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Chern–Weil homomorphism in twisted equivariant cohomology ☆

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ABSTRACT

We describe the Cartan and Weil models of twisted equivariant cohomology together with the Cartan homomorphism among the two, and we extend the Chern–Weil homomorphism to the twisted equivariant cohomology. We clarify that in order to have a cohomology theory, the coefficients of the twisted equivariant cohomology must be taken in the completed polynomial algebra over the dual Lie algebra of G. We recall the relation between the equivariant cohomology of exact Courant algebroids and the twisted equivariant cohomology, and we show how to endow with a generalized complex structure the finite-dimensional approximations of the Borel construction $M \times_G EG_k$, whenever the generalized complex manifold M possesses a Hamiltonian G-action.

1. Introduction

In ordinary equivariant cohomology there are two well-known models, the bigger while more geometrical Weil model and the smaller while more algebraic Cartan model. In [12], the authors defined twisted equivariant cohomology following the Cartan model and showed that the twisted equivariant cohomology satisfies the cohomology axioms under some natural assumptions, for example, that the group G is a compact Lie group. Under these assumptions, the corresponding Cartan model is twisted by a closed equivariant 3-form (Section 4.5). In the non-twisted case, the Cartan homomorphism gives a quasi-isomorphism between the Weil and the Cartan models. In this article, we show (Sections 2, 3) that in the twisted case, the twisted equivariant theory also has the corresponding twisted Weil model and that the Cartan homomorphism could be extended to the twisted models yielding also a quasi-isomorphism between them. Moreover, we show that the Chern–Weil homomorphism can also be extended to the twisted equivariant case, which we use to demonstrate that the twisted equivariant cohomology is isomorphic to the twisted cohomology of the Borel construction; this provides us with an alternative proof of the fact that the twisted cohomology is indeed a cohomology theory (cf. [12]).

One subtlety comes up when defining the equivariant theory in the twisted case. In the ordinary Cartan model, we consider the complex $(\Omega^*(M) \otimes S(\mathfrak{g}^*))^G$ with the equivariant differential d_G. In the twisted case we have to consider the complex $(\Omega^*(M) \otimes \widehat{S}(\mathfrak{g}^*))^G$ where the $\widehat{S}(\mathfrak{g}^*)$ is the completion of the algebra $S(\mathfrak{g}^*)$, because otherwise, the twisted cohomology defined over the uncompleted algebra could be non-finitely generated as we show in Appendix A.

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The twisted equivariant cohomology appeared as the equivariant theory of generalized complex manifolds [6,12–14]. Then in the case of a Hamiltonian G-action on a generalized complex manifold M, when G is compact and the finite approximations BG_k of BG are symplectic, we apply the coupling construction using the principle G-bundle $EG_k \to BG_k$ to show that the finite approximations of the Borel construction $M \times_G EG_k$ are generalized complex as well. The explicit computation of the twisting form here coincides with the result given by the twisted version of the Chern–Weil homomorphism.

The structure of the article is the following. In Section 2 we recall the twisted equivariant cohomology in both the Cartan and the Weil models. We also clarify the subtlety about the completion mentioned above. The Chern–Weil homomorphism and its consequences are shown in Section 3. In Section 4 we recall the basics on the symmetries of the exact Courant algebroids as well as the definition of extended equivariant cohomology from [12]. Then we show that when the Lie group G is compact the extended equivariant cohomology is isomorphic to some twisted equivariant cohomology. In Section 5, we describe the coupling construction for Hamiltonian actions on generalized complex manifolds using principle G-bundles. Appendix A compares the cohomologies of the completed and uncompleted complexes for the twisted equivariant cohomology using Example 4.11.

2. Twisted equivariant cohomology

In as much as the twisted cohomology is defined twisting the differential of the De Rham complex with a closed 3-form, the twisted equivariant cohomology is defined by twisting the equivariant differential with a closed and equivariant 3-form.

We will define the twisted equivariant cohomology using the Cartan model and we will explicitly show the relation with the twisted Weil model. Then we will generalize the Chern–Weil map for twisted equivariant cohomology and we will finish by giving the topological counterpart of the twisted equivariant cohomology.

Remark 2.1. We would like to emphasize the fact that the twisted cohomology is a $\mathbb{Z}_2$-graded theory; therefore all inverse limits that will be carried out in this section will be $\mathbb{Z}_2$-graded. Let us see the difference with a simple example:

> Take the $\mathbb{Z}$-graded rings $H^*(\mathbb{C}P^n) = \mathbb{R}[x]/x^{n+1}$ where $|x| = 2$. The inverse limit of these $\mathbb{Z}$-graded rings is the polynomial algebra $H^*(\mathbb{C}P^\infty) = \mathbb{R}[x]$. Now, if one consider the same rings $H^\bullet(\mathbb{C}P^n) = \mathbb{R}[x]/x^{n+1}$ but $\mathbb{Z}_2$-graded, as in the case of twisted cohomology, the inverse limit of these rings gives the algebra of formal series $H^\bullet(\mathbb{C}P^\infty) = \mathbb{R}[\![x]\!]$.

To distinguish between $\mathbb{Z}$- and $\mathbb{Z}_2$-graded theories we will denote the former with an asterisk H^* and the latter with a bullet $H^\bullet$.

Let us start by recalling the models of Cartan and Weil for the equivariant cohomology (see [18]).

2.1. Equivariant cohomology

Following Weil one introduces a universal model for the curvature and connection on a principal G-bundle. The Weil algebra is then by definition

$$W(\mathfrak{g}) := S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$$

the tensor product of the symmetric algebra and the exterior algebra of $\mathfrak{g}^*$. If we denote with lower case letters $a, b, c, \ldots$ a base for the Lie algebra $\mathfrak{g}$ then we will denote by θ^a the variables dual to a of degree one that generate the exterior algebra, and by Ω^a the variables of degree two that generate the symmetric algebra.

The derivations and contractions on this algebra are generated by

$$\iota_a\theta^b = \delta_{ab}, \qquad \iota_a\Omega^b = 0, \qquad d\theta^a = \Omega^a - \frac{1}{2} f^a_{bc}\theta^b\theta^c, \qquad d\Omega^a = f^a_{bc}\Omega^b\theta^c,$$

where f^a_{bc} are the structural constants: $[b, c] = f^a_{bc}a$.

The tensor product $\Omega^*(M) \otimes W(\mathfrak{g})$ is a differential graded algebra with a $\mathfrak{g}$-action and derivations ι_a satisfying the standard identities of the contractions. The contractions ι_a act on the differential forms $\Omega^*(M)$ by contracting on the direction of the vector field X_a that a generates, but to make the notation less heavy we will simply denote ι_a the operator ι_{X_a}. The Lie derivative $\mathcal{L}_a$ is defined by the Cartan formula $\mathcal{L}_a = d\iota_a + \iota_a d$.

The basic subalgebra

$$\Omega^*_{\mathfrak{g}}(M) := \{\Omega^*(M) \otimes W(\mathfrak{g})\}_{bas} := \bigcap_a (\ker \mathcal{L}_a \cap \ker \iota_a)$$

is a differential graded algebra whose elements are called (Weil) equivariant differential forms and whose cohomology $H^*(\Omega^*_{\mathfrak{g}}(M))$ is the G-equivariant cohomology of M.

Cartan [7] (cf. [18]) showed that there is a smaller model for the equivariant forms which is given by the G-invariant forms on $(\Omega^*(M) \otimes S(\mathfrak{g}^*))^{\mathfrak{g}}$ with differential given by $d_{\mathfrak{g}} = d - \Omega^a\iota_a$. Let us denote the cohomology of this complex by

$$H^*_{\mathfrak{g}}(M) := H^*\big(\big(\Omega^*(M)\otimes S\big(\mathfrak{g}^*\big)\big)^{\mathfrak{g}}, d_{\mathfrak{g}}\big).$$

Cartan showed that there is a homomorphism

$$j : \Omega^*(M)\otimes S\big(\mathfrak{g}^*\big) \to \Omega^*(M)\otimes W(\mathfrak{g}),$$
$$\alpha \mapsto \prod_a \big(1-\theta^a\iota_a\big)\alpha$$

that induces a quasi-isomorphism of complexes

$$j : \big(\Omega^*(M)\otimes S\big(\mathfrak{g}^*\big)\big)^{\mathfrak{g}} \xrightarrow{\sim} \Omega^*_{\mathfrak{g}}(M) \tag{2.1}$$

and therefore the map j induces an isomorphism of equivariant cohomologies

$$j : H^*_{\mathfrak{g}}(M) \xrightarrow{\cong} H^*\big(\Omega^*_{\mathfrak{g}}(M)\big).$$

2.2. Twisted equivariant cohomology

Let us start by taking a closed equivariant 3-form $\mathcal{H} = H + \Omega^a\xi_a$ on the Cartan complex $(\Omega^*(M)\otimes S(\mathfrak{g}^*))^{\mathfrak{g}}$ with H a 3-form and ξ_a 1-forms on M.

