle Du, v\rangle)(x)] \right| \\
&\quad = \left| \langle D(x)\, A(D)u, v\rangle - \frac{1}{t} \int [\langle D(xy)u, v\rangle - \langle D(x)u, v\rangle]\, \mu_t(dy) \right| \\
&\quad = \left| \langle A(D)u, D(x^{-1})v\rangle - \left\langle \frac{1}{t} \int [D(y)u - u]\, \mu_t(dy), D(x^{-1})v \right\rangle \right| \\
&\quad \leqslant \left\| A(D)u - \frac{1}{t}\, [\hat{\mu}_t(D)u - u] \right\| \|v\| \qquad (\text{all } x \in G).
\end{aligned}$$

This proves our assertions. ∎

COROLLARY 3. $\mathscr{H}_0(D) \subset \mathscr{A}(D)$.

Proof. This follows immediately from the definition of $\mathscr{H}_0(D)$, $\mathscr{E}(G) \subset \mathscr{A}$ and Proposition 3.1. ∎

We give an application of Proposition 3.1 needed in the sequel. Let $\mathscr{L}(G)$ be the Lie algebra of G in the sense of Lashof [17] and exp the exponential mapping of $\mathscr{L}(G)$ into G. It is well known that $\mathscr{L}(G)$ is in one-to-one correspondence with the one-parameter subgroups in G. For every $X \in \mathscr{L}(G)$ and $f \in \mathscr{E}(G)$ the limit

$$(Xf)(x) = \lim_{t \to 0}\, (1/t)[f(x \exp tX) - f(x)]$$

exists for all $x \in G$, and $Xf \in \mathscr{E}(G)$ if Xf is bounded.

LEMMA 3.1. *Let $X \in \mathscr{L}(G)$ and $D \in \mathrm{Rep}(G)$. Then we have*

(i) *$X(D)u := \lim_{t \to 0} (1/t)[D(\exp tX) - D(e)]u$ exists for every $u \in \mathscr{H}_0(D)$, and $\langle DX(D)u, v\rangle = X(\langle Du, v\rangle)$ for all $v \in \mathscr{H}(D)$. In particular, $\mathscr{H}_0(D)$ is invariant under $X(D)$.*

(ii) *$X(D)$ is a linear operator on $\mathscr{H}_0(D)$, and $\sqrt{-1}\, X(D)$ is self-adjoint.*

(iii) *For every $f \in \mathscr{D}(G)$ we have $X(D)\, D(f^*) = D((Xf)^*)$. In particular $\mathscr{H}_1(D)$ is invariant under $X(D)$.*

Proof. (i) Let $x_t := \exp tX$ for all $t \in \mathbb{R}$. Then $(\varepsilon_{x_t})_{t>0}$ and $(\varepsilon_{x_{-t}})_{t>0}$ are convolution semigroups in $\mathscr{M}^1(G)$. Thus our assertions are special cases of Proposition 3.1 and its corollaries.

(ii) By (i), we have $\langle DX(D)u, v\rangle = X(\langle Du, v\rangle) \in \mathscr{E}(G)$ for all $v \in \mathscr{H}(D)$, and thus $X(D)u \in \mathscr{H}_0(D)$ $(u \in \mathscr{H}_0(D))$. Since the $D(x)$ are unitary operators we have $X(D)^* = -X(D)$, i.e., $\sqrt{-1}\, X(D)$ is self-adjoint.

(iii) Let $f \in \mathscr{D}(G)$ and $u \in \mathscr{H}(D)$. Since $f \in \mathscr{N}$ we have $\lim_{t \downarrow 0} (1/t)[f_{x_t} - f]^* = (Xf)^*$ uniformly on G and thus in the norm of $L^1(G)$ by Lebesgue's theorem (all functions have their supports in a fixed compact set). Thus by (i) and Lemma 1.1 we can conclude

$$X(D)\, D(f^*)u = \lim_{t \downarrow 0} \frac{1}{t}\, [D(x_t)\, D(f^*) - D(f^*)]u$$

$$= \lim_{t \downarrow 0} D\left(\left(\frac{1}{t}\, [f_{x_t} - f]\right)^*\right) u = D((Xf)^*)u. \quad \blacksquare$$

Remark. Let G be a Lie group and $D \in \mathrm{Rep}(G)$, $u \in \mathscr{H}(D)$. If the function $x \to D(x)u$ is weakly differentiable infinitely often (i.e., $u \in \mathscr{H}_0(D)$) then it is also strongly differentiable infinitely often. (This is immediate by Lemma 3.1; cf. [28, p. 484, Remark].)

PROPOSITION 3.2. *The canonical decomposition* (LC) *of the generating functional A of the convolution semigroup $(\mu_t)_{t>0}$ can be extended to all representations $D \in \mathrm{Rep}(G)$ and all differentiable vectors $u \in \mathscr{H}_0(D)$ in the following way*:

$$A(D)u = A_1(D)u + A_2(D)u + \int_{G^\times} [D(x) - D(e) - \Gamma(D)(x)]u\, d\eta(x).$$

Here $A_1(D)$, $A_2(D)$ and $\Gamma(D)(x)$ are linear mappings of $\mathscr{H}_0(D)$ into $\mathscr{H}(D)$ defined by

$$\langle A_j(D)u, v\rangle := A_j(\langle Du, v\rangle) \quad \text{and} \quad \langle \Gamma(D)(x)u, v\rangle := \Gamma(\langle Du, v\rangle)(x).$$

$\sqrt{-1}\,A_1(D)$, $A_2(D)$ and $\sqrt{-1}\,\Gamma(D)(x)$ are symmetric and closable. Moreover $A_2(D)$ is negative semidefinite.

Proof. Let B be a primitive or quadratic form on $\mathscr{E}(G)$. (A_1 and $\Gamma(\cdot)(x)$ are primitive forms, and A_2 is a quadratic form.) There exists a unique convolution semigroup $(\nu_t)_{t>0}$ in $\mathscr{M}^1(G)$ whose generating functional coincides with B on $\mathscr{E}(G)$ [15, 4.5.8; 24]. Hence by Proposition 3.1 and its Corollary 3 there is defined a linear mapping $B(D)$ of $\mathscr{H}_0(D)$ into $\mathscr{H}(D)$ by $\langle B(D)u, v\rangle = B(\langle Du, v\rangle)$. This yields the existence of $A_1(D)$, $A_2(D)$ and $\Gamma(D)(x)$. Moreover $B(D)$ is closable since it is the restriction to $\mathscr{H}_0(D)$ of the closed infinitesimal generator of the semigroup $(\hat{\nu}_t(D))_{t>0}$. Finally $\sqrt{-1}\,A_1(D)$, $\sqrt{-1}\,\Gamma(D)(x)$ and $A_2(D)$ are symmetric. [By definition, B is real, i.e., $B(\bar{f}) = \overline{B(f)}$ for all $f \in \mathscr{E}(G)$. If B is primitive (resp. quadratic) we have $B(f^*) = -B(f)$ (resp. $B(f^*) = B(f)$) [15, 4.4.7(3)]. For $D \in \operatorname{Rep}(G)$ and $u, v \in \mathscr{H}(D)$ we have $\langle Du, v\rangle^* = \langle u, Dv\rangle$. Combining these facts the statements can be proved easily.] Since A_2 is almost positive [15, 4.4.6] it is not difficult to prove that $A_2(D)$ is negative semidefinite.

Applying formula (LC) to the function $\langle Du, v\rangle$ ($u \in \mathscr{H}_0(D)$, $v \in \mathscr{H}(D)$) we get

$$\langle A(D)u, v\rangle = \langle A_1(D)u, v\rangle + \langle A_2(D)u, v\rangle + \int_{G^\times} \langle [D(x) - D(e) - \Gamma(D)(x)]u, v\rangle \, d\eta(x).$$

From this formula we conclude the existence of the vector $\int_{G^\times} [D(x) - D(e) - \Gamma(D)(x)]u \, d\eta(x)$ in $\mathscr{H}(D)$ and thus the desired decomposition. ∎

COROLLARY. *If G is a Lie group the canonical decomposition* (LLC) *extends to $D \in \operatorname{Rep}(G)$ and $u \in \mathscr{H}_0(D)$ in the following way*:

$$A(D)u = \sum_{i=1}^{p} a_i X_i(D)u + \sum_{i,j=1}^{p} a_{ij} X_i(D) X_j(D)u + \int_{G^\times} [D(x) - D(e) - \sum_{i=1}^{p} x_i(x)\, X_i(D)]u \, d\eta(x).$$

Proof. This follows immediately from (LLC) and Lemma 3.1 together with Proposition 3.1 and its Corollary 3. ∎

4. *The Differentiable Vectors Determine the Convolution Semigroup*

Let G be a locally compact group and $(\mu_t)_{t>0}$ a convolution semigroup in $\mathscr{M}^1(G)$ with generating functional $(A, \mathscr{A})$. We are going to show that $(\mu_t)_{t>0}$ is uniquely determined by the family $(A(D) \mid \mathscr{H}_1(D))_{D \in \operatorname{Irr}(G)}$. The crucial point of the proof is the following.

PROPOSITION 4.1. *Let $(N, \mathcal{N})$ be the infinitesimal generator of the associated semigroup $(T_{\mu_t})_{t>0}$ of convolution operators on $\mathscr{C}_u(G)$.*

If $f \in \mathscr{D}(G)$ we have $(Nf)^* \in L^1(G)$ and $A(D)\, D(f^*) = D((Nf)^*)$ for all $D \in \operatorname{Rep}(G)$.

Proof. The proof will be divided into several steps. Let $D \in \operatorname{Rep}(G)$ be fixed.

1. Let A be bounded; i.e., there exists a constant $c > 0$ such that $|A(f)| \leqslant c \|f\|_\infty$ for all $f \in \mathscr{A}$. Then we have $\mathscr{A} = \mathscr{C}^b(G)$ and $A(f) = \int [f - f(e)]\, d\kappa$ for some $\kappa \in \mathscr{M}_+^b(G)$ (all $f \in \mathscr{C}^b(G)$) [11, p. 32].

Let $f \in \mathscr{D}(G)$. Then $f^* \in L^1(G)$ and $\kappa * f^* \in L^1(G)$ [12, Vol. I, (20.12)]. Moreover

$$\begin{aligned}
&(\kappa * f^*)(x) - \kappa(G) f^*(x) \\
&= \int [f^*(y^{-1}x) - f^*(x)]\, d\kappa(y) \\
&= \int [f(x^{-1}y) - f(x^{-1})]\, d\kappa(y) = A({}_{x^{-1}}f) \\
&= (Nf)(x^{-1}) = (Nf)^*(x).
\end{aligned}$$

Thus we have $(Nf)^* \in L^1(G)$. Let $u \in \mathscr{H}(D)$. Applying Fubini's theorem (κ is bounded) we get

$$\begin{aligned}
A(D)\, D(f^*)u &= \int [D(y) - D(e)]\, D(f^*)u\, d\kappa(y) \\
&= \int [\kappa * f^*(x) - \kappa(G) f^*(x)]\, D(x)u\, dx \\
&= \int (Nf)^*(x)\, D(x)u\, dx = D((Nf)^*)u.
\end{aligned}$$

2. Let G_1 be an open Lie projective subgroup of G and $\mathscr{L}(G)$ the Lie algebra of G. We assume that A admits the canonical decomposition (A_1, A_2, η), where the support of η is contained in a compact neighborhood $U \in \mathfrak{U}(G)$. We fix $f \in \mathscr{D}(G)$. Then there exists a compact normal subgroup H in G_1 such that G_1/H is a Lie group and f lies in the space $\mathscr{E}_H(G)$ of all functions in $\mathscr{E}(G)$ that are constant on the right cosets modulo H (definition of $\mathscr{D}(G)$). An easy calculation shows (together with Lemma 1.1) that $\langle DD(f^*)u, v\rangle \in \mathscr{E}_H(G)$ for all $u, v \in \mathscr{H}(D)$.

(a) There exist vectors $X_1, \ldots, X_p \in \mathscr{L}(G)$, functions $x_1, \ldots, x_p \in \mathscr{D}(G)$ and real numbers a_i, a_{ij} $(1 \leqslant i, j \leqslant p)$ such that

$$A_1(g) = \sum_{i=1}^{p} a_i(X_i g)(e), \qquad A_2(g) = \sum_{i,j=1}^{p} a_{ij}(X_i X_j g)(e)$$

and

$$A_\eta(g) := A(g) - A_1(g) - A_2(g) = \int_{G^x} \left[g - g(e) - \sum_{i=1}^{p} x_i(X_i g)(e) \right] d\eta$$

for all $g \in \mathscr{E}_H(G)$ [22, p. 328].

With the aid of Lemma 3.1 we get

$$\begin{aligned}
&\langle (A_1 + A_2)(D)\, D(f^*)u, v \rangle \\
&\quad = (A_1 + A_2)(\langle DD(f^*)u, v \rangle) \\
&\quad = \sum_{i=1}^{p} a_i X_i(\langle DD(f^*)u, v \rangle)(e) \\
&\qquad + \sum_{i,j=1}^{p} a_{ij} X_i X_j(\langle DD(f^*)u, v \rangle)(e) \\
&\quad = \sum_{i=1}^{p} a_i \langle X_i(D)\, D(f^*)u, v \rangle \\
&\qquad + \sum_{i,j=1}^{p} a_{ij} \langle X_i(D)\, X_j(D)\, D(f^*)u, v \rangle \\
&\quad = \left\langle D\left(\left(\sum_{i=1}^{p} a_i X_i f + \sum_{i,j=1}^{p} a_{ij} X_i X_j f\right)^*\right) u, v \right\rangle \qquad (u, v \in \mathscr{H}(D)).
\end{aligned}$$

(b) Let $h(x, y) := f(yx) - f(y) - \sum_{i=1}^{p} x_i(x)(X_i f)(y)$ (all $x, y \in G$). Obviously there exists a constant $d > 0$ such that $|h(x, y)| \leqslant d$ for all $x, y \in G$. Moreover we have

$$h(x,y) = {}_y f(x) - {}_y f(e) - \sum_{i=1}^{p} x_i(x)(X_i({}_y f))(e).$$

Lifting the Taylor formula from G_1/H to G_1 there exists a neighbourhood $U_0 \in \mathfrak{U}(G)$, $U_0 \subset U$, such that

$$\begin{aligned}
|h(x, y)| &\leqslant \frac{1}{2} \psi(x) \sum_{i,j=1}^{p} |(X_i X_j({}_y f))(\xi(x, y))| \\
&= \frac{1}{2} \psi(x) \sum_{i,j=1}^{p} |(X_i X_j f)(y\xi(x, y))|
\end{aligned}$$

for $x \in U_0$, $y \in G$, where $\xi(x, y) \in U_0$ and $\psi = \sum_{i=1}^{p} x_i^2$. Thus there exists a constant $c > 0$ such that $|h(x, y)| \leqslant c\psi(x)$ for all $x \in U_0$, $y \in G$.