The fact that $\mathcal{H}$ is closed implies that

$$d_{\mathfrak{g}}\mathcal{H} = \big(d-\Omega^b\iota_b\big)\big(H+\Omega^a\xi_a\big) = dH + \Omega^a(d\xi_a - \iota_a H) - \Omega^a\Omega^b\iota_b\xi_a = 0$$

which happens if and only if

$$dH = 0, \qquad d\xi_a - \iota_a H = 0, \quad \text{and} \quad \iota_a\xi_b = -\iota_b\xi_a.$$

The fact that $\mathcal{H}$ is equivariant implies that for all $b\in\mathfrak{g}$

$$\mathcal{L}_b\mathcal{H} = \mathcal{L}_b\mathcal{H} + \big(\mathcal{L}_b\Omega^a\big)\xi_a + \Omega^a(\mathcal{L}_b\xi_a) = \mathcal{L}_b\mathcal{H} + f^a_{cb}\Omega^c\xi_a + \Omega^a(\mathcal{L}_b\xi_a) = 0;$$

this happens if and only if $\mathcal{H}$ is $\mathfrak{g}$ invariant and $\xi_{[b,a]} = \mathcal{L}_b\xi_a$.

Let us now see what is the image in the Weil model of the three form $\mathcal{H}$. We need this in order to have a very explicit description of the twisted Chern–Weil homomorphism. Fortunately the expression of the three form turned out to be very simple, its proof not; we will reproduce here the calculations.

Proposition 2.2. *The image of $\mathcal{H}$ in $\Omega^*_{\mathfrak{g}}(M)$ under the quasi-isomorphism j defined in (2.1) is the basic three form*

$$\mathbf{H} := j(\mathcal{H}) = H + d\big(\theta^a\xi_a\big) - \frac{1}{2}d\big(\theta^p\theta^q\iota_q\xi_p\big).$$

Proof. We will proceed by expanding the derivations in the three form $\mathbf{H}$ and we will compare them with the expansion of $j(\mathcal{H})$.

Let us start by expanding $\mathbf{H}$:

$$\begin{aligned}
\mathbf{H} &= H + d\big(\theta^a\xi_a\big) - \frac{1}{2}d\big(\theta^p\theta^q\iota_q\xi_p\big)\\
&= H + \Omega^a\xi_a - \frac{1}{2}f^a_{bc}\theta^b\theta^c\xi_a - \theta^a d\xi_a - \frac{1}{2}\Omega^p\theta^q\iota_q\xi_p + \frac{1}{4}f^p_{rs}\theta^r\theta^s\theta^q\iota_q\xi_p\\
&\quad + \frac{1}{2}\theta^p\Omega^q\iota_q\xi_p - \frac{1}{4}f^q_{tu}\theta^p\theta^t\theta^u\iota_q\xi_p - \frac{1}{2}\theta^p\theta^q d\iota_b\xi_p\\
&= H + \Omega^a\xi_a - \frac{1}{2}\theta^b\theta^c\xi_{[b,c]} - \theta^a d\xi_a - \Omega^p\theta^q\iota_q\xi_p + \frac{1}{2}\theta^r\theta^s\theta^q\iota_q\xi_{[r,s]} - \frac{1}{2}\theta^p\theta^q d\iota_b\xi_p.
\end{aligned} \tag{2.2}$$

Numerate from left to right the expressions in line (2.2). We will match them with the expansion of $j(H_G)$. Let us calculate then:

$$\begin{aligned}
j(H_G) &= \big(1-\theta^e\iota_e\big)\big(1-\theta^c\iota_c\big)\big(1-\theta^b\iota_b\big)\big(H+\Omega^a\xi_a\big)\\
&= H + \Omega^a\xi_a - \theta^b\iota_b\big(H+\Omega^a\xi_a\big) + \frac{1}{2}\theta^b\theta^c\iota_c\iota_b\big(H+\Omega^a\xi_a\big) - \frac{1}{6}\theta^b\theta^c\theta^e\iota_e\iota_c\iota_b\big(H+\Omega^a\xi_a\big)\\
&= H + \Omega^a\xi_a - \theta^b d\xi_b - \theta^b\Omega^a\iota_b\xi_a + \frac{1}{2}\theta^b\theta^c\iota_c d\xi_b - \frac{1}{6}\theta^b\theta^c\theta^e\iota_e\iota_c d\xi_b.
\end{aligned}$$

As we have that

$$\iota_c d\xi_b = \mathcal{L}_c\xi_b - d\iota_c\xi_b = \xi_{[c,b]} - d\iota_c\xi_b,$$

then

$$j(H_G) = H + \Omega^a\xi_a - \theta^b d\xi_b - \theta^b\Omega^a\iota_b\xi_a + \frac{1}{2}\theta^b\theta^c\xi_{[c,b]} - \frac{1}{2}\theta^b\theta^c d\iota_c\xi_b - \frac{1}{6}\theta^b\theta^c\theta^e\iota_e\iota_c d\xi_b. \tag{2.3}$$

We see that the first 6 terms in (2.3) match all but the sixth term in (2.2). Let us then expand the last term in (2.3):

$$\begin{aligned}
-\frac{1}{6}\theta^b\theta^c\theta^e\iota_e\iota_c d\xi_b &= -\frac{1}{6}\theta^b\theta^c\theta^e\iota_e(\mathcal{L}_c\xi_b - d\iota_c\xi_b)\\
&= -\frac{1}{6}\theta^b\theta^c\theta^e(\iota_e\xi_{[c,b]} - \mathcal{L}_e\iota_c\xi_b)\\
&= -\frac{1}{6}\theta^b\theta^c\theta^e(\iota_e\xi_{[c,b]} - \iota_c\mathcal{L}_e\xi_b - \iota_{[e,c]}\xi_b)\\
&= -\frac{1}{6}\theta^b\theta^c\theta^e(\iota_e\xi_{[c,b]} - \iota_c\xi_{[e,b]} + \iota_b\xi_{[e,c]})\\
&= -\frac{1}{2}\theta^b\theta^c\theta^e\iota_e\xi_{[c,b]}
\end{aligned}$$

and we can see that it matches the sixth term in (2.2). Here we have used the fact that

$$\iota_{[e,c]}\xi_b = f^a_{ec}\iota_a\xi_b = -f^a_{ec}\iota_b\xi_a = -\iota_b\xi_{[e,c]}. \qquad \square$$

Let us note that as $\mathbf{H}$ is basic we have that $\iota_a\mathbf{H} = 0$, $\mathcal{L}_a\mathbf{H} = 0$ and $d\mathbf{H} = 0$.

Having in hand the closed three forms defined previously, we can now change the differential in the Cartan model as well as in the Weil model. But as the twisted equivariant cohomology is a $\mathbb{Z}_2$-graded theory, we first need to complete the symmetric algebra $S(\mathfrak{g}^*)$ in both the Cartan and the Weil models. The algebra $\widehat{S}(\mathfrak{g}^*)$ is the $\mathfrak{a}$-adic completion of the symmetric algebra $S(\mathfrak{g}^*)$ where $\mathfrak{a}$ is the ideal generated by all elements without constant term (see [2, Ch. 10]); in the next section it will become clear why it is necessary to complete the symmetric algebra. If $\{a, b, c\ldots\}$ is a base of $\mathfrak{g}$ and $\Omega^a, \Omega^b, \ldots$ are dual elements of even degree then $S(\mathfrak{g}^*) = \mathbb{R}[\Omega^a, \Omega^b, \ldots]$ and $\widehat{S}(\mathfrak{g}^*) = \mathbb{R}[\![\Omega^a, \Omega^b, \ldots]\!]$. The completion of the Weil algebra, let us denote by $\widehat{W}(\mathfrak{g}) = \widehat{S}(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$.

Now, in the Cartan model define the twisted equivariant differential as

$$d_{\mathfrak{g},\mathcal{H}} := d_{\mathfrak{g}} - \mathcal{H}\wedge$$

and in the Weil model as

$$d_{\mathbf{H}} := d - \mathbf{H}\wedge.$$

As we have that $(d_{\mathfrak{g},\mathcal{H}})^2 = 0$ and $(d_{\mathbf{H}})^2 = 0$, we can define:

Definition 2.3. The (Cartan) twisted equivariant cohomology is the cohomology of the complex of $\mathfrak{g}$-invariant forms of $\Omega^\bullet(M) \otimes \widehat{S}(\mathfrak{g}^*)$ and the twisted differential $d_{\mathfrak{g},\mathcal{H}}$, i.e.

$$H^\bullet_{\mathfrak{g}}(M, \mathcal{H}) := H^\bullet\big(\big(\Omega^\bullet(M) \otimes \widehat{S}\big(\mathfrak{g}^*\big)\big)^{\mathfrak{g}}; d_{\mathfrak{g},\mathcal{H}}\big).$$

The (Weil) twisted equivariant cohomology is the cohomology of the basic forms $\widehat{\Omega}^\bullet_{\mathfrak{g}}(M) = (\Omega^\bullet(M) \otimes \widehat{W}(\mathfrak{g}))_{bas}$ with the twisted differential $d_{\mathbf{H}}$, i.e.

$$H^\bullet\big(\widehat{\Omega}^\bullet_{\mathfrak{g}}(M), \mathbf{H}\big) := H^\bullet\big(\widehat{\Omega}^\bullet_{\mathfrak{g}}(M); d_{\mathbf{H}}\big).$$

As the map $j : \Omega^\bullet(M) \otimes \widehat{S}(\mathfrak{g}^*) \to \Omega^\bullet(M) \otimes \widehat{W}(\mathfrak{g})$ induces a quasi-isomorphism of complexes $j : (\Omega^\bullet(M) \otimes \widehat{S}(\mathfrak{g}^*))^{\mathfrak{g}} \to \widehat{\Omega}^\bullet_{\mathfrak{g}}(M)$ and $j(H_G) = \mathbf{H}$, then we can conclude that j also induces a quasi-isomorphism of twisted complexes. So we have