If $C := \operatorname{supp}(f)$ then $h(x, y) = 0$ for $x \in U$ and $y \notin CU^{-1}$. Since η has its support in U and is a Lévy measure we conclude

$$\int_{G^x} \int_G |h(x, y^{-1})| \, d\omega(y) \, d\eta(x)$$

$$\leqslant \omega(CU^{-1}) \left[c \int_{U_0 \setminus \{e\}} \psi(x) \, d\eta(x) + d \cdot \eta(U \setminus U_0) \right] < \infty.$$

Thus the application of Fubini's theorem in the following calculation is justified; and we have $(\int_{G^x} h(x, \cdot\,) d\eta(x))^* \in L^1(G)$.

$$\begin{aligned}
&\langle A_\eta(D) \, D(f^*)u, v\rangle \\
&\quad = A_\eta(\langle DD(f^*)u, v\rangle) \\
&\quad = \int_{G^x} \left[\langle DD(f^*)u, v\rangle - \langle D(f^*)u, v\rangle \right. \\
&\qquad \left. - \sum_{i=1}^p x_i(X_i(\langle DD(f^*)u, v\rangle))(e) \right] d\eta \\
&\quad = \int_{G^x} \left\langle \left[D((f_x)^*) - D(f^*) - \sum_{i=1}^p x_i(x) \, D((X_i f)^*) \right] u, v \right\rangle d\eta(x) \\
&\quad = \int_{G^x} \left\langle D\left(\left[f_x - f - \sum_{i=1}^p x_i(x) X_i f \right]^* \right) u, v \right\rangle d\eta(x) \\
&\quad = \int_{G^x} \langle D(h(x, \cdot\,)^*)u, v\rangle d\eta(x) \\
&\quad = \int_{G^x} \left\{ \int_G h(x, y^{-1}) \langle D(y)u, v\rangle dy \right\} d\eta(x) \\
&\quad = \int_G \left\{ \int_{G^x} h(x, y^{-1}) d\eta(x) \right\} \langle D(y)u, v\rangle dy \\
&\quad = \int_G \left(\int_{G^x} h(x, \cdot\,) d\eta(x) \right)^* \langle D(y)u, v\rangle dy \\
&\quad = \left\langle D\left(\left(\int_{G^x} h(x, \cdot\,) d\eta(x) \right)^* \right) u, v \right\rangle \qquad (u, v \in \mathscr{H}(D)).
\end{aligned}$$

(c) Since $(Nf)(y) = A({}_y f)$ and ${}_y f \in \mathscr{E}_H(G)$ for all $y \in G$, we have

$$Nf = \sum_{i=1}^p a_i X_i f + \sum_{i,j=1}^p a_{ij} X_i X_j f + \int_{G^x} h(x, \cdot\,) d\eta(x).$$

Combining (a) and (b) we thus get $(Nf)^* \in L^1(G)$ and

$$\begin{aligned} A(D)\,D(f^*)u &= (A_1 + A_2)(D)\,D(f^*)u + A_\eta(D)\,D(f^*)u \\ &= D((Nf)^*)u. \end{aligned}$$

3. Now let A be arbitrary. It is an easy consequence of the canonical decomposition (LC) that A can be written as a sum of a bounded generating functional and a generating functional as in 2 (cf. [11, p. 34]). Combining the results in 1 and 2 we arrive at our assertion. ∎

Remark. The formula $A(D)\,D(f^*) = D((Nf)^*)$ in Proposition 4.1 has been stated without proof in [7]. For an Abelian group G and for $D \in \operatorname{Irr}(G)$ it is proved in [2, Theorem 12.16].

PROPOSITION 4.2. *Let $(A, \mathscr{A})$ be the generating functional of the convolution semigroup $(\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$. Then $(\mu_t)_{t>0}$ is uniquely determined by the family $(A(D) \mid \mathscr{H}_1(D))_{D \in \operatorname{Irr}(G)}$.*

Proof. Let $(\nu_t)_{t>0}$ be a second convolution semigroup in $\mathscr{M}^1(G)$ with the generating functional $(B, \mathscr{B})$ and the infinitesimal generator $(M, \mathscr{M})$ for $(T_{\nu_t})_{t>0}$ such that $A(D) \mid \mathscr{H}_1(D) = B(D) \mid \mathscr{H}_1(D)$ for all $D \in \operatorname{Irr}(G)$.

For $f \in \mathscr{D}(G)$ we have $D(f^*)u \in \mathscr{H}_1(D)$ and thus $D((Nf)^*)u = A(D)\,D(f^*)u = B(D)\,D(f^*)u = D((Mf)^*)u$ for all $u \in \mathscr{H}(D)$ (Proposition 4.1). Hence $D((Nf)^* - (Mf)^*) = 0$ for all $D \in \operatorname{Irr}(G)$. This yields $(Nf)^* = (Mf)^*$ [18, p. 271]. In particular, we get $A \mid \mathscr{D}(G) = B \mid \mathscr{D}(G)$. But this implies $\mu_t = \nu_t$ for all $t \geqslant 0$ [15, 4.5.6]. ∎

We shall give a simple application of Proposition 4.2 that will be needed later on. For this let us call a convolution semigroup $(\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ *degenerate* (resp. *trivial*) if any μ_t is degenerate (resp. if $\mu_t = \varepsilon_e$ for all $t \geqslant 0$).

PROPOSITION 4.3. *Let $S = (\mu_t)_{t>0}$ be a convolution semigroup in $\mathscr{M}^1(G)$ with generating functional A.*

(i) *S is a degenerate semigroup iff $\operatorname{Re}\langle A(D)u, u\rangle = 0$ for all $D \in \operatorname{Irr}(G)$ and $u \in \mathscr{H}_0(D)$.*

(ii) *S is the trivial semigroup iff $\langle A(D)u, u\rangle = 0$ for all $D \in \operatorname{Irr}(G)$ and $u \in \mathscr{H}_0(D)$.*

Proof. (i) Let (A_1, A_2, η) be the canonical decomposition of A. It is easy to see that S is degenerate iff $A_2 = 0$ and $\eta = 0$. Let S be degenerate. Then $A = A_1$. By Proposition 3.2 we have $A_1(D)^* = -A_1(D)$ and thus $(*)$ $\operatorname{Re}\langle A(D)u, u\rangle = 0$ $(D \in \operatorname{Irr}(G),\ u \in \mathscr{H}_0(D))$.

Conversely let $(*)$ be satisfied. Again by Proposition 3.2 we have $0 = \operatorname{Re}\langle A(D)u, u\rangle = \langle A_2(D)u, u\rangle + \int_{G^\times} [\operatorname{Re}\langle Du, u\rangle - \langle u, u\rangle]\, d\eta$, and the last two terms are not positive, hence zero. From $\langle A_2(D)u, u\rangle = 0$ for all $u \in \mathscr{H}_0(D)$ we obtain $A_2(D) = 0$ since $A_2(D)$ is symmetric. From $\int_{G^\times} [\mathrm{Re}\langle Du, u\rangle - \langle u, u\rangle]\, d\eta = 0$ for all $D \in \mathrm{Irr}(G)$, $u \in \mathscr{H}_0(D)$, we obtain $\eta = 0$ since the only common zero of the non-negative functions $(\langle u, u\rangle - \mathrm{Re}\langle Du, u\rangle)$ is the identity e of G (cf. the proof of Lemma 5.2). Consequently $A(D) \mid \mathscr{H}_0(D) = A_1(D)$ for all $D \in \mathrm{Irr}(G)$. Since A_1 is the generating functional of a degenerate semigroup the semigroup $(\mu_t)_{t>0}$ is degenerate by Proposition 4.2.

(ii) S is trivial iff $A = 0$. Thus in view of Proposition 4.2 we only have to show that $\langle A(D)u, u\rangle = 0$ for all $u \in \mathscr{H}_0(D)$ implies $A(D) \mid \mathscr{H}_0(D) = 0$. But by (i) A is a primitive form. Hence $\sqrt{-1}A(D)$ is symmetric (Proposition 3.2). This proves our statement. ∎

5. *Differentiable Vectors on Lie Groups*

Let G be a Lie group of dimension $p \geqslant 1$. We choose a basis $\{X_1, \ldots, X_p\}$ of its Lie algebra $\mathscr{L}(G)$ and a system $\{x_1, \ldots, x_p\}$ of canonical coordinates (in $\mathscr{D}(G)$) adapted to this basis and valid in the neighborhood $U_0 \in \mathfrak{U}(G)$. Finally let φ be a Hunt function for G; i.e., φ is a symmetric function in $\mathscr{E}_+(G)$ bounded away from zero on $G \backslash U$ for any $U \in \mathfrak{U}(G)$, and $\varphi(x) = x_1(x)^2 + \cdots + x_p(x)^2$ for all $x \in U_0$ [15, p. 260].

LEMMA 5.1. *Let $D \in \mathrm{Rep}(G)$ and $u \in \mathscr{H}_0(D)$.*

(i) *For the mapping $x \to D(x)u$ the* Taylor formula

$$D(x)u = u + \sum_{i=1}^{p} x_i(x)\, X_i(D)u + \frac{1}{2} \sum_{i,j=1}^{p} x_i(x) x_j(x)\, T(D)(x)\, X_i(D) X_j(D)u$$

is valid for all $x \in U_0$. Here each $T(D)(x)$ is a bounded linear operator on $\mathscr{H}(D)$ such that $\| T(D)(x)\| \leqslant 1$.

(ii) *The following estimation holds for all $x \in U_0$:*

$$\Big\| D(x)u - u - \sum_{i=1}^{p} x_i(x)\, X_i(D)u \Big\| \leqslant \frac{1}{2}\varphi(x) \sum_{i,j=1}^{p} \| X_i(D)\, X_j(D)u\|.$$

Proof. (i) Let $x \in U_0$ be fixed and define $X := \sum_{i=1}^{p} x_i(x) X_i$. The mapping f of $\mathbb{R}$ into $\mathscr{H}(D)$, defined by $f(t) := D(\exp tX)u$ is infinitely differentiable (cf. the remark after Lemma 3.1). We have $f'(t) =$

$D(\exp tX)X(D)u$ and $f''(t) = D(\exp tX)\,X(D)\,X(D)u$. The Taylor formula for vector-valued functions yields

$$f(t) = f(0) + tf'(0) + t^2 \int_0^1 (1-s) f''(st)\,ds \qquad (\text{all } t \in \mathbb{R}).$$

In particular, we get for $t = 1$,

$$D(x)u = u + X(D)u + \int_0^1 (1-s)\, D(\exp sX)\, X(D)\, X(D)u\, ds.$$

Since $[0, 1]$ is compact and $\mathscr{H}(D)$ is complete $T(D)(x)v := 2\int_0^1 (1-s) D(\exp sX) v\, ds$ is an element of $\mathscr{H}(D)$ for any $v \in \mathscr{H}(D)$ [4, Chap. III, Sect. 3,3]. Obviously $T(D)(x)$ is a linear operator on $\mathscr{H}(D)$ such that $\| T(D)(x) \| \leqslant 1$. Finally we have $X(D) = \sum_{i=1}^p x_i(x) X_i(D)$. Hence our first statement is proved.

(ii) This follows directly from (i) since we have $|x_i(x) x_j(x)| \leqslant x_1(x)^2 + \cdots + x_p(x)^2 = \varphi(x)$ for all $i, j = 1, \ldots, p$ and $x \in U_0$. ∎

We are now going to prove two lemmata which will enable us to compare the coordinate functions $x_1, \ldots, x_p$ and the Hunt function φ with the coefficient functions $\langle Du, v \rangle$ for representations $D \in \operatorname{Irr}(G)$ and vectors $u, v \in \mathscr{H}_0(D)$.

LEMMA 5.2. *There exist a neighborhood $U \in \mathfrak{U}(G)$, a constant $c > 0$, representations $D_1, \ldots, D_n \in \operatorname{Irr}(G)$ and vectors $u_j \in \mathscr{H}_0(D_j)$ $(1 \leqslant j \leqslant n)$ such that*

$$\varphi(x) \leqslant c \sum_{j=1}^{n} \operatorname{Re}[\langle u_j, u_j \rangle - \langle D_j(x) u_j, u_j \rangle] \quad \textit{for all } x \in U.$$

Proof. For any $D \in \operatorname{Irr}(G)$ and $u \in \mathscr{H}_0(D)$ we define

$$H_{D,u} := \{x \in G \mid \langle D(x)u, u \rangle = \langle u, u \rangle\}.$$

We have $x \in H_{D,u}$ iff $D(x)u = u$. [$D(x)u = u$ obviously implies $x \in H_{D,u}$. Conversely let $x \in H_{D,u}$ $(u \neq 0)$. Then $\langle D(x)u, u \rangle = \|u\|^2 = \|D(x)u\|\,\|u\|$. Schwarz' inequality implies $D(x)u = du$ for some $d \in \mathbb{C}$. But from $d\langle u, u \rangle = \langle D(x)u, u \rangle = \langle u, u \rangle$ we conclude that $d = 1$.] Thus any $H_{D,u}$ is a closed subgroup in G. We have $\bigcap_{u \in \mathscr{H}_0(D)} H_{D,u} = \ker(D)$. [Obviously $\ker(D)$ is contained in this intersection. Conversely, $x \in H_{D,u}$ for all $u \in \mathscr{H}_0(D)$ implies

$D(x)u = u$ for all $u \in \mathscr{H}(D)$ by Lemma 1.1. Therefore $D(x) = D(e)$, i.e., $x \in \ker(D)$.] This gives us

$$\bigcap_{D \in \mathrm{Irr}(G), u \in \mathscr{H}_0(D)} H_{D,u} = \bigcap_{D \in \mathrm{Irr}(G)} \ker(D) = \{e\} \qquad (*)$$

[12, Vol. I, (22.12)].

We choose a symmetric and open neighborhood $U \in \mathfrak{U}(G)$ such that U has compact closure and contains no proper subgroup of G (G is a Lie group!). Then $C := (U^2)^- \setminus U$ is compact and $e \notin C$. Thus by (*) there exist $D_1, \ldots, D_n \in \mathrm{Irr}(G)$ and $u_j \in \mathscr{H}_0(D_j)$ $(1 \leqslant j \leqslant n)$ such that $H \cap C = \varnothing$, where $H := \bigcap_{1 \leqslant j \leqslant n} H_{D_j, u_j}$. Hence $K := H \cap U$ is a subgroup of G. [For $x, y \in K$ we have $xy^{-1} \in H \cap (U^2)^-$. But $H \cap (U^2)^- = H \cap (C \cup U) = (H \cap C) \cup (H \cap U) = H \cap U = K$.] From $K \subset U$ we conclude $K = \{e\}$.

Let $f(x) := \sum_{j=1}^n \mathrm{Re}[\langle u_j, u_j \rangle - \langle D_j(x) u_j, u_j \rangle]$ for all $x \in G$. Obviously we have $f \geqslant 0$. Let $x \in U$ such that $f(x) = 0$. Then $\|u_j\|^2 = \mathrm{Re}\langle D_j(x) u_j, u_j \rangle$ and consequently $\|u_j\|^2 = \langle D_j(x) u_j, u_j \rangle$ for $j = 1, \ldots, n$. Therefore $x \in H$. But $H \cap U = \{e\}$ implies $x = e$. Thus f has a strict local minimum in e. Moreover $(XXf)(e) \neq 0$ for all $X \in \mathscr{L}(G)$ as can be easily seen. Since φ has the same property our assertion follows. ∎

LEMMA 5.3. *There exist representations* $D'_1, \ldots, D'_m \in \mathrm{Irr}(G), D_0 \in \mathrm{Rep}(G)$ *and vectors* $v, w_1, \ldots, w_p \in \mathscr{H}_0(D_0)$ *such that* $D_0 = D'_1 \oplus \cdots \oplus D'_m$ *and* $\langle X_i(D_0) v, w_j \rangle = \delta_{ij}$ *for* $1 \leqslant i, j \leqslant p$.