Proposition 2.4. *The Cartan and the Weil twisted equivariant cohomologies are isomorphic,*

$$j : H^\bullet_{\mathfrak{g}}(M, \mathcal{H}) \xrightarrow{\cong} H^\bullet\big(\widehat{\Omega}^\bullet_{\mathfrak{g}}(M), \mathbf{H}\big).$$

In [12] the last two authors have shown that the twisted equivariant cohomology possesses all the properties of a cohomology theory (we will provide an alternative proof in the next section). Moreover, as the twisted equivariant cohomology is a module over the symmetric algebra $S(\mathfrak{g}^*)^{\mathfrak{g}}$ (the equivariant cohomology of a point) then the last two authors have shown that a generalization of the localization theorem of Atiyah–Bott [1] in the case of torus actions holds for the twisted equivariant cohomology; namely, for $F = M^G$ the fixed point set of the action, and $i : F \to M$ the inclusion, one has that in a suitable localization, for all classes $x \in H^\bullet_{\mathfrak{g}}(M, \mathcal{H})$ the following formula holds:

$$x = \sum_{Z \subset F} i^Z_* \big(i^*_Z(x)\big) \wedge e_G(\nu_Z)^{-1}$$

where Z runs over the connected components of F, and $e_G(\nu_Z)$ is the equivariant Euler class of ν_Z the normal bundle of Z in M (see Theorem 7.2.5 in [12]).

3. Twisted Chern–Weil homomorphism

In this section we will extend the Chern–Weil homomorphism for the twisted case. Then let us start by recalling the basics of Chern–Weil theory. From now on the Lie group G will be compact and therefore all equivariant cohomologies will have the subscript G.

Let P be a principal G-bundle together with its connection and curvature

$$\theta \in \big(\Omega^1(P) \otimes \mathfrak{g}\big)^G, \qquad \Omega \in \big(\Omega^2(P) \otimes \mathfrak{g}\big)^G,$$

satisfying the identities

$$\iota_X \theta = X, \qquad \iota_X \Omega = 0 \quad (X \in \mathfrak{g}), \qquad \Omega = d\theta + \frac{1}{2}[\theta, \theta] \quad \text{and} \quad d\Omega = [\Omega, \theta].$$

The connection and curvature determine maps

$$\mathfrak{g}^* \to \Omega^1(P), \qquad \mathfrak{g}^* \to \Omega^2(P)$$

that induce a homomorphism of graded algebras

$$W(\mathfrak{g}) \to \Omega^*(P)$$

which is the unique homomorphism carrying the universal connection and curvature to the connection and curvature of P. This homomorphism is called the Weil homomorphism.

If M is a manifold with a G-action, then the Weil homomorphism for the G-principal bundle $M \times P \to M \times_G P$ (where G-acts diagonally) combined with the lifting of forms from M to $P \times M$ determine a homomorphism

$$w : \Omega^*(M) \otimes W(\mathfrak{g}) \to \Omega^*(M \times P)$$

which induces a map of basic subalgebras

$$\overline{w} : \Omega^*_G(M) \to \Omega^*(M \times P)_{bas} \cong \Omega^*(M \times_G P)$$

which is a homomorphism of differential graded algebras. This map is known as the Chern–Weil homomorphism determined by the connection in $M \times P$.

The induced homomorphism in cohomologies

$$\overline{w} : H^*\big(\Omega^*_G(M)\big) \to H^*(M \times_G P)$$

is independent of the connection in P. Following [9, Th. 2.5.1, Pr. 2.5.5], we can choose finite-dimensional manifolds EG_k with free G-actions, and equivariant inclusions $EG_k \to EG_{k+1}$ in such a way that $EG = \lim_{\to} EG_k$ becomes a model for the universal G principal bundle with EG contractible. Then we have that $\Omega^*(EG) = \lim_{\leftarrow} \Omega^*(EG_k)$ and that this complex is acyclic. Moreover, the Chern–Weil map for each k

$$\overline{w}_k : \Omega^*_G(M) \to \Omega^*(M \times_G EG_k)$$

induces a map

$$\overline{w} : \Omega^*_G(M) \to \Omega^*(M \times_G EG) = \lim_{\leftarrow} \Omega^*(M \times_G EG_k) \tag{3.1}$$

which becomes a quasi-isomorphism of complexes and therefore an isomorphism of cohomologies

$$H^*\big(\Omega^*_G(M)\big) \cong H^*(M \times_G EG).$$

Let us show that this result can be generalized to the twisted case.

Theorem 3.1. *Let* $\mathbf{H}$ *be a closed (Weil) equivariant 3-form in* $\Omega_G^3(M)$ *and consider* $H_k := \overline{w}_k(\mathbf{H})$ *and* $H = \lim_{\leftarrow} H_k$*. Then the twisted Chern–Weil homomorphisms*

$$\phi_k : \big(\widehat{\Omega}_G^\bullet(M), d_{\mathbf{H}}\big) \to \big(\Omega^\bullet(M \times_G EG_k), d_{H_k}\big)$$

induce a homomorphism

$$\phi : \big(\widehat{\Omega}_G^\bullet(M), d_{\mathbf{H}}\big) \to \big(\Omega^\bullet(M \times_G EG), d_H\big) := \lim_{\leftarrow k}\big(\Omega^\bullet(M \times_G EG_k), d_{H_k}\big)$$

which induces an isomorphism in twisted cohomologies

$$\phi : H^\bullet\big(\widehat{\Omega}_G^\bullet(M), \mathbf{H}\big) \cong H^\bullet(M \times_G EG, H) := H^\bullet\big(\Omega^\bullet(M \times_G EG), d_H\big).$$

Proof. Even though the complexes $\widehat{\Omega}_G^\bullet(M)$ and $\Omega^\bullet(M \times_G EG)$ are $\mathbb{Z}_2$-graded, we could use the $\mathbb{Z}$-grading of their untwisted versions to define the filtration $F^p\widehat{\Omega}_G^\bullet(M)$ and $F^p\Omega^\bullet(M \times_G EG)$ of all forms of degree greater or equal than p. The map ϕ becomes a homomorphism of filtered complexes and therefore it gives rise to a map of spectral sequences with p-th term $\phi_p : E_p^{*,*} \to \overline{E}_p^{*,*}$.

The first terms of these spectral sequences are

$$E_1^{*,*} = \big(\Omega_G^*(M), d\big) \quad \text{and} \quad \overline{E}_1^{*,*} = \big(\Omega^*(M \times_G EG), d\big)$$

and ϕ_1 is simply the Chern–Weil map $\overline{w}$ of (3.1). By the equivariant de Rham theorem the second term becomes isomorphic

$$\phi_2 : E_2^{*,*} = H^*\big(\Omega_G^*(M)\big) \xrightarrow{\cong} \overline{E}_2^{*,*} = H^*(M \times_G EG),$$

then by Theorem 3.9 of [17] if the filtrations are exhaustive and complete we would have that ϕ induces an isomorphism of twisted cohomologies

$$\phi : H^\bullet\big(\widehat{\Omega}_G^\bullet(M), \mathbf{H}\big) \cong H^\bullet(M \times_G EG, H).$$

Let us finish the proof by showing that both filtrations are complete. The filtrations are exhaustive because the filtrations were defined by the degree.

The twisted cohomology $H^\bullet(\widehat{\Omega}_G^\bullet(M), \mathbf{H})$ is complete because the twisted complex is complete; this follows from the following equalities

$$\Omega(M) \otimes \widehat{W}(\mathfrak{g}) = \lim_{\leftarrow} \Omega(M) \otimes \widehat{W}(\mathfrak{g})/F^p\widehat{S}\big(\mathfrak{g}^*\big) = \lim_{\leftarrow} \Omega(M) \otimes \widehat{W}(\mathfrak{g})/\big(F^p\Omega(M) \otimes \widehat{W}(\mathfrak{g})\big).$$

For the twisted cohomology $H^\bullet(M \times_G EG, H)$ we will also show its completeness by showing it at the level of the twisted complex. For this we just need to show that the induced map

$$\psi : \lim_{\leftarrow k} \Omega^\bullet(M_k) \to \lim_{\leftarrow p}\lim_{\leftarrow k} \Omega^\bullet(M_k)/F^p\Omega^\bullet(M_k)$$

is an isomorphism, where we have denoted $M_k = M \times_G EG_k$ to simplify the notation. As the filtration is exhaustive, then the map ψ is injective. Now let us show that is surjective.

Any element

$$\alpha \in \lim_{\leftarrow p}\lim_{\leftarrow k} \Omega^\bullet(M_k)/F^p\Omega^\bullet(M_k)$$

consists of a sequence $\alpha = \{\alpha_p\}_p$ where $\alpha_p \in \lim_{\leftarrow k} \Omega^\bullet(M_k)/F^p\Omega^\bullet(M_k)$ and $\alpha_{p+1} \mapsto \alpha_p$. Each α_p consists also of a sequence $\alpha_p = \{\alpha_{p,k}\}_k$ where $\alpha_{p,k} \in \Omega^\bullet(M_k)/F^p\Omega^\bullet(M_k)$ and $\alpha_{p,k+1} \mapsto \alpha_{p,k}$. Therefore we have that the α's satisfy

$$\begin{tikzcd}
\alpha_{p+1,k+1} \arrow[r, mapsto] \arrow[d, mapsto] & \alpha_{p+1,k} \arrow[d, mapsto] \\
\alpha_{p,k+1} \arrow[r, mapsto] & \alpha_{p,k}.
\end{tikzcd}$$

Note that if $d(k)$ is the dimension of $M_k = M \times_G EG_k$ then for $p > d(k)$ we have that

$$\Omega^\bullet(M_k)/F^p\Omega^\bullet(M_k) = \Omega^\bullet(M_k),$$

and therefore for all $p > d(k)$, $\alpha_{p,k} = \alpha_{d(k)+1,k}$.

So, define $\beta_k := \alpha_{d(k)+1,k}$ and consider the element

$$\beta = \{\beta_k\}_k \in \lim_{\leftarrow k} \Omega^\bullet(M_k).$$

We claim that $\psi(\beta) = \alpha$. For this let us see that for p fixed $\beta \mapsto \alpha_p$, and this easy to check because for $d(k) < p$ then $\alpha_{p,k} = \alpha_{d(k)+1,k} = \beta_k$, and for $d(k) \geqslant p$ then $\beta_k = \alpha_{d(k)+1,k} \mapsto \alpha_{p,k}$.

Then as the map ψ is an isomorphism the complex $\Omega^\bullet(M \times_G EG)$ is complete and therefore the twisted cohomology $H^\bullet(M \times_G EG, H)$ is complete. □

We have the following useful corollaries:

Corollary 3.2. *If the equivariant cohomology class of twisting form* **H** *is zero, then*

$$H^0\big(\widehat{\Omega}^\bullet_G(M), \mathbf{H}\big) \cong \prod_{i=0}^{\infty} H^{2i}_G(M) \quad \text{and} \quad H^1\big(\widehat{\Omega}^\bullet_G(M), \mathbf{H}\big) \cong \prod_{i=0}^{\infty} H^{2i+1}_G(M)$$

where $H^*_G(M)$ is the equivariant cohomology of M.

Proof. Because **H** is cohomologous to zero we have that

$$H^\bullet\big(\widehat{\Omega}^\bullet_G(M), \mathbf{H}\big) \cong H^\bullet\big(\widehat{\Omega}^\bullet_G(M), 0\big)$$

and by the Chern–Weill homomorphism we have that

$$H^\bullet\big(\widehat{\Omega}^\bullet_G(M), 0\big) \cong \varprojlim_{k} H^\bullet(M \times_G EG_k, 0)$$

as the cohomology commutes with the inverse limit.

The spaces $M \times_G EG_k$ are finite-dimensional manifolds, then their twisted cohomologies twisted by zero are isomorphic to their cohomologies but $\mathbb{Z}_2$-graded, i.e.

$$H^0(M \times_G EG_k, 0) \cong \bigoplus_{i=0}^{\infty} H^{2i}(M \times_G EG_k),$$
$$H^1(M \times_G EG_k, 0) \cong \bigoplus_{i=0}^{\infty} H^{2i+1}(M \times_G EG_k).$$

Taking the inverse limit we have then

$$H^0(M \times_G EG, 0) \cong \prod_{i=0}^{\infty} H^{2i}(M \times_G EG),$$
$$H^1(M \times_G EG, 0) \cong \prod_{i=0}^{\infty} H^{2i+1}(M \times_G EG);$$

the result now follows from the Chern–Weil homomorphism for the untwisted case. □

Corollary 3.3. *If G acts freely on M, then the twisted equivariant cohomology is isomorphic to the twisted cohomology of M/G.*

Proof. This follows from the fact that twisted cohomology is a cohomology theory (see [3]) and the fact that $M \times_G EG$ and M/G are homotopically equivalent. Therefore the twisted cohomology of $M \times_G EG$ is isomorphic to the twisted cohomology of M/G. Note that in this case the twisted equivariant cohomology is finitely generated. This is because M/G is a manifold and its twisted cohomology is finitely generated. □

Corollary 3.4. *The twisted equivariant cohomology satisfies all the axioms of a cohomology theory.*

Proof. Because of the Chern–Weil isomorphism, the twisted equivariant cohomology is isomorphic to the twisted cohomology of the space $M \times_G EG$. Because the twisted cohomology satisfies all the axioms of cohomology, the result follows. □

Example 3.5. Let us calculate the twisted equivariant cohomology of the trivial action of $G = S^1$ on $M = S^1$ with the equivariant three form $[d\theta\Omega] \in H^3(\Omega^\bullet_{S^1}(S^1))$ where $[d\theta]$ generates $H^1(S^1)$ and $\mathbb{R}[\Omega]$ is the symmetric algebra on one generator, using the twisted Chern–Weil homomorphism of Theorem 3.1. Taking $S^{2k+1} \subset \mathbb{C}^{k+1}$ as the set ES^1_k, then we have that $M \times_G EG_k = S^1 \times \mathbb{C}P^k$. The cohomology of $S^1 \times \mathbb{C}P^k$ is

$$H^*\big(S^1 \times \mathbb{C}P^k\big) = \Lambda[d\theta] \otimes \mathbb{R}[\Omega]/\big\langle \Omega^{k+1}\big\rangle$$

and the three form induced by the Chern–Weil map is $d\theta\Omega$.

As both manifolds S^1 and $\mathbb{C}P^k$ are formal, the twisted cohomology could be calculated from the cohomology of $S^1 \times \mathbb{C}P^k$ and the operator $-d\theta\Omega\wedge$, i.e.

$$H^\bullet\big(S^1 \times \mathbb{C}P^k; d\theta\Omega\big) = H^\bullet\big(H^\bullet\big(S^1 \times \mathbb{C}P^k\big); -d\theta\Omega\wedge\big).$$

Then we have that

$$H^\bullet\big(S^1 \times \mathbb{C}P^k; d\theta\Omega\big) = \mathbb{R}\langle d\theta\rangle \oplus \mathbb{R}\big\langle \Omega^k\big\rangle.$$

Now, if we take the inverse limit of these cohomologies we get

$$\lim_{\leftarrow k} H^\bullet\big(S^1 \times \mathbb{C}P^k; d\theta\Omega\big) = \lim_{\leftarrow k} \mathbb{R}\langle d\theta\rangle \oplus \mathbb{R}\big\langle \Omega^k\big\rangle = \mathbb{R}\langle d\theta\rangle,$$

which implies that

$$H^\bullet\big(S^1 \times \mathbb{C}P^\infty; d\theta\Omega\big) = \mathbb{R}\langle d\theta\rangle$$

and therefore

$$H^\bullet\big(\widehat{\Omega}^\bullet_{S^1}\big(S^1\big); d\theta\Omega\big) = \mathbb{R}\langle d\theta\rangle.$$

4. Extended equivariant cohomology

In this section we show how the twisted equivariant cohomology arises as the equivariant theory for compact group actions on exact Courant algebroids. We do not claim any originality in this section as most of the results (except the last example) have appeared in a disorganized way in various other articles [5,6,12–15]. What we accomplish is to present all the details in a clear and succinct way for the equivalence between the twisted equivariant cohomology and the extended equivariant cohomology for compact Lie groups.

4.1. Exact Courant algebroid

An exact Courant algebroid is a Courant algebroid $\mathcal{T}M$ over a manifold M that fits into the exact sequence

$$0 \longrightarrow T^*M \longrightarrow \mathcal{T}M \xrightarrow{a} TM \longrightarrow 0.$$

The Courant algebroid $\mathcal{T}M$ is endowed with a nondegenerate symmetric bilinear form and a skew symmetric bracket called Courant bracket (see [5,16]). One can always choose a splitting $s : TM \to \mathcal{T}M$ with isotropic image (see [20,21]) such that one can identify the extended Courant algebroid with the direct sum of the tangent and the cotangent bundles of M, namely

$$\mathcal{T}M \overset{\cong}{\to} \mathbb{T}M := TM \oplus T^*M,$$
$$\mathfrak{X} \mapsto \big(a(\mathfrak{X}), \mathfrak{X} - s\big(a(\mathfrak{X})\big)\big),$$
$$\mathfrak{X} \mapsto (X + \xi).$$

With this identification the Courant bracket on $\mathcal{T}M$ becomes the H-twisted Courant bracket on $\mathbb{T}M$, where H is a closed three form determined by the splitting:

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(\iota_X\eta - \iota_Y\xi) - \iota_X\iota_Y H;$$

and the bilinear form becomes:

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\iota_X\eta + \iota_Y\xi).$$

Let us emphasize that the splitting of the Courant algebroid is not canonical; for any two form B, the B-field transformation of the splitting

$$e^B(X+\xi) = X + \xi + \iota_X B$$

gives another splitting of $\mathcal{T}M$ with twisting three form $H - dB$. The cohomology class $[H] \in H^3(M;\mathbb{R})$ is called the Ševera class of $\mathcal{T}M$.

From now on we will work with a chosen splitting of the Courant algebroid $\mathcal{T}M$. Hence, the three form H will also be fixed.