Proof. We keep the notations of Lemma 5.2 and define $D := D_1 \oplus \cdots \oplus D_n$ and $u := u_1 \oplus \cdots \oplus u_n$. Then $D(x) u \neq u$ for all $x \in U \setminus \{e\}$. Since $u \in \mathscr{H}_0(D)$ we can define $v_i := X_i(D) u$ $(1 \leqslant i \leqslant p)$. Thus the Taylor formula from Lemma 5.1 takes the form

$$D(x)u = u + \left(\sum_{i=1}^p x_i(x) v_i \right) + \frac{1}{2} T(D)(x) \sum_{j=1}^p (x_j(x) X_j(D)) \left(\sum_{i=1}^p x_i(x) v_i \right)$$

for all $x \in U$ (without loss of generality $U \subset U_0$). Thus $\sum_{i=1}^p x_i(x) v_i \neq 0$ for all $x \in U \setminus \{e\}$. Since U is a coordinate neighborhood of e this yields the linear indepedence over $\mathbb{R}$ of the vectors $v_1, \ldots, v_p$. Since $v_i = X_i(D) u$ the linear operators $X_1(D), \ldots, X_p(D)$ on $\mathscr{H}_0(D)$ are linearly independent over $\mathbb{R}$ too. But we can even show that they are linearly independent over $\mathbb{C}$. [We have $X_j(D)^* = -X_j(D)$ (Lemma 3.1). Thus $\sum a_j X_j(D) = 0$ $(a_1, \ldots, a_p \in \mathbb{C})$ implies $0 = (\sum a_j X_j(D))^* = \sum \overline{a_j} X_j(D)^* = -\sum \overline{a_j} X_j(D)$.]

We need the following simple

LEMMA. *Let $A_1,\dots,A_p$ be linearly independent linear mappings of a vector space $\mathscr{V}$ over $\mathbb{C}$ into itself. Then there exist elements $z_1,\dots,z_l \in \mathscr{V}$ $(l \leqslant p)$ such that the vectors $(A_1 z_j)_{1<j<l},\dots,(A_p z_j)_{1<j<l}$ of $\mathscr{V}^l$ are linearly independent* (*over* $\mathbb{C}$).

(The proof is carried out by induction on p. Compare with [12, Vol. II, (28.14)].)

Applying this lemma to $X_1(D),\dots,X_p(D)$ and $\mathscr{H}_0(D)$ we obtain vectors $v_1',\dots,v_l' \in \mathscr{H}_0(D)$ such that $(X_1(D)v_j')_{1<j<l},\dots,(X_p(D)v_j')_{1<j<l}$ are linearly independent over $\mathbb{C}$. Let D_0 be the direct sum of l copies of D and $v := v_1' \oplus \cdots \oplus v_l' \in \mathscr{H}_0(D_0)$. Then $X_1(D_0)v,\dots,X_p(D_0)v$ are linearly independent over $\mathbb{C}$. Let $\mathscr{W}$ be the (finite-dimensional) subspace of $\mathscr{H}_0(D_0)$ generated by these vectors (cf. Lemma 3.1(ii)). By $L_j(\sum a_i X_i(D_0)v) := a_j$ for all $a_1,\dots,a_p \in \mathbb{C}$ there is given a continuous linear functional L_j on $\mathscr{W}$ $(1 \leqslant j \leqslant p)$. Since $\mathscr{W}$ is a Hilbert space (as a closed subspace of $\mathscr{H}(D_0)$) there exist vectors $w_1,\dots,w_p$ in $\mathscr{W}$ (and thus in $\mathscr{H}_0(D_0)$) such that $L_j w = \langle w, w_j\rangle$ for all $w \in \mathscr{W}$. By construction we have $\langle X_i(D_0)v, w_j\rangle = \delta_{ij}$ $(1 \leqslant i, j \leqslant p)$. Obviously D_0 is a direct sum of finitely many irreducible representations. ∎

Remark 1. If G is a maximally almost periodic Lie group (i.e., the finite-dimensional representations separate the points of G) then Lemmata 5.2 and 5.3 remain true if they are restricted to finite-dimensional representations (cf. [15, 4.3.6, 4.3.7; 24]).

Remark 2. Let $\mathscr{H}$ be a Hilbert space over $\mathbb{C}$ with scalar product $\langle\cdot,\cdot\rangle$ and let T be a linear operator on $\mathscr{H}$. Then we have for all $u, v \in \mathscr{H}$ the following identity, which we will have to use several times:

$$\langle Tu, v\rangle = \tfrac{1}{2}[\langle T(u+v), u+v\rangle + i\langle T(u+iv), u+iv\rangle]$$
$$-\frac{1+i}{2}[\langle Tu, u\rangle + \langle Tv, v\rangle].$$

LEMMA 5.4. *Let $(\kappa_\alpha)_{\alpha\in I}$ be a net in $\mathscr{M}_+^b(G)$. Then the following assertions are equivalent*:

(i) $\overline{\lim}_\alpha \|\hat{\kappa}_\alpha(D)u - \kappa_\alpha(G)u\| < \infty$ *for all* $D \in \mathrm{Irr}(G)$, $u \in \mathscr{H}_0(D)$, *and* $\overline{\lim}_\alpha \kappa_\alpha(\complement U) < \infty$ *for all* $U \in \mathfrak{U}(G)$.

(ii) $\overline{\lim}_\alpha |\langle \hat{\kappa}_\alpha(D)u - \kappa_\alpha(G)u, u\rangle| < \infty$ *for all* $D \in \mathrm{Irr}(G)$, $u \in \mathscr{H}_0(D)$, and $\overline{\lim}_\alpha \kappa_\alpha(\complement U) < \infty$ *for all* $U \in \mathfrak{U}(G)$.

(iii) $\overline{\lim}_\alpha (\int \varphi\, d\kappa_\alpha) < \infty$ *and* $\overline{\lim}_\alpha (|\int x_i\, d\kappa_\alpha|) < \infty$ *for* $i = 1,\dots,p$.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) 1. We keep the notations of Lemma 5.2. Then we have the following estimation:

$$\begin{aligned}
\int \varphi\, d\kappa_\alpha &= \int_U \varphi\, d\kappa_\alpha + \int_{\complement U} \varphi\, d\kappa_\alpha \\
&\leqslant c \sum \operatorname{Re} \int \langle u_j - D_j(x) u_j, u_j\rangle \kappa_\alpha(dx) + \|\varphi\|_\infty \kappa_\alpha\left(\complement U\right) \\
&= c \sum \operatorname{Re}\langle \kappa_\alpha(G) u_j - \hat{\kappa}_\alpha(D_j) u_j, u_j\rangle + \|\varphi\|_\infty \kappa_\alpha\left(\complement U\right) \\
&\leqslant c \sum |\langle \kappa_\alpha(G) u_j - \hat{\kappa}_\alpha(D_j) u_j, u_j\rangle| + \|\varphi\|_\infty \kappa_\alpha\left(\complement U\right).
\end{aligned}$$

(One observes $\operatorname{Re}\langle u_j - D_j u_j, u_j\rangle \geqslant 0$.) Our assumptions yield $\overline{\lim}_\alpha \left(\int \varphi\, d\kappa_\alpha\right) < \infty$.

2. We keep the notations of Lemma 5.3. The Taylor formula (Lemma 5.1) yields for all $x \in U_0$,

$$\begin{aligned}
x_k(x) &= \left\langle \sum_i x_i(x) X_i(D_0) v, w_k \right\rangle \\
&= \langle D_0(x) v - v, w_k\rangle \\
&\quad - \frac{1}{2} \sum_{i,j} x_i(x) x_j(x) \langle T(D_0)(x) X_i(D_0) X_j(D_0) v, w_k\rangle. \qquad (*)
\end{aligned}$$

Applying the identity of Remark 2 to $T = \hat{\kappa}_\alpha(D_0) - \kappa_\alpha(G) D_0(e)$ our first assumption yields

$$\overline{\lim_\alpha} \left| \int \langle D_0(x) v - v, w_k\rangle \kappa_\alpha(dx) \right| < \infty \qquad (**)$$

since D_0 is the direct sum of finitely many $D_j' \in \operatorname{Irr}(G)$. Furthermore we have

$$\left| \int_{U_0} \left[\sum_{i,j} x_i(x) x_j(x) \langle T(D_0)(x) X_i(D_0) X_j(D_0) v, w_k\rangle \right] \kappa_\alpha(dx) \right|$$
$$\leqslant \sum_{i,j} \|X_i(D_0) X_j(D_0) v\| \, \|w_k\| \int_{U_0} \varphi\, d\kappa_\alpha =: d_k \int_{U_0} \varphi\, d\kappa_\alpha.$$

With the aid of (*) we can now conclude

$$\begin{aligned}
&\left|\int_{U_0} x_k \, d\kappa_\alpha\right| \\
&\quad \leqslant \left|\int_G \langle D_0(x)v - v, w_k\rangle \, \kappa_\alpha(dx)\right| \\
&\qquad + \left|\int_{G\setminus U_0} \langle D_0(x)v - v, w_k\rangle \kappa_\alpha(dx)\right| + d_k \int_{U_0} \varphi \, d\kappa_\alpha \\
&\quad \leqslant \left|\int_G \langle D_0(x)v - v, w_k\rangle \, \kappa_\alpha(dx)\right| \\
&\qquad + 2\,\|v\|\,\|w_k\|\,\kappa_\alpha(G\setminus U_0) + d_k \int_{U_0} \varphi \, d\kappa_\alpha .
\end{aligned}$$

By (**), the second assumption and part 1 we have $\overline{\lim}_\alpha \left|\int_{U_0} x_k \, d\kappa_\alpha\right| < \infty$ and finally $\overline{\lim}_\alpha \left|\int x_k \, d\kappa_\alpha\right| < \infty$.

(iii) $\Rightarrow$ (i) $\overline{\lim}_\alpha \left(\int \varphi \, d\kappa_\alpha\right) < \infty$ obviously implies $\overline{\lim}_\alpha \kappa_\alpha(\complement U) < \infty$ for all $U \in \mathfrak{U}(G)$. The inequalities

$$\left\|\int_{\complement U_0} [D(x)u - u]\kappa_\alpha(dx)\right\| \leqslant 2\,\|u\|\,\kappa_\alpha\left(\complement\, U_0\right)$$

and

$$\begin{aligned}
&\left\|\int_{U_0} [D(x)u - u]\,\kappa_\alpha(dx)\right\| \\
&\qquad \leqslant \sum_i \|X_i(D)u\| \left|\int_{U_0} x_i \, d\kappa_\alpha\right| + \frac{1}{2}\sum_{i,j} \|X_i(D)X_j(D)u\| \int_{U_0} \varphi \, d\kappa_\alpha
\end{aligned}$$

(Taylor's formula) prove the second statement. ∎

Corollary. *If we have* $\kappa_\alpha = \sum_{k=1}^{k(\alpha)} \mu_{\alpha k}$, $\mu_{\alpha k} \in \mathscr{M}^1(G)$ *for all* $k = 1, \ldots, k(\alpha)$ *and all* $\alpha \in I$, *then the following assertions are equivalent*:

(i) $\overline{\lim}_\alpha \sum_{k=1}^{k(\alpha)} \|\hat{\mu}_{\alpha k}(D)u - u\| < \infty$ *for all* $D \in \mathrm{Irr}(G)$, $u \in \mathscr{H}_0(D)$, *and* $\overline{\lim}_\alpha \sum_{k=1}^{k(\alpha)} \mu_{\alpha k}(\complement U) < \infty$ *for all* $U \in \mathfrak{U}(G)$.

(ii) $\overline{\lim}_\alpha \sum_{k=1}^{k(\alpha)} |\langle \hat{\mu}_{\alpha k}(D)u - u, u\rangle| < \infty$ *for all* $D \in \mathrm{Irr}(G)$, $u \in \mathscr{H}_0(D)$, *and* $\overline{\lim}_\alpha \sum_{k=1}^{k(\alpha)} \mu_{\alpha k}(\complement U) < \infty$ *for all* $U \in \mathfrak{U}(G)$.

(iii) $\overline{\lim}_\alpha \sum_{k=1}^{k(\alpha)} \left(\int \varphi \, d\mu_{\alpha k}\right) < \infty$ *and* $\overline{\lim}_\alpha \sum_{k=1}^{k(\alpha)} \left|\int x_i \, d\mu_{\alpha k}\right| < \infty$ *for* $i = 1, \ldots, p$.

Proof. This follows by the same arguments as those in the proof of Lemma 5.4. ∎

II. Limit Theorems for Distributions on Locally Compact Groups

6. *Convergence of Convolution Semigroups: Necessary Conditions*

Let G be a locally compact group. By $\mathfrak{S}(G)$ we denote the system of all convolution semigroups in $\mathcal{M}^1(G)$. Clearly $\mathfrak{S}(G)$ can be considered as a subspace of the space $\mathfrak{F}(G)$ of all continuous mappings of $\mathbb{R}_+$ into $(\mathcal{M}^1(G), \mathscr{E}_w)$. We equip $\mathfrak{F}(G)$ with the topology $\mathscr{E}_c$ of compact convergence. Obviously $\mathfrak{S}(G)$ is a $\mathscr{E}_c$-closed subspace of $\mathfrak{F}(G)$.

Let I be an index set directed by $>$. For any $\alpha \in I$ let there be given a (fixed) convolution semigroup $S_\alpha := (\mu_t^{(\alpha)})_{t>0}$ in $\mathcal{M}^1(G)$ with generating functional A_α and Lévy measure η_α. In this and the next section we shall study the $\mathscr{E}_c$-convergence of the net $(S_\alpha)_{\alpha \in I}$ in terms of the A_α and η_α.

If $S = (\mu_t)_{t>0}$ is a further convolution semigroup in $\mathcal{M}^1(G)$ we have $\mathscr{E}_c$-$\lim S_\alpha = S$ iff $\mathscr{E}_w$-$\lim_\alpha \mu_t^{(\alpha)} = \mu_t$ uniformly in $t \in [0, d]$ for all $d > 0$. Since the topologies $\mathscr{E}_w$ and $\mathscr{E}_v$ coincide on $\mathcal{M}^1(G)$ this is equivalent with $\mathscr{E}_v$-$\lim_\alpha \mu_t^{(\alpha)} = \mu_t$ uniformly in $t \in [0, d]$ for all $d > 0$ (cf. [16, Chap. 7]).

The $\mathscr{E}_c$-convergence of *sequences* in $\mathfrak{S}(G)$ admits the following convenient characterization:

Proposition 6.1. *Let* $I = \mathbb{N}$. *Then the following assertions are equivalent*:

(i) *The sequence* $(S_n)_{n \geq 1}$ *is* $\mathscr{E}_c$-*convergent.*

(ii) *For each* $t \in \mathbb{R}_+$ *there exists a measure* $\mu_t \in \mathcal{M}^1(G)$ *such that* $\mathscr{E}_f$-$\lim_n \mu_t^{(n)} = \mu_t$.

Proof. We only have to prove (ii) $\Rightarrow$ (i). Let $T_t^{(n)}$ and T_t be the convolution operators corresponding to $\mu_t^{(n)}$ and μ_t, resp. By Lemma 2.1 we have $\mathscr{E}_w$-$\lim_n \mu_t^{(n)} = \mu_t$ and thus $\lim_n T_t^{(n)} = T_t$ strongly on $\mathscr{C}^0(G)$ (cf. Section 2). Therefore $(T_t)_{t>0}$ is a strongly measurable semigroup and thus even strongly continuous [30, p. 233, proof of the theorem]. Consequently $(\mu_t)_{t \geq 0}$ is a convolution semigroup.