4.2. Symmetries of the exact Courant algebroid

The group $\operatorname{Diff} M \ltimes \Omega^2(M)$ with composition

$$(\lambda, \alpha) \circ (\mu, \beta) = \left(\lambda\mu, \mu^*\alpha + \beta\right)$$

acts on $\mathbb{T}M$ in the following way

$$(\lambda, \alpha) \circ (X + \xi) = \lambda_* X + \left(\lambda^{-1}\right)^* (\xi + \iota_X \alpha) = \lambda_* X + \left(\lambda^{-1}\right)^* \xi + \iota_{\lambda_* X} \alpha$$

and its induced action on the twisted Courant bracket becomes

$$(\lambda, \alpha) \circ [X + \xi, Y + \eta]_H = \left[(\lambda, \alpha) \circ (X + \xi), (\lambda, \alpha) \circ (Y + \eta)\right]_{(\lambda^{-1})^*(H - d\alpha)}.$$

Hence, the group of symmetries of the exact Courant algebroid $\mathbb{T}M$ is

$$\mathcal{G}_H := \left\{(\lambda, \alpha) \in \operatorname{Diff} M \ltimes \Omega^2(M) \mid H = \left(\lambda^{-1}\right)^* (H - d\alpha)\right\}$$

The Lie algebra $\mathcal{X}_H$ of $\mathcal{G}_H$ is then

$$\mathcal{X}_H = \left\{(X, A) \in \Gamma(TM) \oplus \Omega^2(M) \mid dA = -\mathcal{L}_X H = -d\iota_X H\right\}$$

with Lie bracket

$$\left[(X, A), (Y, B)\right] = \left([X, Y], \mathcal{L}_X B - \mathcal{L}_Y A\right),$$

and with infinitesimal action on $\mathbb{T}M$ given by

$$(X, A) \circ (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - \iota_Y A.$$

Note then that (X, A) belongs to $\mathcal{X}_H$ if and only if $d(A + \iota_X H) = 0$ (this equation will be of use later).

Definition 4.1. A Lie group G acts on the exact Courant algebroid $\mathbb{T}M$ if there is a homomorphism

$$G \to \mathcal{G}_H, \qquad g \mapsto (\lambda_g, \alpha_g),$$

and a Lie algebra $\mathfrak{g}$ acts infinitesimally on $\mathbb{T}M$ if there is a Lie algebra homomorphism

$$\mathfrak{g} \to \mathcal{X}_H, \qquad a \mapsto (X_a, A_a).$$

4.3. Extended symmetries

Note that there is a homomorphism of algebras

$$\kappa : \Gamma(\mathbb{T}M) \to \mathcal{X}_H, \qquad (X + \xi) \mapsto (X, d\xi - \iota_X H)$$

that sends the Courant bracket to the Lie bracket on $\mathcal{X}_H$. With this in mind we have an action of $\Gamma(\mathbb{T}M)$ on itself given by the formula

$$(X + \xi) \circ (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - \iota_Y (d\xi - \iota_X H).$$

Definition 4.2. We will say that the Lie Group G (or the Lie algebra $\mathfrak{g}$) acts by extended symmetries on $\mathbb{T}M$ whenever:

- the infinitesimal action factors through $\Gamma(\mathbb{T}M)$ as algebras, i.e. there is an algebra homomorphism

  $$\delta : \left(\mathfrak{g}, [,]\right) \to \left(\Gamma(\mathbb{T}M), [,]_H\right),$$
  $$a \mapsto (X_a + \xi_a)$$

  that makes the infinitesimal action be $a \mapsto (X_a, d\xi_a - \iota_{X_a} H)$, and
- the image of $\mathfrak{g}$ in $\Gamma(\mathbb{T}M)$ is an isotropic subspace, in other words, for every $a, b \in \mathfrak{g}$

  $$\langle X_a + \xi_a, X_b + \xi_b \rangle = 0,$$

  and what is the same

  $$\iota_{X_a} \xi_b = -\iota_{X_b} \xi_a \quad \text{and} \quad \iota_{X_a} \xi_a = 0.$$

Remark 4.3. For an extended action $\delta : \mathfrak{g} \to \Gamma(\mathbb{T}M)$ the three form $H + \Omega^a \xi_a$, with H the twisting form and $\xi_a = \delta(a) - X_a$, would be closed and equivariant if and only if $d\xi_a = -\iota_a H$, as this would imply that $\mathcal{L}_a H = d\iota_a H = -dd\xi_a = 0$.

Remark 4.4. We would like to point out that an "extended action", as we have defined above and in [12], is equivalent to a "lifted action" as is defined in Section 2.3 of [4] (we use the skew-symmetric version of the Courant bracket while in [4] is used the non-skew-symmetric version, but as in both cases the image of the Lie algebra must be isotropic, then the two definitions agree). In [4,5] the authors call "extended action" a more general type of construction that includes Courant algebras and Courant algebra morphisms $\mathfrak{a} \to \Gamma(\mathbb{T}M)$ that extend the Lie algebra morphism $\mathfrak{g} \to \Gamma(TM)$.

4.4. Extended equivariant cohomology

In [12] the last two authors have defined an equivariant cohomology for extended actions. Let us recall the construction.

Consider the complex of differential forms $\Omega^\bullet(M) := \Omega^{\text{even}}(M) \oplus \Omega^{\text{odd}}(M)$ but with $\mathbb{Z}_2$ grading given by the parity of the degree and odd differential $d_H := d - H\wedge$ where H is a closed 3-form on M. The cohomology of this complex $H^\bullet(M, H)$ is known as the H-twisted cohomology of M.

For $\mathfrak{X} = X + \xi \in \Gamma(\mathbb{T}M)$ consider the even operator $\mathcal{L}_{\mathfrak{X}}$ and the odd operator $\iota_{\mathfrak{X}}$ that act on $\Omega^\bullet(M)$ in the following way: for $\rho \in \Omega^\bullet(M)$ we have

$$\iota_{\mathfrak{X}}\rho = \iota_X \rho + \xi \wedge \rho \quad \text{and} \quad \mathcal{L}_{\mathfrak{X}}\rho = \mathcal{L}_X \rho + (d\xi - \iota_X H) \wedge \rho.$$

If we consider any two elements $\mathfrak{X}, \mathfrak{Y}$ that lie in the image of the map $\delta : \mathfrak{g} \to \Gamma(\mathbb{T}M)$, then the operators $\mathcal{L}$, ι and d_H behave in the following way with respect to the graded commutators (see [12, Thm. 4.4.3]):

$$[d_H, \iota_{\mathfrak{X}}] = \mathcal{L}_{\mathfrak{X}}, \qquad [\mathcal{L}_{\mathfrak{X}}, \mathcal{L}_{\mathfrak{Y}}] = \mathcal{L}_{[\mathfrak{X},\mathfrak{Y}]_H}, \qquad [\iota_{\mathfrak{X}}, \iota_{\mathfrak{Y}}] = 0,$$
$$[\mathcal{L}_{\mathfrak{X}}, \iota_{\mathfrak{Y}}] = \iota_{[\mathfrak{X},\mathfrak{Y}]_H}, \qquad [d_H, \mathcal{L}_{\mathfrak{X}}] = 0, \quad \text{and} \quad [d_H, d_H] = 0.$$

Following the definition of the (Cartan) twisted equivariant complex done before, we will consider the algebra $\Omega^\bullet(M) \otimes \widehat{S}(\mathfrak{g}^*)$ of formal series on $\mathfrak{g}$ with values in $\Omega^\bullet(M)$.

If we consider an extended action $\delta : \mathfrak{g} \to \Gamma(\mathbb{T}M)$, we can extend the action of the operators $\mathcal{L}_{\delta(a)}$ and $\iota_{\delta(a)}$ on the generators $\Omega^\bullet(M) \otimes \widehat{S}(\mathfrak{g}^*)$ in the natural way, namely

$$\mathcal{L}_{\delta(a)}\big(\rho \otimes \Omega^b\big) := (\mathcal{L}_{\delta(a)}\rho) \otimes \Omega^b + \rho \otimes \mathcal{L}_a \Omega^b \quad \text{and} \quad \iota_{\delta(a)}\big(\rho \otimes \Omega^b\big) := (\iota_{\delta(a)}\rho) \otimes \Omega^b,$$

and we can define the extended equivariant differential

$$d_{\mathfrak{g},\delta} : \Omega^\bullet(M) \otimes \widehat{S}\big(\mathfrak{g}^*\big) \to \Omega^\bullet(M) \otimes \widehat{S}\big(\mathfrak{g}^*\big)$$

as the odd operator

$$d_{\mathfrak{g},\delta} := d_H \otimes 1 + \Omega^a \iota_{\delta(a)}$$

where the sum goes over a base of $\mathfrak{g}$ and we are using the repeated index convention.

It is easy to check that

$$(d_{\mathfrak{g},\delta})^2 \rho = -\Omega^a \mathcal{L}_{\delta(a)}\rho$$

and therefore the second two authors have proposed the following definition (see [12, Def. 5.1.1]):

Definition 4.5. Let $\delta : \mathfrak{g} \to \Gamma(\mathbb{T}M)$ be an extended action, then the $\mathfrak{g}$-extended equivariant complex of $\mathbb{T}M$ is the $\mathbb{Z}_2$-graded complex

$$C^\bullet_{\mathfrak{g}}(\mathbb{T}M; \delta) := \big\{\rho \in \Omega^\bullet(M) \otimes \widehat{S}\big(\mathfrak{g}^*\big) \mid \mathcal{L}_{\delta(a)}\rho = 0 \text{ for all } a \in \mathfrak{g}\big\}$$

with differential $d_{\mathfrak{g},\delta}$. The cohomology of $H^\bullet_{\mathfrak{g}}(\mathbb{T}M; \delta)$ of the complex $C^\bullet_{\mathfrak{g}}(\mathbb{T}M; \delta)$ is the extended $\mathfrak{g}$-equivariant De Rham cohomology of $\mathbb{T}M$ under the extended action defined by δ.

Let us note that the extended $\mathfrak{g}$-equivariant cohomology does not depend on the choice of splitting for $\mathcal{T}M$ and its isomorphism class depends only on the Ševera class of the Courant algebroid. If one performs a B-field transform, the action transforms δ to $\delta'(a) = X_a + \xi_a + \iota_{X_a} B$, one gets the isomorphism $e^B : \Omega^\bullet(M) \to \Omega^\bullet(M)$ and therefore one obtains a quasi-isomorphism of complexes $C^\bullet_{\mathfrak{g}}(\mathbb{T}M; \delta) \to C^\bullet_{\mathfrak{g}}(\mathbb{T}M; \delta')$ with $d_{\mathfrak{g},\delta'} := d_{H-dB} \otimes 1 + \Omega^a \iota_{\delta'(a)}$ that induces an isomorphism of cohomologies $H^\bullet_{\mathfrak{g}}(\mathbb{T}M; \delta) \cong H^\bullet_{\mathfrak{g}}(\mathbb{T}M; \delta')$.

Also, if we take the Cartan complex

$$C^*_{\mathfrak{g}}(M) = \big\{\rho \in \Omega^*(M) \otimes S\big(\mathfrak{g}^*\big) \mid \mathcal{L}_{X_a}\rho = 0 \text{ for all } a \in \mathfrak{g}\big\}$$

with differential $d_{\mathfrak{g}}\rho = d\rho - \Omega^a \iota_{X_a}\rho$, then the extended $\mathfrak{g}$-equivariant complex $C^\bullet_{\mathfrak{g}}(\mathbb{T}M;\delta)$ becomes a module over $C^*_{\mathfrak{g}}(M)$ and therefore the extended $\mathfrak{g}$-equivariant cohomology $H^\bullet_{\mathfrak{g}}(\mathbb{T}M;\delta)$ is a module over the equivariant cohomology $H^*_{\mathfrak{g}}(M)$. In particular we have that $H^\bullet_{\mathfrak{g}}(\mathbb{T}M;\delta)$ is also a module over $S(\mathfrak{g}^*)^{\mathfrak{g}} = H^*_{\mathfrak{g}}(\cdot)$.

So far we do not know whether the extended equivariant cohomology fulfills all the properties of an equivariant cohomology theory, nor if it has a topological counterpart. Nevertheless, in the case that the group G is compact, the extended equivariant cohomology turns out to be equivalent to what is known as twisted equivariant cohomology. This will be the subject of what follows.

4.5. Extended equivariant cohomology for compact Lie groups

Now we will show that in the case of an extended action of a compact Lie group, one can find an appropriate choice of splitting of the exact Courant algebroid in such a way that the extended equivariant cohomology becomes isomorphic to the twisted equivariant cohomology defined previously.

Let us first show how the group of symmetries $\mathcal{G}_H$ change after performing a B-field transform, thus changing the splitting of $\mathbb{T}M$. For $(\lambda,\alpha)\in\mathcal{G}_H$ it is easy to check that

$$\big(\lambda, \alpha+\lambda^*B-B\big)\circ e^B(X+\xi) = e^B\big((\lambda,\alpha)\circ(X+\xi)\big),$$

and therefore the group of symmetries changes to

$$\begin{aligned} &e^B : \mathcal{G}_H \to \mathcal{G}_{H-dB}, \\ &(\lambda,\alpha)\mapsto \big(\lambda, \alpha+\lambda^*B-B\big). \end{aligned} \tag{4.1}$$

Let us denote $\overline{H} := H - dB$.

Lemma 4.6. *(See [5, Proposition 2.11].) Consider the action of a compact Lie group G on the extended Courant algebroid $\mathbb{T}M$ given by the map $G\to\mathcal{G}_H$, $g\mapsto(\lambda_g,\alpha_g)$. Then there exists a 2-form B such that*

$$\alpha_g + \lambda_g^* B - B = 0 \quad \textit{for all } g\in G;$$

and therefore after changing the splitting with B (see Eq. (4.1)), the action of G is only given by diffeomorphisms, i.e. $G\to\mathcal{G}_{\overline{H}}$, $g\mapsto(\lambda_g,0)$. Moreover, the 3-form $\overline{H}=H-dB$ is G-invariant.

Proof. For $g\in G$ and $B\in\Omega^2(M)$ let us define the 2-form

$$g\cdot B := \big(\lambda_g^{-1}\big)^*(B-\alpha_g).$$

This becomes an action of G on $\Omega^2(M)$ as we have that

$$\begin{aligned}(hg)\cdot B &= \big(\lambda_{hg}^{-1}\big)^*(B-\alpha_{hg}) = \big(\lambda_h^{-1}\big)^*\big(\lambda_g^{-1}\big)^*\big(B-\lambda_g^*\alpha_h-\alpha_g\big)\\ &= \big(\lambda_h^{-1}\big)^*\big[\big(\lambda_g^{-1}\big)^*(B-\alpha_g)-\alpha_h\big] = h\cdot(g\cdot B).\end{aligned}$$

Now, taking a G-invariant metric $d\mu$ with total volume 1, we can define the 2-form

$$B := \int_G (h\cdot 0)\,d\mu(h) = \int_G \big(\lambda_h^{-1}\big)^*\alpha_h\,d\mu(h)$$

which clearly satisfies $g\cdot B = B$ for all $g\in G$, and therefore we have that $\lambda_g^*B = B-\alpha_g$.

The fact the $\overline{H}$ is invariant follows from the definition of $\mathcal{G}_{\overline{H}}$; for $(\lambda,\overline{\alpha})\in\mathcal{G}_{\overline{H}}$ we have that $\lambda^*\overline{H} = H - d\overline{\alpha}$. But for all $g\in G$ we have that $\overline{\alpha}_g=0$, then $\lambda_g^*\overline{H}=\overline{H}$. □

In the case that the compact Lie group G acts by extended symmetries

$$\begin{aligned}&\delta : \mathfrak{g}\to\Gamma(\mathbb{T}M)\to\mathcal{X}_H,\\ &a\mapsto(X_a,\xi_a)\mapsto(d\xi_a-\iota_{X_a}H),\end{aligned}$$

the change of splitting of Lemma 4.6 defines a map $\bar\delta:\mathfrak{g}\to\Gamma(\mathbb{T}M)$,

$$\bar\delta(a) = X_a + \bar\xi_a = X_a + \xi_a + \iota_{X_a}B,$$

and as the action is only given by diffeomorphisms we have that the 2-forms $d\bar\xi_a - \iota_{X_a}\overline{H}$ are all equal to zero. Therefore we can conclude:

Corollary 4.7. *If the compact Lie group acts by extended symmetries and we perform the B-field transform of Lemma 4.6, then for all $a \in \mathfrak{g}$ we have that*

$$d\bar{\xi}_a - \iota_{X_a}\overline{H} = 0$$

where the infinitesimal action is given by $\bar{\delta}(a) = X_a + \bar{\xi}_a$.

Lemma 4.8. *(See [5, Theorem 2.13].) Let the compact Lie group G act on $\mathbb{T}M$ by extended symmetries. Then, after performing the B-field transform of Lemma 4.6, the 3-form*

$$\overline{\mathcal{H}} := \overline{H} + \Omega^a\bar{\xi}_a \in \Omega^*(M) \otimes S\big(\mathfrak{g}^*\big)$$

becomes G-invariant and d_G-closed. Hence, it defines an equivariant De Rham class $[\overline{\mathcal{H}}] \in H^3_G(M)$.

Proof. Let us first check that $\overline{\mathcal{H}}$ is G-invariant, so for $a, b, c \ldots$ a base of $\mathfrak{g}$ we have that

$$\begin{aligned}
\mathcal{L}_b\overline{\mathcal{H}} &= \mathcal{L}_b\overline{H} + \big(\mathcal{L}_b\Omega^a\big)\bar{\xi}_a + \Omega^a(\mathcal{L}_b\bar{\xi}_a) \\
&= \mathcal{L}_b\overline{H} + \big(f^a_{cb}\Omega^c\big)\bar{\xi}_a + \Omega^a(\bar{\xi}_{[b,a]}) \\
&= \mathcal{L}_b\overline{H} + \Omega^c\bar{\xi}_{[c,b]} + \Omega^a\bar{\xi}_{[b,a]} \\
&= 0
\end{aligned}$$

where we have used that $\mathcal{L}_b\bar{\xi}_a = \bar{\xi}_{[b,a]}$. This last equality follows from the fact that the map $\mathfrak{g} \to \Gamma(\mathbb{T}M)$ is an algebra map, and so we have

$$\begin{aligned}
\bar{\xi}_{[a,b]} &= \mathcal{L}_{X_a}\bar{\xi}_b - \mathcal{L}_{X_b}\bar{\xi}_a - \frac{1}{2}d(\iota_{X_a}\bar{\xi}_b - \iota_{X_b}\bar{\xi}_a) + \iota_{X_b}\iota_{X_a}\overline{H} \\
&= \mathcal{L}_{X_a}\bar{\xi}_b - d\iota_{X_b}\bar{\xi}_a - \iota_{X_b}d\bar{\xi}_a + d\iota_{X_b}\bar{\xi}_a + \iota_{X_b}d\bar{\xi}_a \\
&= \mathcal{L}_{X_a}\bar{\xi}_b.
\end{aligned}$$