Let $\lambda > 0$ and $f \in \mathscr{C}^0(G)$ be fixed. Define

$$f_n(t) := \| e^{-\lambda t} T_t^{(n)} f - e^{-\lambda t} T_t f \|_\infty \qquad (t \geq 0).$$

Then we have $\lim f_n(t) = 0$ for all $t \geq 0$. Furthermore $f_n(t) \leq 2 \|f\|_\infty e^{-\lambda t}$ for all $n \in \mathbb{N}$. By Lebesgue's theorem we conclude

$$\overline{\lim_n} \left\| \int_0^\infty e^{-\lambda t} T_t^{(n)} f\,dt - \int_0^\infty e^{-\lambda t} T_t f\,dt \right\|$$

$$\leqslant \lim_n \int_0^\infty f_n(t)\,dt = 0.$$

Thus $\lim R_\lambda^{(n)} f = R_\lambda f$ for all $f \in \mathscr{C}^0(G)$. (Here $R_\lambda^{(n)}$ and R_λ are the resolvents corresponding to the semigroups $(T_t^{(n)})_{t>0}$ and $(T_t)_{t>0}$, resp.) The Trotter–Kato theorem [30, p. 269] yields $\lim T_t^{(n)} = T_t$ strongly on $\mathscr{C}^0(G)$ and uniformly in $t \in [0, d]$ $(d > 0)$. This proves (i). ∎

Remark. Proposition 6.1 seems to be well known. For the sake of completeness we have included a proof.

LEMMA 6.1. *Let* $D \in \operatorname{Rep}(G)$ *and* $u \in \mathscr{H}_0(D)$ *such that* $\overline{\lim}_\alpha \|A_\alpha(D)u\| < \infty$. *Then we have:*

(i) *There exist* $\alpha_0 \in I$ *and* $c > 0$ *such that* $\|\hat{\mu}_s^{(\alpha)}(D)u - \hat{\mu}_t^{(\alpha)}(D)u\| \leqslant c\,|s-t|$ *for all* $s, t \geqslant 0$ *and* $\alpha > \alpha_0$.

(ii) *If* $\mathscr{C}_w$-$\lim_\alpha \mu_t^{(\alpha)} = \mu_t \in \mathscr{M}^1(G)$ *exists for all* $t \in \mathbb{R}_+$ *then* $\lim_\alpha \|\hat{\mu}_t^{(\alpha)}(D)u - \hat{\mu}_t(D)u\| = 0$ *uniformly in* $t \in [0, d]$ *for all* $d > 0$.

Proof. (i) There exist $\alpha_0 \in I$ and $c > 0$ such that $\|A_\alpha(D)u\| \leqslant c$ for all $\alpha > \alpha_0$. Thus by Corollary 1 of Proposition 3.1 we have, for $\alpha > \alpha_0$ and $s, t \geqslant 0$,

$$\begin{aligned}\|\hat{\mu}_s^{(\alpha)}(D)u - \hat{\mu}_t^{(\alpha)}(D)u\| &= \left\| \int_s^t \hat{\mu}_r^{(\alpha)}(D) A_\alpha(D)u\,dr \right\| \\ &\leqslant |s-t|\,\|A_\alpha(D)u\| \leqslant |s-t|\,c.\end{aligned}$$

(ii) This is an immediate consequence of (i). ∎

PROPOSITION 6.2. *Let there exist a convolution semigroup* $(\mu_t)_{t>0}$ *in* $\mathscr{M}^1(G)$ *with generating functional A such that* $\mathscr{C}_w$-$\lim_\alpha \mu_t^{(\alpha)} = \mu_t$ *for all* $t \geqslant 0$.
Then for each representation $D \in \operatorname{Rep}(G)$ *such that* $\overline{\lim}_\alpha \|A_\alpha(D)u\| < \infty$ *and* $\overline{\lim}_\alpha \|\tilde{A}_\alpha(D)u\| < \infty$ *for all* $u \in \mathscr{H}_0(D)$, *we have* $\lim_\alpha \langle A_\alpha(D)u, v\rangle = \langle A(D)u, v\rangle$ *for all* $u \in \mathscr{H}_0(D)$, $v \in \mathscr{H}(D)$.

Proof. Let us fix $u, v \in \mathscr{H}_0(D)$. We have $\overline{\lim}_\alpha |\langle A_\alpha(D)u, v\rangle| \leqslant \overline{\lim}_\alpha \|A_\alpha(D)u\|\,\|v\| < \infty$. Substituting I by a universal subnet we may assume that $a := \lim_\alpha \langle A_\alpha(D)u, v\rangle$ exists.

By assumption there exist $\alpha_0 \in I$ and $c > 0$ such that $\|A_\alpha(D)u\| \leqslant c$ for all $\alpha > \alpha_0$. Thus by Corollary 1 of Proposition 3.1 we obtain the following estimation (for some $t > 0$):

$$\left| \frac{1}{t} \left[\langle \hat{\mu}_t^{(\alpha)}(D)u, v\rangle - \langle u, v\rangle\right] - \langle A_\alpha(D)u, v\rangle \right|$$

$$= \frac{1}{t} \left| \int_0^t \int_G \left[\langle D(y)A_\alpha(D)u, v\rangle - \langle A_\alpha(D)u, v\rangle\right] \mu_s^{(\alpha)}(dy)\, ds \right|$$

$$= \frac{1}{t} \left| \int_0^t \langle [\hat{\mu}_s^{(\alpha)}(D) - D(e)] A_\alpha(D)u, v\rangle\, ds \right|$$

$$= \frac{1}{t} \left| \int_0^t \langle A_\alpha(D)u, [\hat{\tilde{\mu}}_s^{(\alpha)}(D) - D(e)]v\rangle\, ds \right|$$

$$\leqslant \|A_\alpha(D)u\| \frac{1}{t} \int_0^t \|[\hat{\tilde{\mu}}_s^{(\alpha)}(D) - D(e)]v\|\, ds$$

$$\leqslant \frac{c}{t} \int_0^t \|\hat{\tilde{\mu}}_s^{(\alpha)}(D)v - v\|\, ds.$$

Lemma 6.1(ii) applied to the adjoint measures $\tilde{\mu}_t^{(\alpha)}$ and $\tilde{\mu}_t$ yields $\lim_\alpha \|\hat{\tilde{\mu}}_s^{(\alpha)}(D)v - \hat{\tilde{\mu}}_s(D)v\| = 0$ uniformly in $s \in [0, t]$. Going with α to the limit in the inequality above yields

$$\left| \frac{1}{t} \left[\langle \hat{\mu}_t(D)u, v\rangle - \langle u, v\rangle\right] - a \right|$$

$$\leqslant \frac{c}{t} \int_0^t \|\hat{\tilde{\mu}}_s(D)v - v\|\, ds \leqslant c \sup_{0 \leqslant s \leqslant t} \|\hat{\tilde{\mu}}_s(D)v - v\|.$$

Since $(\tilde{\mu}_t)_{t \geqslant 0}$ is a convolution semigroup we get

$$a = \lim_{t \downarrow 0} (1/t)[\langle \hat{\mu}_t(D)u, v\rangle - \langle u, v\rangle] = \langle A(D)u, v\rangle,$$

i.e., $\lim_\alpha \langle A_\alpha(D)u, v\rangle = \langle A(D)u, v\rangle$. Since this limit is independent of the universal subset chosen above the result holds for the original net too.

Since $\overline{\lim}_\alpha \|A_\alpha(D)u\| < \infty$ and since $\mathscr{H}_0(D)$ is dense in $\mathscr{H}(D)$ we finally have $\lim_\alpha \langle A_\alpha(D)u, v\rangle = \langle A(D)u, v\rangle$ for all $v \in \mathscr{H}(D)$ (and $u \in \mathscr{H}_0(D)$). ∎

Now let G be a Lie group of dimension $p \geqslant 1$, $\{X_1, \ldots, X_p\}$ a basis of $\mathscr{L}(G)$ and $\{x_1, \ldots, x_p\}$ a system of canonical coordinates in $\mathscr{D}(G)$ adapted to this basis and valid in the neighbourhood $U_0 \in \mathfrak{U}(G)$. Finally let φ be a Hunt function for G (cf. Section 5). Let

$$A_\alpha(f) = \sum_{i=1}^{p} a_i^{(\alpha)}(X_i f)(e) + \sum_{i,j=1}^{p} a_{ij}^{(\alpha)}(X_i X_j f)(e)$$
$$+ \int_{G^\times} \left[f(y) - f(e) - \sum_{i=1}^{p} x_i(y)(X_i f)(e) \right] d\eta_\alpha(y)$$

($f \in \mathscr{E}(G)$) be the canonical decomposition of A_α (cf. (LLC) in Section 3).

PROPOSITION 6.3. *Let there exist a convolution semigroup $S = (\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ such that $\mathscr{C}_c$-$\lim S_\alpha = S$.*
Then we have

$$\overline{\lim_\alpha} \left[\sum_{i=1}^{p} |a_i^{(\alpha)}| + \sum_{i,j=1}^{p} |a_{ij}^{(\alpha)}| + \int_{G^\times} \varphi \, d\eta_\alpha \right] < \infty.$$

Proof. 1. Let us denote by d_α the maximum of the numbers $|a_i^{(\alpha)}|$, $|a_{ij}^{(\alpha)}|$, $\int_{G^\times} \varphi \, d\eta_\alpha$ $(1 \leqslant i, j \leqslant p)$. We have to show $\overline{\lim}_\alpha d_\alpha < \infty$.

Let us suppose $\overline{\lim}_\alpha d_\alpha = \infty$. Selecting an appropriate subnet we can even assume $\lim_\alpha d_\alpha = \infty$. Let $c_\alpha := d_\alpha^{-1}$. Then we have $\lim_\alpha c_\alpha = 0$. We define $b_i^{(\alpha)} := c_\alpha a_i^{(\alpha)}$, $b_{ij}^{(\alpha)} := c_\alpha a_{ij}^{(\alpha)}$, $\xi_\alpha := c_\alpha \eta_\alpha$ and $B_\alpha := c_\alpha A_\alpha$. Then B_α is the generating functional of the (unique) convolution semigroup $(\nu_t^{(\alpha)})_{t>0}$ in $\mathscr{M}^1(G)$ such that $\nu_t^{(\alpha)} = \mu_{c_\alpha t}^{(\alpha)}$, and B_α admits the canonical decomposition $(b_i^{(\alpha)}, b_{ij}^{(\alpha)}, \xi_\alpha)_{1 \leqslant i,j \leqslant p}$. Since $\lim_\alpha c_\alpha = 0$ and $\mathscr{C}_c$-$\lim_\alpha S_\alpha = S$ we have $\mathscr{C}_w$-$\lim_\alpha \nu_t^{(\alpha)} = \varepsilon_e$ for all $t \geqslant 0$.

2. Let $D \in \operatorname{Rep}(G)$ and $u \in \mathscr{H}_0(D)$. By the corollary of Proposition 3.2 we have

$$B_\alpha(D)u = \sum_{i=1}^{p} b_i^{(\alpha)} X_i(D)u + \sum_{i,j=1}^{p} b_{ij}^{(\alpha)} X_i(D) X_j(D)u$$
$$+ \int_{G^\times} \left[D(x) - D(e) - \sum_{i=1}^{p} x_i(x) X_i(D) \right] u \, d\xi_\alpha(x)$$

and thus

$$\|B_\alpha(D)u\| \leqslant \sum_{i=1}^{p} \|X_i(D)u\| + \sum_{i,j=1}^{p} \|X_i(D) X_j(D)u\| + c(D,u)$$

with some constant $c(D,u) > 0$, i.e., $\overline{\lim}_\alpha \|B_\alpha(D)u\| < \infty$. [We have $|b_i^{(\alpha)}| \leqslant 1$, $|b_{ij}^{(\alpha)}| \leqslant 1$ and $\int_{G^\times} \varphi \, d\xi_\alpha \leqslant 1$. The definition of φ yields $\xi_\alpha(\complement\, U_0) < \infty$. Together with Lemma 5.1(ii) this proves the existence of $c(D,u)$.]

The generating functional $\tilde{B}_\alpha$ of the adjoint convolution semigroup $(\tilde{\nu}_t^{(\alpha)})_{t>0}$ admits the canonical decomposition $(-b_i^{(\alpha)}, b_{ij}^{(\alpha)}, \tilde{\xi}_\alpha)_{1 \leqslant i,j \leqslant p}$. Thus the inequalities $|-b_i^{(\alpha)}| \leqslant 1$, $|b_{ij}^{(\alpha)}| \leqslant 1$, $\int_{G^\times} \varphi \, d\tilde{\xi}_\alpha = \int_{G^\times} \varphi \, d\xi_\alpha \leqslant 1$ are fulfilled again. As above we conclude $\overline{\lim}_\alpha \|\tilde{B}_\alpha(D)u\| < \infty$. Now Proposition 6.2 yields $\overline{\lim}_\alpha \langle B_\alpha(D)u, v\rangle = 0$ for all $v \in \mathscr{H}(D)$ (the generating functional of the trivial convolution semigroup (ε_e) is zero).

3. Let us define $f := \mathrm{Re}(\langle Du, u\rangle)$. ($D$ and u will be appropriately chosen later on) f is real valued and positive definite. Therefore it attains its maximum at e. This yields $(X_i f)(e) = 0$ for $i = 1,\ldots, p$. By $Q_\alpha(g) := \sum_{i,j} b_{ij}^{(\alpha)}(X_i X_j g)(e)$ $(g \in \mathscr{E}(G))$ there is defined a quadratic form Q_α on $\mathscr{E}(G)$. It follows: $\mathrm{Re}\langle B_\alpha(D)u, u\rangle = \mathrm{Re}(B_\alpha(\langle Du, u\rangle)) = B_\alpha(f) = Q_\alpha(f) + \int_{G^\times} [f - f(e)]\, d\xi_\alpha$. But $f \leqslant f(e)$ implies $Q_\alpha(f) \leqslant 0$ and $\int_{G^\times}[f - f(e)]\, d\xi_\alpha \leqslant 0$ $(\alpha \in I)$. Thus $\lim_\alpha \langle B_\alpha(D)u, u\rangle = 0$ yields $\lim_\alpha Q_\alpha(f) = 0$ and $\lim_\alpha \int_{G^\times}[f(e) - f]\, d\xi_\alpha = 0$.

(a) By Lemma 5.2 there exist $D \in \mathrm{Rep}(G)$ and $u \in \mathscr{H}_0(D)$ such that $\varphi(x) \leqslant c(f(e) - f(x))$ for all $x \in U$, where $U \in \mathfrak{U}(G)$ and $c > 0$ are chosen appropriately. Since Q_α is a quadratic form we get $0 \leqslant Q_\alpha(\varphi) \leqslant -cQ_\alpha(f)$ and thus $\lim_\alpha Q_\alpha(\varphi) = 0$. But $Q_\alpha(\varphi) = 2\sum_i b_{ii}^{(\alpha)}$ and the positive semi-definiteness of $(b_{ij}^{(\alpha)})_{1 \leqslant i,j \leqslant p}$ yield $\lim_\alpha b_{ij}^{(\alpha)} = 0$ $(1 \leqslant i, j \leqslant p)$. Moreover we have $\lim_\alpha \int_{U\setminus\{e\}} \varphi\, d\xi_\alpha = 0$.

(b) Let us choose for D the left regular representation L of G and for u a function $g \in \mathscr{D}_+(G)$ such that $\int g^2\, d\omega = 1$ and $\mathrm{supp}(g) \subset V$, where $V \in \mathfrak{U}(G)$, $VV^{-1} \subset U$; then we have $\mathrm{supp}(f) \subset U$ (cf. Section 1, example a). Therefore $\xi_\alpha(\complement U) \leqslant \int_{G^\times}[f(e) - f]\, d\xi_\alpha$. Thus $\lim_\alpha \xi_\alpha(\complement U) = 0$ and taking into account (a), finally $\lim_\alpha \int_{G^\times} \varphi\, d\xi_\alpha = 0$.