Now let us calculate $d_G\overline{\mathcal{H}}$:

$$d_G\overline{\mathcal{H}} = dH + \Omega^a(d\bar{\xi}_a - \iota_a\overline{H}) + \Omega^b\Omega^c(\iota_b\bar{\xi}_a + \iota_a\bar{\xi}_b)$$

and as H is closed, $d\bar{\xi}_a - \iota_a\overline{H} = 0$ because of Lemma 4.7 and

$$\iota_b\bar{\xi}_a + \iota_a\bar{\xi}_b = \langle X_a + \bar{\xi}_a, X_b + \overline{\xi_b}\rangle = 0$$

as the action is isotropic (see Definition 4.2), then $d_G\overline{\mathcal{H}} = 0$. □

Knowing that the 3-form $\overline{\mathcal{H}}$ is invariant and closed, we can conclude this section with the following result:

Theorem 4.9. *(See [12, Proposition 6.1.2].) Let the compact Lie group G act on $\mathbb{T}M$ by extended symmetries. Then the extended equivariant cohomology $H^\bullet_{\mathfrak{g}}(\mathbb{T}M, \delta)$ is isomorphic to the twisted equivariant cohomology $H^\bullet_G(M, \overline{\mathcal{H}})$ where $\overline{\mathcal{H}}$ is the equivariant closed 3-form of Lemma 4.8.*

Proof. Let us perform the B-field transform of Lemma 4.6. Then we have a quasi-isomorphism of extended complexes $C^\bullet_{\mathfrak{g}}(\mathbb{T}M, \delta) \to C^\bullet_{\mathfrak{g}}(\mathbb{T}M, \bar{\delta})$ that induces an isomorphism of cohomologies

$$H^\bullet_{\mathfrak{g}}(\mathbb{T}M, \delta) \cong H^\bullet_{\mathfrak{g}}(\mathbb{T}M, \bar{\delta}).$$

Now for $\rho \in \Omega^\bullet(M)$, we have that

$$\mathcal{L}_{\bar{\delta}(a)}\rho = \mathcal{L}_a\rho + (d\bar{\xi}_a - \iota_a\overline{H}) \wedge \rho = \mathcal{L}_a\rho,$$

and therefore the extended complex becomes

$$C^\bullet_{\mathfrak{g}}(\mathbb{T}M, \bar{\delta}) = \big\{\rho \in \Omega^\bullet(M) \otimes \widehat{S}\big(\mathfrak{g}^*\big) \mid \mathcal{L}_a\rho = 0 \text{ for all } a \in \mathfrak{g}\big\}$$

with derivative $d_{\mathfrak{g},\bar{\delta}} := d_{\overline{H}} + \Omega^a\iota_{\bar{\delta}(a)}$ which can be expanded and transformed into

$$\begin{aligned} d_{\mathfrak{g},\bar\delta} &= d - \overline{H}\wedge + \Omega^a\iota_a + \Omega^b\bar\xi_b\wedge \\ &= d + \Omega^a\iota_a - \left(\overline{H} - \Omega^b\bar\xi_b\right)\wedge \\ &= d_{\mathfrak{g}} - \mathcal{H}\wedge \\ &= d_{\mathfrak{g},\overline{\mathcal{H}}}. \end{aligned}$$

Thus the extended complex $C^\bullet_{\mathfrak{g}}(\mathbb{T}M,\bar\delta)$ is the same as the twisted complex

$$C^\bullet_G(M,\overline{\mathcal{H}}) = \left(\left(\Omega^\bullet(M)\otimes\widehat{S}\left(\mathfrak{g}^*\right)\right)^G; d_{\mathfrak{g},\overline{\mathcal{H}}}\right).$$

Therefore the cohomologies $H^\bullet_{\mathfrak{g}}(\mathbb{T}M,\bar\delta)$ and $H^\bullet_G(M,\overline{\mathcal{H}})$ are the same. The theorem follows. □

Remark 4.10. Corollary 3.4 together with Theorem 4.9 imply that extended equivariant cohomology for compact Lie groups satisfies the properties of a cohomology theory as are the Mayer–Vietoris long exact sequence, excision and the Thom isomorphism. This was also proved in [12] using equivariant methods.

Let us finish this section with an example.

Example 4.11. Let $G = S^1$ and $M = S^1$ where G acts trivially on M, but with extended action given by the map

$$\delta : \mathbb{R} \to \Gamma\left(\mathbb{T}S^1\right), \qquad \delta(1) = d\theta$$

where $d\theta \in \Omega^1(S^1)$. The extended equivariant cohomology $H^\bullet_{\mathfrak{g}}(\mathbb{T}M,\delta)$ becomes the cohomology of the complex

$$\Omega^\bullet\left(S^1\right)\otimes\mathbb{R}\left[[\Omega]\right] \quad \text{with differential } d_{\mathfrak{g},\delta} = d - d\theta\,\Omega\wedge$$

which is the twisted equivariant cohomology $H^\bullet_{S^1}(M, d\theta\,\Omega)$.

One can check easily that the closed forms are the odd forms. Now let us see which odd forms are exact. Consider the odd form $f_i\,d\theta\,\Omega^i$ and the even form $g_j\Omega^j$. The equation $d_{\mathfrak{g},\delta}(g_j\Omega^j) = f_i\,d\theta\,\Omega^i$ is equivalent to the equations $dg_0 = f_0\,d\theta$ and $dg_i - g_{i-1}d\theta = f_i\,d\theta$ for $i > 0$. If $\int_{S^1} f_0\,d\theta = 0$ these equations are solved inductively starting from 0 and making sure that one chooses the g_i such that their integral satisfies $\int_{S^1}(g_i + f_{i+1})\,d\theta = 0$.

Then the twisted equivariant cohomology is

$$H^1_{S^1}(M, d\theta\,\Omega) = \mathbb{R} \quad \text{and} \quad H^0_{S^1}(M, d\theta\,\Omega) = 0,$$

which agrees with the calculations done in Example 3.5 using the twisted Chern–Weil isomorphism.

With this particular example one can show what happens in the case that the twisted cohomology is defined without completing the symmetric algebra. The calculations will be done in Appendix A.

5. Hamiltonian actions on generalized complex manifolds

In this last section we will show how to induce a generalized complex structure on $M \times_G P$ whenever we have a Hamiltonian G action on the generalized complex manifold M and $P \to Q$ is a G-principal bundle over a generalized complex manifold Q.

Let us start by recalling the definitions and theorems of generalized complex geometry that will be used in what follows (see [5,8,10,11,15,22]).

Definition 5.1. A generalized complex manifold is a manifold M together with one of the following equivalent structures:

- An endomorphism $\mathbb{J} : \mathbb{T}M \to \mathbb{T}M$ such that $\mathbb{J}^2 = -1$, orthonormal with respect to the inner product $\langle\,,\rangle$ and such that the $\sqrt{-1}$-eigenbundle $L < \mathbb{T}M\otimes\mathbb{C}$ is involutive with respect to the H-twisted Courant bracket, i.e. $[L,L]_H \subset L$.
- A maximal isotropic subbundle $L < \mathbb{T}M\otimes\mathbb{C}$ which is involutive with respect to the H-twisted Courant bracket and such that $L\cap\overline{L} = \{0\}$.
- A line bundle U in $\wedge^*T^*M\otimes\mathbb{C}$ generated locally by a form of the form $\rho = e^{B+\sqrt{-1}\omega}\Omega$ such that Ω is a decomposable complex form, B and ω are real 2-forms and $\Omega\wedge\overline{\Omega}\wedge\omega^{n-k} \neq 0$ at the points where $\deg(\Omega) = k$; together with a section $\mathfrak{X} = X + \xi \in \Gamma(\mathbb{T}M\otimes\mathbb{C})$ such that

$$d_H\rho = \iota_{\mathfrak{X}}\rho = \iota_X\rho + \xi\wedge\rho.$$

The equivalence of these three definitions can be found in Gualtieri's thesis [10]. Let us only note that the elements of $\mathbb{T}M \otimes \mathbb{C}$ that annihilate the form ρ of the third definition define the subbundle L. The line bundle U is called the *canonical line bundle* of the generalized complex structure.

Definition 5.2. We say that a Lie group G acts on the generalized complex manifold $(M, \mathbb{J})$ if the group G acts by extended symmetries on $\mathbb{T}M$ and if the generalized complex structure $\mathbb{J}$ is preserved by the action.

The action is **Hamiltonian** if there is an equivariant moment map $\mu : M \to \mathfrak{g}^*$ such that $\mathbb{J}(d\mu_a) = X_a + \xi_a$ for all $a \in \mathfrak{g}$ and $\mu_a(m) := \mu(m)(a)$.

The condition on the moment map could be rephrased as

$$\iota_{\mathfrak{X}_a}\rho = \iota_{X_a}\rho + \xi_a \wedge \rho = -\sqrt{-1}\, d\mu_a \wedge \rho$$

whenever ρ is a local section of the canonical line bundle.

Let us now state the main theorem of this section.

Theorem 5.3. *Let G be a compact Lie group, $P \to Q$ a G-principal bundle such that the base Q is compact and is endowed with a generalized complex structure, and a Hamiltonian action of G on the generalized complex manifold M. Then the manifold $M \times_G P$ admits a generalized complex structure.*

Proof. Let us start by choosing the splitting of Lemma 4.8 for the manifold M thus defining a G-invariant twisting form $H_1 \in \Omega^3(M)$ and a moment map $\sigma : M \to \mathfrak{g}^*$ with $\mathbb{J}(d\sigma_a) = X_a + \xi_a$. Let $H_2 \in \Omega^3(Q)$ be the twisting form of Q.