4. By Lemma 5.3 there exist a representation $D \in \mathrm{Rep}(G)$ and vectors v, $w_j \in \mathscr{H}_0(D)$ such that $\langle X_i(D)v, w_j\rangle = \delta_{ij}$ $(1 \leqslant i, j \leqslant p)$. If $f_k := \langle Dv, w_k\rangle$ we have

$$\langle B_\alpha(D)v, w_k\rangle = B_\alpha(\langle Dv, w_k\rangle) = B_\alpha(f_k)$$

$$= b_k^{(\alpha)} + Q_\alpha(f_k) + \int_{G^\times} [f_k - f_k(e) - x_k]\, d\xi_\alpha.$$

Applying parts 2 and 3(a, b) we get $\lim_\alpha b_k^{(\alpha)} = 0$ $(1 \leqslant k \leqslant p)$.

5. But now we have arrived at a contradiction to

$$\max\left\{|b_i^{(\alpha)}|, |b_{ij}^{(\alpha)}|, \int_{G^\times} \varphi\, d\xi_\alpha \mid 1 \leqslant i, j \leqslant p\right\} = c_\alpha d_\alpha = 1$$

for all $\alpha \in I$. Our assumption $\overline{\lim}_\alpha d_\alpha = \infty$ was incorrect. This finishes our proof. ∎

Let us return to an arbitrary locally compact group G. By $\mathrm{Fac}(G)$ we denote the system of all representations $D \in \mathrm{Rep}(G)$ with the following property: There exists a compact normal subgroup K in G (depending on D)

such that G/K is a Lie group and a representation $D' \in \mathrm{Rep}(G/K)$ such that $D = D' \circ \pi$, where π is the canonical mapping of G onto G/K. Obviously $D' \circ \pi \in \mathrm{Fac}(G)$ for all $D' \in \mathrm{Rep}(G/K)$. For a Lie projective group G we have $\mathrm{Irr}(G) \subset \mathrm{Fac}(G)$ [3, p. 247, Korollar 2].

PROPOSITION 6.4. *Let G be a Lie projective group and let there exist a convolution semigroup $S = (\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ with generating functional A and Lévy measure η. We assume $\mathscr{T}_c$-$\lim_\alpha S_\alpha = S$. Then following assertions are valid:*

(i) $\overline{\lim}_\alpha \|A_\alpha(D)u\| < \infty$ *and* $\overline{\lim}_\alpha \|\tilde{A}_\alpha(D)u\| < \infty$ *for all* $D \in \mathrm{Fac}(G)$ *and* $u \in \mathscr{H}_0(D)$.

(ii) $\lim_\alpha \langle A_\alpha(D)u, v\rangle = \langle A(D)u, v\rangle$ *for all* $D \in \mathrm{Fac}(G)$, $u \in \mathscr{H}_0(D)$ *and* $v \in \mathscr{H}(D)$.

(iii) *There exists a basis $\mathfrak{U}$ of closed neighbourhoods for $e \in G$ such that*

$$\mathscr{T}_w\text{-}\lim_\alpha \eta_\alpha \mid \complement U = \eta \mid \complement U \qquad \textit{for all} \quad U \in \mathfrak{U}$$

In particular, we have $\mathscr{T}_v$-$\lim_\alpha \eta_\alpha = \eta$.

Proof. (i) By the definition of $\mathrm{Fac}(G)$ we may assume without loss of generality that G is a Lie group. But then the statement follows immediately from Proposition 6.3, (LLC) in Section 3 and Lemma 5.1(ii).

(ii) This statement follows from (i) and Proposition 6.2.

(iii) Since G is a Lie projective group there exists for any $U \in \mathfrak{U}(G)$ a compact normal subgroup K in G such that G/K is a Lie group and a neighbourhood $V \in \mathfrak{U}(G)$ such that $VK \subset U$. Thus we may assume again that G is a Lie group. In this case we have $\mathrm{Fac}(G) = \mathrm{Rep}(G)$.

1. We first prove $\mathscr{T}_v$-$\lim_\alpha \eta_\alpha = \eta$. [Let $h \in \mathscr{K}(G^x)$ be real valued and $C := \mathrm{supp}(h)$. Since $e \notin C$ there exists an open neighbourhood $U \in \mathfrak{U}(G)$ such that $e \notin CU$. Let $\varepsilon > 0$. Then there exists a compact neighbourhood $V \in \mathfrak{U}(G)$ such that $V \subset U$, $V = V^{-1}$ and $|h^*(x) - h^*(y^{-1}x)| < \varepsilon$ for all $y \in V$ and $x \in G$. Let us choose a function $g \in \mathscr{D}_+(G)$ such that $\mathrm{supp}(g) \subset V$ and $\int g\,d\omega = 1$. Then we have, for any $x \in G$,

$$\begin{aligned}
&| h^*(x) - (g * h^*)(x)| \\
&\quad = \left| \int_V h^*(x)\, g(y)\, dy - \int h^*(y^{-1}x)\, g(y)\, dy \right| \\
&\quad \leqslant \int_V g(y) \mid h^*(x) - h^*(y^{-1}x) \mid dy \leqslant \varepsilon \int_V g(y)\, dy = \varepsilon.
\end{aligned}$$

Since $(g * h^*)(x^{-1}) = \langle L(x) g, h\rangle$ we have $|h(x) - \langle L(x)g, h\rangle| \leqslant \varepsilon$ for all $x \in G$. Furthermore $\operatorname{supp}(\langle Lg, h\rangle) \subset \operatorname{supp}(h)(\operatorname{supp}(g))^{-1} \subset CV$. Since CV is compact and $e \notin CV$ there exists a constant $c > 0$ such that $\varphi(x) \geqslant c$ for all $x \in CV$. Thus we get $|h - \langle Lg, h\rangle| \leqslant \varepsilon c^{-1}\varphi$.

By Proposition 6.3 there exist a constant $d > 0$ and an index $\alpha_0 \in I$ such that $\int_{G^x} \varphi \, d\eta_\alpha \leqslant d$ for all $\alpha > \alpha_0$, and $\int_{G^x} \varphi \, d\eta \leqslant d$.

$e \notin \operatorname{supp}(\langle Lg, h\rangle)$ and $g \in \mathscr{H}_0(L)$ imply $\langle A_\alpha(L)g, h\rangle = \int_{G^x} \langle Lg, h\rangle \, d\eta_\alpha$ and $\langle A(L)g, h\rangle = \int_{G^x} \langle Lg, h\rangle d\eta$ (cf. (LLC) in Section 3). Thus we can estimate $|\int_{G^x} h \, d\eta_\alpha - \langle A_\alpha(L)g, h\rangle| = |\int_{G^x} [h - \langle Lg, h\rangle] \, d\eta_\alpha| \leqslant \varepsilon c^{-1} \int_{G^x} \varphi \, d\eta_\alpha \leqslant \varepsilon c^{-1} d$ for $\alpha > \alpha_0$ and analogously $|\int_{G^x} h \, d\eta - \langle A(L)g, h\rangle| \leqslant \varepsilon c^{-1} d$. Finally by (ii) we have $\lim_\alpha \langle A_\alpha(L) g, h\rangle = \langle A(L)g, h\rangle$. Since $\varepsilon > 0$ was arbitrary we get $\lim_\alpha \int_{G^x} h \, d\eta_\alpha = \int_{G^x} h \, d\eta$.]

2. Let $U \in \mathfrak{U}(G)$ be closed. There exists a function $f \in \mathscr{D}_+(G)$ such that with $g := \langle Lf, f\rangle$ we have $g(e) = 1$, $g^* = g$ and $\operatorname{supp}(g) \subset U$. Since $f \in \mathscr{H}_0(L)$ we have $f \otimes f \in \mathscr{H}_0(L \otimes L)$ [28, 4.4.1.10]. Moreover $(g-1)^2 = g^2 - 2g + 1 = \langle (L \otimes L)(f \otimes f), \ (f \otimes f)\rangle - 2\langle Lf, f\rangle + \langle E1, 1\rangle$. (Here E denotes the trivial representation of G on $\mathbb{C}$.) Thus by (ii) we obtain $\lim_\alpha A_\alpha((g-1)^2) = A((g-1)^2)$.

For any primitive orquadratic form B on $\mathscr{E}(G)$ the defining equation yields $B((g-1)^2) = 0$ [15, 4.4.6]. Therefore we have $A_\alpha((g-1)^2) = \int_{G^x} (g-1)^2 d\eta_\alpha$ and $A((g-1)^2) = \int_{G^x} (g-1)^2 d\eta$. Thus $\lim_\alpha \int_{G^x} (g-1)^2 d\eta_\alpha = \int_{G^x} (g-1)^2 d\eta$. Taking part 1 into account this yields $\mathscr{E}_w\text{-}\lim_\alpha (g-1)^2 \cdot \eta_\alpha = (g-1)^2 \cdot \eta$ [4, Chap. IX, p. 61, Proposition 9].

If the boundary of U has η-measure zero it follows $\lim_\alpha \eta_\alpha(\complement U) = \lim_\alpha \int_{\complement U} (g-1)^2 d\eta_\alpha = \int_{\complement U} (g-1)^2 d\eta = \eta(\complement U)$ [20, p. 40] (we have $g(\dot{x}) = 0$ for all $\dot{x} \in \complement U$).

Since $\complement U$ is open we may consider $\mathscr{K}(\complement U)$ as a subspace of $\mathscr{K}(G^x)$. Thus keeping part 1 in mind we can conclude $\mathscr{E}_w\text{-}\lim_\alpha \eta_\alpha \,|\, \complement U = \eta \,|\, \complement U$.

3. We finish the proof by exhibiting a basis $\mathfrak{U}$ of closed neighbourhoods for e whose boundaries have η-measure zero. [For any $r \in \mathbb{R}_+^*$ the set $U_r := \varphi^{-1}([0, r])$ is a closed neighbourhood for $e \in G$. By the definition of φ the system $(U_r)_{r \in \mathbb{R}_+^*}$ is a basis for $\mathfrak{U}(G)$. Since η is a Lévy measure we have $\varphi(\eta)(]r, \infty[) = \eta(\varphi^{-1}(]r, \infty[)) = \eta(\complement U_r) < \infty$ for all $r \in \mathbb{R}_+^*$. Thus there exists an at most countable subset Z in $\mathbb{R}_+^*$ such that $\varphi(\eta)(\{r\}) = 0$ for all $r \in \mathbb{R}_+^* \backslash Z$. For these r the η-measure of the boundary of U_r is zero since it is contained in $\varphi^{-1}(\{r\})$. Obviously $\mathfrak{U} := (U_r)_{r \in \mathbb{R}_+^* \backslash Z}$ is also a basis for $\mathfrak{U}(G)$.] ∎

7. *Convergence of Convolution Semigroups: Sufficient Conditions*

Again we consider a net $(S_\alpha)_{\alpha \in I}$ in $\mathfrak{S}(G)$ with $S_\alpha = (\mu_t^{(\alpha)})_{t \geqslant 0}$. A_α and η_α denote the generating functional and the Lévy measure of S_α, respectively.

PROPOSITION 7.1. *Let G be a Lie projective group and let the following conditions be satisfied*:

(a) $\overline{\lim}_\alpha |\langle A_\alpha(D)u, u\rangle| < \infty$ *for all* $D \in \operatorname{Irr}(G)$ *and* $u \in \mathscr{H}_0(D)$.

(b) $(\eta_\alpha \mid \complement U)_{\alpha\in I}$ *is a compact net for any open* $U \in \mathfrak{U}(G)$.

Then $(S_\alpha)_{\alpha\in I}$ *is a compact net in* $(\mathfrak{S}(G), \mathscr{E}_c)$. *Moreover* $(\eta_\alpha)_{\alpha\in I}$ *is a compact net in* $(\mathscr{M}_+(G^\times), \mathscr{E}_v)$.

Proof. By Remark 5.2, condition (a) is equivalent with

(a′) $\overline{\lim}_\alpha|\langle A_\alpha(D)u, v\rangle| < \infty$ for all $D \in \operatorname{Irr}(G)$ and $u, v \in \mathscr{H}_0(D)$.

Without loss of generality let I be a universal net. We then have to show that $(S_\alpha)_{\alpha\in I}$ is $\mathscr{E}_c$-convergent and $(\eta_\alpha)_{\alpha\in I}$ is $\mathscr{E}_v$-convergent.

1. Let G be a Lie group with canonical coordinates $\{x_1,..., x_p\}$ valid in $U_0 \in \mathfrak{U}(G)$ adapted to the basis $\{X_1,..., X_p\}$ of $\mathscr{L}(G)$ and with Hunt function φ. Let $(a_i^{(\alpha)}, a_{ij}^{(\alpha)}, \eta_\alpha)_{1\leqslant i,j\leqslant p}$ be the canonical decomposition of A_α. (One should compare the arguments in (a) and (b) infra with those given in parts 3 and 4 of the proof of Proposition 6.3).)

(a) By $Q_\alpha(g) := \sum_{i,j} a_{ij}^{(\alpha)}(X_iX_j g)(e)$ there is defined a quadratic form on $\mathscr{E}(G)$. Let $f := \operatorname{Re}\langle Du, u\rangle$, where D is a finite direct sum of irreducible representations of G and $u \in \mathscr{H}_0(D)$. Then we have

$$-\operatorname{Re}\langle A_\alpha(D)u, u\rangle = Q_\alpha(f(e) - f) + \int_{G^\times} [f(e) - f]\, d\eta_\alpha,$$

and both summands on the right-hand side are non-negative. By condition (a) we have $\overline{\lim}_\alpha |\operatorname{Re}\langle A_\alpha(D)u, u\rangle| < \infty$; hence $\overline{\lim}_\alpha Q_\alpha(f(e) - f) < \infty$ and $\overline{\lim}_\alpha \int_{G^\times} [f(e) - f]\, d\eta_\alpha < \infty$. By a proper choice of D and u (cf. Lemma 5.2) and taking into account condition (b) we obtain

$$\overline{\lim_\alpha}\, Q_\alpha(\varphi) < \infty \qquad \text{and} \qquad \overline{\lim_\alpha} \int_{G^\times} \varphi\, d\eta_\alpha < \infty.$$

But $Q_\alpha(\varphi) = 2\sum_i a_{ii}^{(\alpha)}$ and the positive semi-definiteness of the matrix $(a_{ij}^{(\alpha)})_{1\leqslant i,j\leqslant p}$ yield

$$\overline{\lim_\alpha}\, |a_{ij}^{(\alpha)}| < \infty \qquad \text{for} \quad i, j = 1,..., p.$$

(b) Let us choose D_0, v, w_k $(1 \leqslant k \leqslant p)$ as in Lemma 5.3. If $f_k := \langle D_0 v, w_k\rangle$ we have

$$\langle A_\alpha(D_0)v, w_k\rangle = A_\alpha(f_k) = a_k^{(\alpha)} + Q_\alpha(f_k) + \int_{G^\times} [f_k - f_k(e) - x_k]\, d\eta_\alpha.$$

Applying condition (a) and the results of part (a) of the proof we obtain $\overline{\lim}_\alpha |a_i^{(\alpha)}| < \infty$ for $i = 1,\dots, p$.

(c) Without loss of generality we can assume

$$\sup_\alpha |a_i^{(\alpha)}| < \infty, \sup_\alpha |a_{ij}^{(\alpha)}| < \infty, \sup_\alpha \int_{G^x} \varphi \, d\eta_\alpha < \infty \qquad (1 \leqslant i, j \leqslant p).$$

We recall that I is a universal net. Hence there exist

$$a_i := \lim_\alpha a_i^{(\alpha)}, \qquad b_{ij} := \lim_\alpha a_{ij}^{(\alpha)}, \qquad a := \lim_\alpha \int_{G^x} \varphi \, d\eta_\alpha.$$

Moreover by condition (b) there exists (for all open $U \in \mathfrak{U}(G)$)

$$\mathscr{T}_w\text{-}\lim_\alpha \eta_\alpha \mid \complement U =: \eta_U \in \mathscr{M}_+^b\left(\complement U\right).$$

(d) Let $\bar{\eta}_U := \eta_U \mid \complement(U^-)$. If $U, V \in \mathfrak{U}(G)$ are open and $U \subset V$ we have $\bar{\eta}_U \mid \complement(V^-) = \bar{\eta}_V$. [Let $f \in \mathscr{K}(\complement(V^-))$. Since $\complement(V^-)$ is open $\mathscr{K}(\complement(V^-))$ may be considered as a subspace of $\mathscr{K}(\complement V)$ as well as of $\mathscr{K}(\complement U)$. Hence $\bar{\eta}_V(f) = \eta_V(f) = \lim_\alpha(\eta_\alpha \mid \complement V)(f) = \lim_\alpha(\eta_\alpha \mid \complement U)(f) = \eta_U(f) = \bar{\eta}_U(f)$.]

Hence there exists a unique measure $\eta \in \mathscr{M}_+(G^x)$ such that $\eta \mid \complement(U^-) = \bar{\eta}_U$ for all open $U \in \mathfrak{U}(G)$. We have $\mathscr{T}_v\text{-}\lim_\alpha \eta_\alpha = \eta$. [Let $f \in \mathscr{K}(G^x)$. There exists an open $U \in \mathfrak{U}(G)$ such that $\operatorname{supp}(f) \subset \complement(U^-)$. Considering f also as an element of $\mathscr{K}(\complement(U^-))$ and $\mathscr{K}(\complement U)$ we obtain $\eta(f) = \bar{\eta}_U(f) = \eta_U(f) = \lim_\alpha(\eta_\alpha \mid \complement U)(f) = \lim_\alpha \eta_\alpha(f)$.]

We have $\int_{G^x} \varphi \, d\eta < \infty$; i.e., η is a Lévy measure. [Let $U \in \mathfrak{U}(G)$ be open. Then $\int_{\complement(U^-)} \varphi \, d\eta = \int (\varphi \mid \complement(U^-)) \, d\bar{\eta}_U \leqslant \int (\varphi \mid \complement U) \, d\eta_U = \lim_\alpha \int (\varphi \mid \complement U) \, d(\eta_\alpha \mid \complement U) \leqslant \lim_\alpha \int_{G^x} \varphi \, d\eta_\alpha = a < \infty$. Hence $\int_{G^x} \varphi \, d\eta = \sup\{\int_{\complement(U^-)} \varphi \, d\eta \mid U \in \mathfrak{U}(G) \text{ open}\} \leqslant a$.].

If $U \in \mathfrak{U}(G)$ is open and if the boundary of U has η-measure zero we have $\mathscr{T}_w\text{-}\lim_\alpha \eta_\alpha \mid \complement U = \eta \mid \complement U$. [Let $V \in \mathfrak{U}(G)$ be open and $V^- \subset U$. Then we have $\complement U \subset \complement(V^-) \subset \complement V$ and $U^- \backslash U \subset \complement(V^-)$. Hence $0 = \eta(U^- \backslash U) = \bar{\eta}_V(U^- \backslash U) = \eta_V(U^- \backslash U)$; i.e., the boundary of U has also η_V-measure zero. Therefore $\mathscr{T}_w\text{-}\lim_\alpha \eta_\alpha \mid \complement V = \eta_V$ implies $\mathscr{T}_w\text{-}\lim_\alpha \eta_\alpha \mid \complement U = \eta_V \mid \complement U$ [20, p. 40]. But $\eta \mid \complement(V^-) = \bar{\eta}_V = \eta_V \mid \complement(V^-)$ and $\complement U \subset \complement(V^-)$ finally yield $\eta_V \mid \complement U = \eta \mid \complement U$.]

Finally there exists a basis $\mathfrak{U}$ of open neighbourhoods for $e \in G$ whose boundaries have η-measure zero (cf. part 3 of the proof of Proposition 6.4(iii)).

(e) For any $\varepsilon > 0$ there exists $c_{ij}(\varepsilon) := \lim_\alpha \int_{0 < \varphi \leqslant \varepsilon} x_i x_j \, d\eta_\alpha$ and $|c_{ij}(\varepsilon)| \leqslant c$ for some $c > 0$ $(1 \leqslant i, j \leqslant p)$. [There is a constant $b > 0$ such that $|x_i x_j| \leqslant b\varphi$. Then $\lim_\alpha \int_{G^x} \varphi \, d\eta_\alpha = a$ yields the statements with $c := ab$.]

There exists $c_{ij} := \lim_{\varepsilon\downarrow 0} c_{ij}(\varepsilon)$. [Let $0<\varepsilon'<\varepsilon$. Then there exists $U \in \mathfrak{U}$ such that $C := \{x \in G \mid \varepsilon' \leqslant \varphi(x) \leqslant \varepsilon\} \subset \complement U$. Since $\mathscr{E}_w$-$\lim_\alpha \eta_\alpha \mid \complement U = \eta \mid \complement U$ and since C is closed we obtain $|c(\varepsilon)-c(\varepsilon')| = \lim_\alpha |\int_{\varepsilon'<\varphi\leqslant\varepsilon} x_i x_j \, d\eta_\alpha| \leqslant \lim_\alpha \int_C |x_i x_j| \, d\eta_\alpha \leqslant \int_C |x_i x_j| \, d\eta \leqslant b \int_{0<\varphi\leqslant\varepsilon} \varphi \, d\eta$. But the last term tends to zero if $\varepsilon \downarrow 0$.]

(f) Let $a_{ij} := b_{ij} + \frac{1}{2} c_{ij}$ so that

$$a_{ij} = \lim_{\varepsilon\downarrow 0} \lim_\alpha \left[a_{ij}^{(\alpha)} + \frac{1}{2} \int_{0<\varphi\leqslant\varepsilon} x_i x_j \, d\eta_\alpha \right].$$

Obviously $(a_{ij})_{1\leqslant i,j\leqslant p}$ is a real symmetric positive semidefinite matrix. Let us define (for all $f \in \mathscr{E}(G)$)

$$A(f) := \sum_{i=1}^{p} a_i (X_i f)(e) + \sum_{i,j=1}^{p} a_{ij} (X_i X_j f)(e)$$

$$+ \int_{G^\times} \left[f - f(e) - \sum_{i=1}^{p} x_i (X_i f)(e) \right] d\eta.$$

Then A is the generating functional of a convolution semigroup $S \in \mathfrak{S}(G)$. We have (*) $\sup_\alpha |A_\alpha(f)| < \infty$ and (**) $\lim_\alpha A_\alpha(f) = A(f)$ for all $f \in \mathscr{E}(G)$.

[Let $f \in \mathscr{E}(G)$ be real valued. Then for any sufficiently small $U \in \mathfrak{U}(G)$ we have the Taylor expansion

$$f(x) = f(e) + \sum_{i=1}^{p} x_i(x)(X_i f)(e)$$

$$+ \tfrac{1}{2} \sum_{i,j=1}^{p} x_i(x)\, x_j(x) (X_i X_j f)(\xi(x)),$$

where x, $\xi(x) \in U$. Hence $|f - f(e) - \sum x_i (X_i f)(e)| \leqslant c\varphi$ for some $c > 0$. Now (*) follows immediately from (c).

Condition (**) is a consequence of the definitions of a_i and a_{ij}, of $\mathscr{E}_w$-$\lim_\alpha \eta_\alpha \mid \complement U = \eta \mid \complement U$ for all $U \in \mathfrak{U}$ (cf. (d)) and of the Taylor expansion above. (This has been pointed out in [9, 196 ff.].)]

Now Hazod has proved [11, p. 36] that conditions (*) and (**) imply $\mathscr{E}_c$-$\lim_\alpha S_\alpha = S$. Hence in the Lie group case the proposition is proved.

2. Let G be a Lie projective group. By $\mathfrak{H}$ we denote the system of all compact normal subgroups H in G such that G/H is a Lie group. For $H \in \mathfrak{H}$ let π_H denote the canonical mapping of G onto G/H. Then $\pi_H(S_\alpha) := (\pi_H(\mu_t^{(\alpha)}))_{t>0}$ is a convolution semigroup in $\mathscr{M}^1(G/H)$. If A_α^H is the generating functional of $\pi_H(S_\alpha)$ and η_α^H its Lévy measure we obviously have $A_\alpha^H(D) = A_\alpha(D \circ \pi_H)$ for all $D \in \operatorname{Irr}(G/H)$ and $\int_{(G/H)^\times} f \, d\eta_\alpha^H = \int_{G^\times} (f \circ \pi_H) \, d\eta_\alpha$ for all $f \in \mathscr{K}((G/H)^\times)$. Hence conditions (a) and (b) also hold for the nets $(\pi_H(S_\alpha))_{\alpha\in I}$.

By part 1 there exist convolution semigroups $S_H = (\mu_t^H)_{t>0}$ in $\mathscr{M}^1(G/H)$ with Lévy measures η^H such that $\mathscr{E}_c$-$\lim_\alpha \pi_H(S_\alpha) = S_H$ and $\mathscr{E}_v$-$\lim_\alpha \eta_\alpha^H = \eta^H$ (all $H \in \mathfrak{H}$). Since the family $(\pi_H(S_\alpha))_{H\in\mathfrak{H}}$ is consistent for each $\alpha \in I$ the same holds for the family $(S_H)_{H\in\mathfrak{H}}$ (i.e., for $H, H' \in \mathfrak{H}$, $H' \subset H$ we have $\pi_{HH'}(S_{H'}) = S_H$, where $\pi_{HH'}$ denotes the canonical mapping of G/H' onto G/H). Hence there exists a unique convolution semigroup $S = (\mu_t)_{t\geqslant 0}$ in $\mathscr{M}^1(G)$ with Lévy measure η such that $\pi_H(S) = S_H$ (i.e., $\pi_H(\mu_t) = \mu_t^H$ for all $t \geqslant 0$) for all $H \in \mathfrak{H}$ (cf. [15, 1.2.17, 1.2.18]). Moreover $\int_{G^\times} (f \circ \pi_H)\, d\eta = \int_{(G/H)^\times} f\, d\eta^H$ for $f \in \mathscr{K}((G/H)^\times)$ and $H \in \mathfrak{H}$.

Finally we have $\mathscr{E}_c$-$\lim_\alpha S_\alpha = S$ and $\mathscr{E}_v$-$\lim_\alpha \eta_\alpha = \eta$ since $\{f \circ \pi_H \mid f \in \mathscr{K}(G/H), H \in \mathfrak{H}\}$ is a dense subspace of $(\mathscr{K}(G), \|\cdot\|_\infty)$. Thus the proposition is completely proved. ∎

COROLLARY 1. *Let G be a Lie projective group, $(S_\alpha)_{\alpha\in I}$ a net in $\mathfrak{S}(G)$ and $S \in \mathfrak{S}(G)$. Let A_α and A be the generating functionals and η_α and η the Lévy measures of S_α and S, resp. Then the following assertions are equivalent*:

(i) $\mathscr{E}_c$-$\lim_\alpha S_\alpha = S$.

(ii) (a) $\lim_\alpha \langle A_\alpha(D)u, u\rangle = \langle A(D)u, u\rangle$ *for all* $D \in \operatorname{Irr}(G)$ *and* $u \in \mathscr{H}_0(D)$.

(b) $(\eta_\alpha \mid \complement U)_{\alpha\in I}$ *is a compact net for any open* $U \in \mathfrak{U}(G)$.

Proof. (i) ⇒ (ii) We apply Proposition 6.4. Then condition (a) follows immediately. As for (b) let $U \in \mathfrak{U}(G)$ be open. Then there exists $V \in \mathfrak{U}$ such that $V \subset U$ and $\mathscr{E}_w$-$\lim_\alpha \eta_\alpha \mid \complement V = \eta \mid \complement V$. Hence $(\eta_\alpha \mid \complement V)_{\alpha\in I}$ is a compact and hence tight net (cf. preliminaries). Given $\varepsilon > 0$ there exists a compact set $K \subset \complement V$ such that $\overline{\lim}_\alpha \eta_\alpha(\complement V \setminus K) < \varepsilon$. Then $\overline{\lim}_\alpha \eta_\alpha(\complement U \setminus (K \cap \complement U)) < \varepsilon$ and $K \cap \complement U$ is compact since $\complement U$ is closed. Thus also $(\eta_\alpha \mid \complement U)$ is a tight and hence compact net.

(ii) ⇒ (i) Obviously condition (a) implies condition (a) of Proposition 7.1. Hence $(S_\alpha)_{\alpha\in I}$ is a compact net in $(\mathfrak{S}(G), \mathscr{E}_c)$. Let $(S_{\alpha(j)})_{j\in J}$ be a convergent subnet with limit $S' \in \mathfrak{S}(G)$. If A' is the generating functional of S' we have by condition (a) (together with Remark 5.2) and Proposition 6.4(ii) $\langle A'(D)u, v\rangle = \langle A(D)u, v\rangle$ for all $D \in \operatorname{Irr}(G)$ and $u, v \in \mathscr{H}_0(D)$. Since $\mathscr{H}_0(D)$ is dense in $\mathscr{H}(D)$ (Lemma 1.1(ii)) this implies $A'(D) \mid \mathscr{H}_0(D) = A(D) \mid \mathscr{H}_0(D)$ for all $D \in \operatorname{Irr}(G)$. Proposition 4.2 yields $S' = S$. Thus we must have $\mathscr{E}_c$-$\lim_\alpha S_\alpha = S$. ∎

COROLLARY 2. *We keep the notations of Corollary 1. Then the following assertions are equivalent*:

(i) $(S_\alpha)_{\alpha\in I}$ *is a compact net in* $(\mathfrak{S}(G), \mathscr{T}_c)$.

(ii) (a) $\overline{\lim}_\alpha |\langle A_\alpha(D)u, u\rangle| < \infty$ *for all* $D \in \operatorname{Irr}(G)$ *and* $u \in \mathscr{H}_0(D)$.

(b) $(\eta_\alpha \mid \complement U)_{\alpha\in I}$ *is a compact net for any open* $U \in \mathfrak{U}(G)$.

Proof. (i) ⇒ (ii) By Corollary 1 above conditions (a) and (b) hold for any universal subnet of $(S_\alpha)_{\alpha\in I}$. Since a net of measures is compact iff any of its universal subnets is compact conditions (a) and (b) also hold for $(S_\alpha)_{\alpha\in I}$.

(ii) ⇒ (i) This is Proposition 7.1. ∎

COROLLARY 3. *Let G be a Lie projective group and $\mathfrak{T}$ a subset of $\mathfrak{S}(G)$. For any $S \in \mathfrak{T}$ let A_S denote the generating functional of S and η_S its Lévy measure. Then the following assertions are equivalent*:

(i) $\mathfrak{T}$ *is relatively compact in* $(\mathfrak{S}(G), \mathscr{T}_c)$.

(ii) (a) $\sup\{|\langle A_S(D)u, u\rangle| \mid S \in \mathfrak{T}\} < \infty$ *for all* $D \in \operatorname{Irr}(G)$ *and* $u \in \mathscr{H}_0(D)$.

(b) $\{(\eta_S \mid \complement U) \mid S \in \mathfrak{T}\}$ *is relatively* $\mathscr{T}_w$*-compact for any open* $U \in \mathfrak{U}(G)$.

Proof. This follows immediately by Corollary 2 since a set Y in a completely regular space X is relatively compact iff any net in Y is a compact net. ∎

Remark. On a locally compact Abelian group G our compactness criteria for subsets of $(\mathfrak{S}(G), \mathscr{T}_c)$ yield the compactness conditions of Parthasarathy for infinitely divisible probability measures on G [20, IV, Theorem 9.1]. This can be seen in the following way:

Let $(\mu^{(\alpha)})_{\alpha\in I}$ be a net of infinitely divisible measures $\mu^{(\alpha)} \in \mathscr{M}^1(G)$ without idempotent factors. Using Fourier transformation it can be easily proved that there exists a unique convolution semigroup $S_\alpha = (\nu_t^{(\alpha)})_{t>0} \in \mathfrak{S}(G)$ of symmetric measures such that $\nu_1^{(\alpha)} = \mu^{(\alpha)} * \tilde{\mu}^{(\alpha)}$ $(\alpha \in I)$. Let $\mu^{(\alpha)}$ have the Parthasarathy representation $(x_\alpha, F_\alpha, \varphi_\alpha)$ [20, IV, Theorem 7.1]. If A_α denotes the generating functional of S_α then we obviously have $A_\alpha(\chi) = 2\{\int [\operatorname{Re}\chi - 1]\, dF_\alpha - \varphi_\alpha(\chi)\}$ for all characters χ of G. Moreover $\eta_\alpha = (F_\alpha + \tilde{F}_\alpha) \mid G^x$ is the Lévy measure of S_α.

Let us now assume that there exist elements $y_\alpha \in G$ such that the net $(\mu^{(\alpha)} * \varepsilon_{y_\alpha})_{\alpha\in I}$ $\mathscr{T}_w$-converges to a measure $\mu \in \mathscr{M}^1(G)$ without idempotent factors. Applying Fourier transformation we obtain the existence of a convolution semigroup $S = (\nu_t)_{t>0} \in \mathfrak{S}(G)$ with $\nu_1 = \mu * \tilde{\mu}$ such that $\mathscr{T}_c$-$\lim_\alpha S_\alpha = S$. Hence Corollary 3 of Proposition 7.1 yields Parthasarathy's result [20, IV, Theorem 9.1].

8. *Infinitesimal Systems of Probability Measures*

Let $\mathfrak{I} = (\mu_{nk})_{k=1,\ldots,k_n; n\geqslant 1}$ be a (triangular) system of probability measures on the locally compact group G. For any $n \in \mathbb{N}$ we define $\mu_n := \mu_{n1} * \cdots * \mu_{nk_n}$ and call $(\mu_n)_{n\geqslant 1}$ the sequence of *row products* of $\mathfrak{I}$.

$\mathfrak{I}$ is said to be *infinitesimal* if we have

$$\lim_n \max_{1\leqslant k\leqslant k_n} \mu_{nk}\left(\complement U\right) = 0 \qquad \text{for all} \quad U \in \mathfrak{U}(G).$$

$\mathfrak{I}$ is said to be *commutative* if we have

$$\mu_{nk} * \mu_{nl} = \mu_{nl} * \mu_{nk} \qquad \text{for } k, l = 1,\ldots, k_n \text{ and for all } n \in \mathbb{N}.$$

$\mathfrak{I}$ is said to be *convergent* with limit μ if we have

$$\mu \in \mathscr{M}^1(G) \qquad \text{and} \qquad \mathscr{E}_w\text{-}\lim_n \mu_n = \mu.$$

LEMMA 8.1. *$\mathfrak{I}$ is infinitesimal if and only if for any $D \in \operatorname{Irr}(G)$ and for all $u \in \mathscr{H}_0(D)$ we have*

$$\text{(I)} \qquad \lim_n \max_{1\leqslant k\leqslant k_n} \|\hat{\mu}_{nk}(D)u - u\| = 0.$$

Proof. 1. Let $\mathfrak{I}$ be infinitesimal. For $D \in \operatorname{Irr}(G)$ and $u \in \mathscr{H}_0(D)$ we define $l_n \in \{1,\ldots, k_n\}$ by $\|\hat{\mu}_{nl_n}(D)u - u\| = \max\{\|\hat{\mu}_{nk}(D)u - u\| \mid 1 \leqslant k \leqslant k_n\}$(all $n \in \mathbb{N}$). By assumption we have $\lim_n \mu_{nl_n}(\complement U) = 0$ for all $U \in \mathfrak{U}(G)$, i.e., $\mathscr{E}_w$-$\lim_n \mu_{nl_n} = \varepsilon_e$. But this implies $\lim_n \|\hat{\mu}_{nl_n}(D)u - u\| = 0$.

2. We assume (I). Let $U \in \mathfrak{U}(G)$. We define $l_n \in \{1,\ldots, k_n\}$ by $\mu_{nl_n}(\complement U) := \max\{\mu_{nk}(\complement U) \mid 1 \leqslant k \leqslant k_n\}$ (all $n \in \mathbb{N}$). By (I) we have $\lim_n \|\hat{\mu}_{nl_n}(D)u - u\| = 0$ for all $D \in \operatorname{Irr}(G)$ and $u \in \mathscr{H}_0(D)$. Lemma 2.1 yields $\mathscr{E}_w$-$\lim_n \mu_{nl_n} = \varepsilon_e$ and thus $\lim_n \mu_{nl_n}(\complement U) = 0$. ∎

We define $\lambda_{nk} := \mu_{nk} - \varepsilon_e$, $\nu_{nk} := \exp(\lambda_{nk})$ $(1 \leqslant k \leqslant k_n)$ and $\nu_n := \nu_{n1} * \cdots * \nu_{nk_n}$, $\lambda_n := \sum_{k=1}^{k_n} \lambda_{nk}$, $\kappa_n := \sum_{k=1}^{k_n} \mu_{nk}$. $\mathfrak{I}_a := (\nu_{nk})_{k=1,\ldots,k_n; n\geqslant 1}$ is called the *accompanying system* of $\mathfrak{I}$; its sequence of row products is $(\nu_n)_{n\geqslant 1}$. If $\mathfrak{I}$ is commutative $\mathfrak{I}_a$ is commutative too and we have $\nu_n = \exp(\lambda_n)$ $(n \in \mathbb{N})$. $\mathfrak{I}_a$ is infinitesimal if and only if $\mathfrak{I}$ is infinitesimal. [This follows from $\nu_{nk} \geqslant e^{-1}\mu_{nk}$ respectively from $\|\hat{\nu}_{nk}(D)u - u\| \leqslant \|\hat{\mu}_{nk}(D)u - u\|$ together with Lemma 8.1.]

PROPOSITION 8.1. *Let $\mathfrak{I}$ be a commutative and infinitesimal system which satisfies the condition*

$$\text{(B)} \qquad \varlimsup_n \sum_{k=1}^{k_n} \|\hat{\mu}_{nk}(D)u - u\| < \infty \qquad \textit{for all } D \in \operatorname{Irr}(G),\ u \in \mathscr{H}_0(D).$$

Then we have $\lim_n |\langle \hat{\mu}_n(D)u, v\rangle - \langle \hat{\nu}_n(D)u, v\rangle| = 0$ *for all* $D \in \mathrm{Irr}(G)$, $u \in \mathscr{H}_0(D)$ *and* $v \in \mathscr{H}(D)$.

Proof. We fix a representation $D \in \mathrm{Irr}(G)$ and define $T_{nk} := \hat{\mu}_{nk}(D) - D(e)$ (for all k, n). Then we have $\|T_{nk}\| \leqslant 2$. Furthermore with $S_{nk} := \hat{\mu}_{n1}(D) \cdots \hat{\mu}_{n,k-1}(D) \hat{\nu}_{n,k+1}(D) \cdots \hat{\nu}_{nk_n}(D)$ we have $\|S_{nk}\| \leqslant 1$ and (since $\mathfrak{I}$ is commutative)

$$\hat{\mu}_n(D) - \hat{\nu}_n(D) = \sum_{k=1}^{k_n} S_{nk}(D)(\hat{\mu}_{nk}(D) - \hat{\nu}_{nk}(D)).$$

Finally we define

$$A_{nk} := \frac{1}{2!} D(e) + \frac{1}{3!} T_{nk} + \frac{1}{4!} T_{nk}^2 + \cdots .$$

We have $\|A_{nk}\| \leqslant 1/2! + (1/3!)2 + (1/4!)2^2 + \cdots \leqslant e^2$ and $\hat{\mu}_{nk}(D) - \hat{\nu}_{nk}(D) = \hat{\mu}_{nk}(D) - [D(e) + T_{nk} + (1/2!)T_{nk}^2 + \cdots] = -A_{nk}T_{nk}^2$. Obviously A_{nk}, S_{nk} and T_{nk} are pairwise permutable ($1 \leqslant k \leqslant k_n$).

Let $u \in \mathscr{H}_0(D)$, $v \in \mathscr{H}(D)$. We have the following estimation:

$$\begin{aligned}
&|\langle [\hat{\mu}_n(D) - \hat{\nu}_n(D)]u, v\rangle| \\
&= \left| \sum_{k=1}^{k_n} \langle S_{nk}[\hat{\mu}_{nk}(D) - \hat{\nu}_{nk}(D)]u, v\rangle \right| \\
&\leqslant \sum_{k=1}^{k_n} |\langle S_{nk}A_{nk}T_{nk}^2 u, v\rangle| = \sum_{k=1}^{k_n} |\langle S_{nk}A_{nk}T_{nk}u, T_{nk}^* v\rangle| \\
&\leqslant \sum_{k=1}^{k_n} \|S_{nk}\| \, \|A_{nk}\| \, \|T_{nk}u\| \, \|T_{nk}^* v\| \leqslant e^2 \sum_{k=1}^{k_n} \|T_{nk}u\| \, \|T_{nk}^* v\|.
\end{aligned}$$

We are left to show $\lim_n \sum_{k=1}^{k_n} \|T_{nk}u\| \, \|T_{nk}^* v\| = 0$: Condition (B) yields the existence of a constant $c > 0$ such that $\sum_{k=1}^{k_n} \|T_{nk}u\| \leqslant c$ for all $n \in \mathbb{N}$. Thus it suffices to prove $\lim_n \max\{\|T_{nk}^* v\| \mid 1 \leqslant k \leqslant k_n\} = 0$: Since $\mathfrak{I}$ is infinitesimal the system $(\tilde{\mu}_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ is infinitesimal too. Now $T_{nk}^* = \hat{\mu}_{nk}(D)^* - D(e) = \hat{\tilde{\mu}}_{nk}(D) - D(e)$ and part 1 of the proof of Lemma 8.1 yield the statement. ▮

COROLLARY. *Let G be a Lie projective group and let $\mathfrak{I}$ be a commutative and infinitesimal system in $\mathscr{M}^1(G)$ satisfying $\overline{\lim}_n \sum_{k=1}^{k_n} \mu_{nk}(\complement\, U) < \infty$ for all $U \in \mathfrak{U}(G)$ and the condition*

(WB) $\displaystyle\overline{\lim_n} \sum_{k=1}^{k_n} |\langle \hat{\mu}_{nk}(D)u - u, u\rangle| < \infty$ *for all* $D \in \mathrm{Irr}(G)$, $u \in \mathscr{H}_0(D)$.

Then $\mathfrak{I}$ satisfies condition (B); *hence the conclusion of Proposition* 8.1 *is valid.*

Proof. This is an immediate consequence of the corollary of Lemma 5.4 since any $D \in \operatorname{Irr}(G)$ can be factorized over a Lie quotient group of G (cf. Section 6). ∎

Remark 1. Let $\mathfrak{J}$ be a commutative and infinitesimal system satisfying condition (B). Then $\mathfrak{J}$ enjoys the following property:

(C) $\mathfrak{J}$ is convergent if and only if the accompanying system $\mathfrak{J}_a$ is convergent, and in the affirmative case their limits coincide.

[This is an immediate consequence of Proposition 8.1 and Lemma 2.1.]

Remark 2. Proposition 8.1 becomes incorrect if we substitute condition (B) by the weaker condition

$$(\mathrm{B}') \quad \overline{\lim_n} \left\| \sum_{k=1}^{k_n} (\hat{\mu}_{nk}(D)u - u) \right\| < \infty \qquad \text{for all } D \in \operatorname{Irr}(G), u \in \mathscr{H}_0(D).$$

Let us consider the following *counterexample*:

Let $G = \mathbb{R}$ and $\mu_{nk} := \varepsilon_{x_{nk}}$, where $x_{nk} := (-1)^k/\sqrt{n}$ for all $k = 1,..., k_n := n$ ($n \in \mathbb{N}$). Obviously the system $\mathfrak{J} := (\mu_{nk})$ is commutative and infinitesimal. We have $x_n := x_{n1} + \cdots + x_{nn} = 0$ for even n and $x_n = -1/\sqrt{n}$ for odd n, thus $\lim x_n = 0$ and consequently $\lim \mu_n = \lim \varepsilon_{x_n} = \varepsilon_0$. Therefore $\mathfrak{J}$ is also convergent. Furthermore we have $\hat{\mu}_{nk}(y) = \exp[iy((-1)^k/\sqrt{n})]$; hence $\sum_{k=1}^n [\hat{\mu}_{nk}(y) - 1] = n[\cos(y/\sqrt{n}) - 1]$ for even n resp. $=(n-1)[\cos(y/\sqrt{n}) - 1] + [\exp(-iy/\sqrt{n}) - 1]$ for odd n. But $\lim n(1 - \cos(y/\sqrt{n})) = y^2/2$. Consequently we have $\lim_n \sum_{k=1}^n [\hat{\mu}_{nk}(y) - 1] = -y^2/2$ for all $y \in \mathbb{R}$.In particular, condition (B′) is satisfied for $\mathfrak{J}$. On the other side we get $\lim \hat{\mu}_n(y) = 1$ and $\lim \hat{\nu}_n(y) = \lim \exp\{\sum_{k=1}^n [\hat{\mu}_{nk}(y) - 1]\} = \exp(-y^2/2)$ for all $y \in \mathbb{R}$. Therefore $\overline{\lim} |\hat{\mu}_n(y) - \hat{\nu}_n(y)| = 1 - \exp(-y^2/2) > 0$ for $y \neq 0$. By Proposition 8.1 condition (B) cannot be satisfied for $\mathfrak{J}$.

Remark 3. Let the system $\mathfrak{J} = (\mu_{nk})$ be *identically distributed*, i.e., $\mu_{n1} = \cdots = \mu_{nk_n}$ for any $n \in \mathbb{N}$, and $\lim_n k_n = \infty$. Obviously $\mathfrak{J}$ is commutative. $\mathfrak{J}$ satisfies condition (B) if it satisfies condition (B′) and in the affirmative case $\mathfrak{J}$ is infinitesimal. [We have $\sum_{k=1}^{k_n} \|\hat{\mu}_{nk}(D)u - u\| = k_n \|\hat{\mu}_{n1}(D)u - u\| = \|\sum_{k=1}^{k_n} (\hat{\mu}_{nk}(D)u - u)\|$, i.e., the equivalence of (B) and (B′) for $\mathfrak{J}$. Furthermore if $\mathfrak{J}$ satisfies (B) we get $\lim \|\hat{\mu}_{n1}(D)u - u\| = 0$ for all $D \in \operatorname{Irr}(G)$, $u \in \mathscr{H}_0(D)$ (observe $\lim k_n = \infty$). Hence $\mathfrak{J}$ is infinitesimal by Lemma 8.1.]

Remark 4. Let G be a Lie group with canonical coordinates $\{x_1,..., x_p\}$ and Hunt function φ as usual (cf. Section 5). Wehn [29] derived

Proposition 8.1 (for a commutative and infinitesimal system $\mathfrak{I} = (\mu_{nk})$) assuming the validity of the following two conditions:

(W1) $$\overline{\lim_n} \sum_{k=1}^{k_n} \int \varphi \, d\mu_{nk} < \infty.$$

(W2) $$\overline{\lim_n} \sum_{k=1}^{k_n} \left| \int x_i \, d\mu_{nk} \right| < \infty \qquad \text{for} \quad i = 1,..., p.$$

But (W1) and (W2) imply condition (B) by the corollary of Lemma 5.4. Thus Wehn's result follows from Proposition 8.1.

Remark 5. Let $\mathfrak{I} = (\mu_{nk})$ be a commutative and infinitesimal system in $\mathscr{M}^1(G)$. One may ask for conditions other than (B) that imply property (C) for $\mathfrak{I}$. There are the following two interesting results: If G is totally disconnected and in addition compact or Abelian then $\mathfrak{I}$ always has property (C). Conversely if any system $\mathfrak{I}$ has property (C) then G is necessarily totally disconnected [10, Theorem 2.1].

We are now going to apply our compactness criteria from Section 7 to triangular systems of probability measures.

PROPOSITION 8.2. *Let G be a Lie projective group and let $\mathfrak{I} = (\mu_{nk})_{k=1,\dots,k_n; n \geq 1}$ be a commutative and infinitesimal system in $\mathscr{M}^1(G)$ satisfying condition* (WB). *Moreover for any open $U \in \mathfrak{U}(G)$ let $(\sum_{k=1}^{k_n} \mu_{nk} \mid \complement U)_{n \geq 1}$ be a compact sequence.*

Then the sequences $(\mu_n)_{n \geq 1}$ and $(\nu_n)_{n \geq 1}$ of row products of $\mathfrak{I}$ and $\mathfrak{I}_a$, respectively, are uniformly tight. Their limit points coincide, and any of these limit points is embeddable into a convolution semigroup.

Proof. Let us define $\kappa_n := \sum_{k=1}^{k_n} \mu_{nk}$ and $\nu_t^{(n)} := \exp t(\kappa_n - \kappa_n(G)\varepsilon_e)$ ($n \in \mathbb{N}$, $t \geq 0$). Obviously $A_n := \kappa_n - \kappa_n(G)\varepsilon_e$ is the generating functional of the Poisson semigroup $S_n := (\nu_t^{(n)})_{t \geq 0}$ and $\kappa_n \mid G^x$ is its Lévy measure. By Proposition 7.1 $(S_n)_{n \geq 1}$ is a compact sequence in $(\mathfrak{S}(G), \mathscr{T}_c)$. Hence the sequence $(\nu_1^{(n)})_{n \geq 1}$ of row products of the accompanying system $\mathfrak{I}_a$ of $\mathfrak{I}$ is compact and thus uniformly tight.

Let us assume that $(\mu_n)_{n \geq 1}$ is uniformly tight too. Then by the corollary of Proposition 8.1 the limit points of (μ_n) and $(\nu_1^{(n)})$ coincide. But since the sequence $(S_n)_{n \geq 1}$ is compact any limit point of $(\nu_1^{(n)})$ lies on a convolution semigroup. Thus we are left to show that $(\mu_n)_{n \geq 1}$ is uniformly tight.

Let H be a compact normal subgroup in G such that G/H is a Lie group and let π denote the canonical mapping of G onto G/H. Since H is compact it suffices to prove that the sequence $(\pi(\mu_n))$ is uniformly tight. Obviously $(\pi(\nu_1^{(n)}))$ is uniformly tight. Thus without loss of generality let G be a Lie group. Then G and therefore also $(\mathscr{M}^1(G), \mathscr{T}_w)$ are metrizable [20, p. 43]. We choose a subsequence $(\mu_{n'})$ of (μ_n). Then there is a further subsequence $(\mu_{n''})$ of $(\mu_{n'})$ such that $\mathscr{C}_w$-$\lim \nu_1^{(n'')} = \nu$ exists. By Proposition 8.1 we have $\lim\langle \hat{\mu}_{n''}(D)u, v\rangle = \lim\langle \hat{\nu}_1^{(n'')}(D)u, v\rangle = \langle \hat{\nu}(D)u, v\rangle$ for all $D \in \operatorname{Irr}(G)$, $u, v \in \mathscr{H}_0(D)$. Now Lemma 2.1 implies $\mathscr{C}_w$-$\lim \mu_{n''} = \nu$. Hence (μ_n) is a compact sequence and therefore uniformly tight. ∎

Remark 6. It should be mentioned that for root compact groups some of our convergence theorems can be slightly improved (cf. [15, 21]). If G is a compact group the last assumption of Proposition 8.2 follows from condition (WB).

9. *Central Limit Theorem and Law of Large Numbers*

Let G be a locally compact group. A family $(\mu_t)_{t>0}$ of non-degenerate measures in $\mathscr{M}^1(G)$ is called a *Gaussian semigroup* if we have $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geqslant 0$, and $\lim_{t\downarrow 0}(1/t)\mu_t(\complement U) = 0$ for all $U \in \mathfrak{U}(G)$. A (non-degenerate) measure $\mu \in \mathscr{M}^1(G)$ is called a *Gaussian measure* if there exists a Gaussian semigroup $(\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ such that $\mu_1 = \mu$ [15, 6.2.1].

Remark 1. Any Gaussian semigroup is a convolution semigroup.

Remark 2. If $(\mu_t)_{t>0}$ is a Gaussian semigroup then any μ_t, $t > 0$ is a Gaussian measure and is supported by the connected component of e in G [15, 6.2.3].

Remark 3. Let $(\mu_t)_{t>0}$ be a non-degenerate convolution semigroup in $\mathscr{M}^1(G)$ with canonical decomposition (A_1, A_2, η). $(\mu_t)_{t>0}$ is a Gaussian semigroup if and only if $\eta = 0$ or equivalently if $\lim_{t\downarrow 0}(1/t)\mu_t(f) = 0$ for all $f \in \mathscr{C}^b(G)$ with $e \notin \operatorname{supp}(f)$ (cf. Section 3).

PROPOSITION 9.1. *Let G be a Lie projective group and let $(S_\alpha)_{\alpha\in I}$ be a net in $\mathfrak{S}(G)$. Let η_α be the Lévy measure of S_α. We assume that the net $(S_\alpha)_{\alpha\in I}$ $\mathscr{C}_c$-converges to a non-degenerate semigroup $S \in \mathfrak{S}(G)$. Then the following assertions are equivalent*:

(i) *S is a Gaussian semigroup.*

(ii) $\lim_\alpha \eta_\alpha(\complement U) = 0$ *for all $U \in \mathfrak{U}(G)$.*

Proof. This follows immediately from Proposition 6.4(iii) and Remark 3 above. ∎

COROLLARY. *Let G be a locally compact group and $(S_\alpha)_{\alpha\in I}$ a net of Gaussian semigroups in $\mathfrak{S}(G)$ that $\mathscr{C}_c$-converges to a non-degenerate semigroup $S \in \mathfrak{S}(G)$. Then S is a Gaussian semigroup too.*

Proof. By Remark 2 we can assume that G is connected and hence Lie projective. But then Proposition 9.1 and Remark 3 apply. ∎

Remark 4. This corollary was first proved by Hazod (cf. [15, 6.2.23]).

PROPOSITION 9.2. *Let G be a Lie projective group and let $(\kappa_\alpha)_{\alpha\in I}$ a net in $\mathscr{M}^b_+(G)$ which satisfies the following two conditions*:

(a) $\overline{\lim}_\alpha |\langle\hat{\kappa}_\alpha(D)u - \kappa_\alpha(G)u, u\rangle| < \infty$ *for all* $D \in \mathrm{Irr}(G)$, $u \in \mathscr{H}_0(D)$.

(b) $\lim_\alpha \kappa_\alpha(\complement\, U) = 0$ *for all* $U \in \mathfrak{U}(G)$.

For any $\alpha \in I$ we define the Poisson semigroup $S_\alpha = (\nu_t^{(\alpha)})_{t>0}$ by $\nu_t^{(\alpha)} := \exp t(\kappa_\alpha - \kappa_\alpha(G)\varepsilon_e)$ (all $t \geqslant 0$).

Then $(S_\alpha)_{\alpha\in I}$ is a compact net in $(\mathfrak{S}(G), \mathscr{T}_c)$. Any of its limit points is a Gaussian semigroup or a degenerate semigroup.

Proof. $A_\alpha := \kappa_\alpha - \kappa_\alpha(G)\varepsilon_e$ is the generating functional of S_α and $\kappa_\alpha \mid G^x$ is its Lévy measure. Hence by Proposition 7.1 $(S_\alpha)_{\alpha\in I}$ is a compact net in $(\mathfrak{S}(G), \mathscr{T}_c)$.

Without loss of generality, now let $(S_\alpha)_{\alpha\in I}$ be a universal net that $\mathscr{T}_c$-converges to $S \in \mathfrak{S}(G)$. Let η be the Lévy measure of S. By condition (b) and Proposition 6.4(iii) we have $\eta = 0$. Hence by Remark 3 the semigroup S is Gaussian or degenerate. ∎

Now we are ready for a version of the central limit theorem.

PROPOSITION 9.3. *Let G be a Lie projective group and $\mathfrak{I} = (\mu_{nk})_{k=1,\dots,k_n;n\geqslant 1}$ a commutative system in $\mathscr{M}^1(G)$ satisfying condition* (WB). *Moreover we assume*

$$\text{(G)} \qquad \lim_n \sum_{k=1}^{k_n} \mu_{nk}\left(\complement\, U\right) = 0 \qquad \textit{for all} \quad U \in \mathfrak{U}(G).$$

Then the sequence $(\mu_n)_{n\geqslant 1}$ of row products of $\mathfrak{I}$ is uniformly tight and any of its non-degenerate limit points is a Gaussian measure.

Proof. First of all we remark that condition (G) implies the infinitesimality of $\mathfrak{I}$.

Let us define $\kappa_n := = \sum_{k=1}^{k_n} \mu_{nk}$, $\nu_t^{(n)} := \exp t(\kappa_n - \kappa_n(G)\varepsilon_e)$ and $S_n := (\nu_t^{(n)})_{t>0}$ $(n \in \mathbb{N},\ t \geqslant 0)$. By Proposition 8.2 the sequence $(\mu_n)_{n\geqslant 1}$ is uniformly tight and has the same limit points as $(\nu_1^{(n)})_{n\geqslant 1}$. But by Proposition 9.2 the sequence (S_n) is compact and its limit points are Gaussian or degenerate semigroups. ∎

Remark 5. Proposition 9.3 admits the following partial converse: Let G be a locally compact group and $\nu \in \mathscr{M}^1(G)$ a Gaussian measure. Then there exists a commutative system $(\mu_{nk})_{k=1,\dots,k_n;n\geqslant 1}$ in $\mathscr{M}^1(G)$ satisfying conditions (B) and (G) and convergent with limit ν. [There exists a Gaussian semigroup $(\nu_t)_{t>0}$ in $\mathscr{M}^1(G)$ with generating functional A such that $\nu_1 = \nu$. We define $\mu_{nk} := \nu_{1/n}$ for $k = 1,\dots, k_n := n$ (all $n \in \mathbb{N}$). Since $\lim n[\hat{\nu}_{1/n}(D)u - u] = A(D)u$ $(D \in \mathrm{Irr}(G),\ u \in \mathscr{H}_0(D))$, $\lim n\nu_{1/n}(\complement\, U) = 0$

$(U \in \mathfrak{U}(G))$ and $\nu_{1/n}^n = \nu (n \in \mathbb{N})$ the system (μ_{nk}) obviously has the desired properties.]

Our version of the law of large numbers is merely a special case of Proposition 9.3.

PROPOSITION 9.4. *Let G be a Lie projective group and $\mathfrak{I} = (\mu_{nk})_{k=1,\dots,k_n; n \geqslant 1}$ a commutative system in $\mathcal{M}^1(G)$ satisfying conditions* (WB), (G) *and*

$$\text{(D)} \quad \lim_n \sum_{k=1}^{k_n} \operatorname{Re}\langle \hat{\mu}_{nk}(D)u - u, u \rangle = 0 \qquad \textit{for all} \quad D \in \operatorname{Irr}(G), u \in \mathscr{H}_0(D).$$

Then the sequence $(\mu_n)_{n \geqslant 1}$ of row products of $\mathfrak{I}$ is uniformly tight and has only degenerate limit points.

Proof. We keep the notations of the proof of Proposition 9.3. But by this very proposition and its proof the sequence $(\mu_n)_{n \geqslant 1}$ is uniformly tight and has the same limit points as the sequence $(\nu_1^{(n)})_{n \geqslant 1}$. Hence it suffices to show that any limit point of the sequence $(S_n)_{n \geqslant 1}$, $S_n := (\nu_t^{(n)})$ in $\mathfrak{S}(G)$ is a degenerate semigroup.

Let $(n(\alpha))_{\alpha \in I}$ be a universal subnet of $\mathbb{N}$ and $S := \mathscr{E}_c\text{-}\lim_\alpha S_{n(\alpha)}$. If $A_n = \kappa_n - \kappa_n(G)\varepsilon_e$ and A denote the generating functionals of S_n and S, resp., we have $\lim_\alpha \langle A_{n(\alpha)}(D)u, u \rangle = \langle A(D)u, u \rangle$ for all $D \in \operatorname{Irr}(G)$, $u \in \mathscr{H}_0(D)$ (Proposition 6.4(ii)). On the other hand condition (D) yields $\lim_\alpha \operatorname{Re}\langle A_{n(\alpha)}(D)u, u \rangle = 0$. Hence $\operatorname{Re}\langle A(D)u, u \rangle = 0$ for all $D \in \operatorname{Irr}(G)$, $u \in \mathscr{H}_0(D)$. By Proposition 4.3(i) S is a degenerate semigroup. ∎

COROLLARY. *Let G be a Lie projective group and $\mathfrak{I} = (\mu_{nk})_{k=1,\dots,k_n; n \geqslant 1}$ a commutative system in $\mathcal{M}^1(G)$ satisfying conditions* (WB), (G) *and*

$$\text{(D}'\text{)} \quad \lim_n \sum_{k=1}^{k_n} \langle \hat{\mu}_{nk}(D)u - u, u \rangle = 0 \qquad \textit{for all} \quad D \in \operatorname{Irr}(G), u \in \mathscr{H}_0(D).$$

Then we have $\mathscr{E}_w\text{-}\lim \mu_n = \varepsilon_e$ for the sequence $(\mu_n)_{n \geqslant 1}$ of row products of $\mathfrak{I}$.

Proof. Keeping the notations of the proof of Proposition 9.4 we observe that condition (D′) and Proposition 4.3(ii) imply $S = (\varepsilon_e)$. Hence the sequence (S_n) has only one limit point, namely, the trivial semigroup, i.e., $\mathscr{E}_c$-$\lim S_n = (\varepsilon_e)$. ∎

Remark 6. If the system $\mathfrak{I}$ is identically distributed (Remark 8.3) then condition (D′) obviously implies condition (WB).

Remark 7. On Lie groups a central limit theorem and a law of large numbers have already been proved by Wehn [29]. The results are contained in [9, Theorems 4.4.2, 4.3.1a]. The corollary of Lemma 5.4 shows at once that the conditions posed on the system $\mathfrak{J}$ by Wehn imply our conditions (WB) and (G) in Proposition 9.3, respectively, (WB), (G) and (D) in Proposition 9.4.

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