Following Weinstein's construction in [23] (cf. [19, Theorem 6.10]), if we consider the cotangent vertical bundle of P

$$T^{*v}P := P \times_G T^*G,$$

then every connection 1-form A of P induces an equivariant map $\phi^A : T^{*v}P \to T^*P$ such that the two form $\omega_A := (\phi^A)^*\omega_{\text{can}} \in \Omega^2(T^{*v}P)$ is G-invariant and restricts to the canonical symplectic form of the fibers of the bundle $p : T^{*v}P \to Q$ (the fibers are isomorphic to T^*G). Moreover, as the map $\mu_P : T^*P \to \mathfrak{g}^*$, $\mu_P(p, v^*) = -L_p^* v^*$ is Hamiltonian in the usual sense, with $L_p^* : T_p^*P \to \mathfrak{g}^*$ the dual of the linear map $L_p : \mathfrak{g} \to T_pP$, $L_p\xi = p \cdot \xi$ then the composition

$$\mu = \mu_P \circ \phi^A : T^{*v}P \to \mathfrak{g}^*$$

is a moment map for the G action on the fibers of $T^{*v}P$.

If ρ is the local form defining the generalized complex structure in Q, it was shown in the proof of Theorem 2.2 of [8] that as Q is compact there exists an $\epsilon > 0$ such that the local form

$$\overline{\rho} = e^{\sqrt{-1}\epsilon\omega_A} \wedge p^*\rho$$

defines a generalized complex structure on $T^{*v}P$ with twisting form p^*H_2.

The action of G on $T^{*v}P$ is also Hamiltonian with respect to the generalized complex structure defined by $\overline{\rho}$ and with moment map $\mu^\epsilon : T^{*v}P \to \mathfrak{g}^*$, $\mu^\epsilon(\cdot) := \epsilon\mu(\cdot)$; let us see this. Let $\mathfrak{g} \to T(T^{*v}P)$, $a \mapsto Y_a$, be the infinitesimal action; for the action to be Hamiltonian we need the following equation to be satisfied

$$\iota_{Y_a}\overline{\rho} = -\sqrt{-1}\, d\mu_a^\epsilon \wedge \overline{\rho},$$

which follows from the following set of equalities

$$\begin{aligned}\iota_{Y_a}\overline{\rho} &= \iota_{Y_a}\big(e^{-\sqrt{-1}\epsilon\omega_A} \wedge p^*\rho\big)\\ &= -\sqrt{-1}\epsilon(\iota_{Y_a}\omega_A) \wedge e^{-\sqrt{-1}\epsilon\omega_A} \wedge p^*\rho\\ &= -\sqrt{-1}\epsilon(d\mu_a) \wedge \overline{\rho}\\ &= -\sqrt{-1}\, d\mu_a^\epsilon \wedge \overline{\rho}.\end{aligned}$$

Notice that the equation $\iota_{Y_a}\omega_A = d\mu_a$ is the restriction to $T^{*v}P$ of the equation

$$\iota_{\phi_*^A Y_a}\omega_{\text{can}} = d(\mu_P)_a$$

which follows from the fact that μ_P is a moment map.

We now have that the action of G is Hamiltonian in both generalized complex manifolds $T^{*v}P$ and M. Then we can consider the product $M \times T^{*v}P$ together the generalized complex structure induced by M and $T^{*v}P$ and whose twisting form is $H_1 + p^*H_2$. The diagonal action of G on $M \times T^{*v}P$ is also Hamiltonian (see [11, Prop. 3.9]) and its moment map is $\overline{\mu} = \sigma \oplus \mu$. Because 0 is a regular value of μ_P, and therefore of μ, then 0 is a regular value of $\overline{\mu}$. The group G acts freely on $T^{*v}P$ and therefore it acts freely on $\overline{\mu}^{-1}(0)$. By the reduction theorem for Hamiltonian actions on generalized complex manifolds (see [5,11,15,22]) we have that the manifold $\overline{\mu}^{-1}(0)/G$ possesses a generalized complex structure. The manifold $\overline{\mu}^{-1}(0)$ can be easily identified with $M \times P$ and therefore the quotient $M \times_G P$ becomes a generalized complex manifold.

The twisting form for the generalized complex structure on $M \times_G P$ (following [11, Corollary 4.7]) is the basic 3-form

$$H_1 + p^* H_2 + d\big(\theta^a \xi_a\big) - \frac{1}{2} d\big(\theta^b \theta^c \iota_c \xi_b\big) \in \Omega^3(M \times P)_{bas}$$

where $\theta^a, \theta^b \ldots$ are the connection 1-forms for the principal G-bundle $M \times P \to M \times_G P$. □

Remark 5.4. Let us finish by noting that in the case that the manifolds BG_k are symplectic (for example when $G = U(n)$ and the BG_k's are the complex grassmanians) then the manifolds $M \times_G EG_k$ would acquire a generalized complex structure with twisting form

$$H_1 + d\big(\theta^a \xi_a\big) - \frac{1}{2} d\big(\theta^b \theta^c \iota_c \xi_b\big).$$

This three form is the same one we used in Theorem 3.1 to construct the twisted Chern–Weil homomorphism.

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Appendix A

Let us consider the extended action of the circle S^1 on the manifold S^1 as in Example 4.11, and let us calculate the cohomology of the uncompleted complex $\Omega^\bullet(S^1) \otimes \mathbb{R}[\Omega]$ with differential $d_{\mathfrak{g},\delta} = d - d\theta\, \Omega \wedge$.

The complex is therefore given by even forms $\sum_{i=0}^n f_i \Omega^i$ where $f_i \in C^\infty(S^1)$, and by odd forms $\sum_{j=0}^m g_j\, d\theta \Omega^j$ with $g_j\, d\theta \in \Omega^1(S^1)$. It follows that the closed forms are all the odd forms.

Now let us find the cohomology of the complex; for this we need to understand which odd forms are cohomologous. If we consider the equality

$$d_{\mathfrak{g},\delta}\big(f_j \Omega^j\big) = \frac{\partial f_j}{\partial \theta}\, d\theta\, \Omega^j - f^j\, d\theta\, \Omega^{j+1} \tag{A.1}$$

we can see that the following odd forms are all cohomologous:

$$\begin{aligned}
\sum_{i=0}^n g_i\, d\theta\, \Omega^i &\simeq g_0\, d\theta + g_1\, d\theta \Omega + \cdots + g_{n-2}\, d\theta \Omega^{n-2} + \left(\frac{\partial g_n}{\partial \theta} + g_{n-1}\right) d\theta \Omega^{n-1} \\
&\simeq g_0\, d\theta + g_1\, d\theta \Omega + \cdots + g_{n-3}\, d\theta \Omega^{n-3} + \left(\frac{\partial^2 g_n}{\partial \theta^2} + \frac{\partial g_{n-1}}{\partial \theta} + g_{n-2}\right) d\theta \Omega^{n-2} \\
&\simeq \left(\frac{\partial^n g_n}{\partial \theta^n} + \frac{\partial^{n-1} g_{n-1}}{\partial \theta^{n-1}} + \cdots + \frac{\partial g_1}{\partial \theta} + g_0\right) d\theta \Omega^0.
\end{aligned}$$

So we can focus only on the odd forms that have only non-zero component in the coefficient of Ω^0. Let us see which odd forms in $\Omega^1(S^1) \cong C^\infty(S^1) \otimes_{\mathbb{R}\mathbb{R}} [d\theta]$ are exact. We have that

$$d_{\mathfrak{g},\delta}\left(\sum_{j=0}^m f_j u^j\right) = g\, d\theta$$

for f_j and g in $C^\infty(S^1)$. This equation implies the following set of equalities:

$$\begin{aligned}
&df_0 = g\, d\theta, \\
&df_1 = f_0\, d\theta, \\
&\ldots \\
&df_m = f_{m-1}\, d\theta, \\
&0 = f_m\, d\theta
\end{aligned}$$

and if we check these equations starting from bottom to top, we see that $0 = f_m = f_{m-1} = \cdots = f_0$ and therefore none of the forms of the type $g d\theta$ are exact. Then we can conclude that the cohomology of the uncompleted complex is equal to

$$H^0\big(\Omega^\bullet\big(S^1\big) \otimes \mathbb{R}[\Omega]; d_{\mathfrak{g},\delta}\big) = 0, \qquad H^1\big(\Omega^\bullet\big(S^1\big) \otimes \mathbb{R}[\Omega]; d_{\mathfrak{g},\delta}\big) \cong \Omega^1\big(S^1\big).$$

Let us see what is the $H^*(BS^1) = \mathbb{R}[\Omega]$ module structure. We just need to check what happens with the forms whose only non-zero coefficient is the one of Ω^0. Then $\Omega \cdot (g\, d\theta) = g\, d\theta\, \Omega$, and from Eq. (A.1) we have that $\Omega \cdot g\, d\theta \simeq \frac{\partial g}{\partial \theta}\, d\theta$. Then, the $\mathbb{R}[\Omega]$ module structure on $H^1(\Omega^\bullet(S^1) \otimes \mathbb{R}[\Omega]; d_{\mathfrak{g},\delta}) \cong C^\infty(S^1) \otimes \mathbb{R}\langle d\theta\rangle$ is given by the operator $\frac{\partial}{\partial\theta} : C^\infty(S^1) \to C^\infty(S^1)$,

$$H^1\big(\Omega^\bullet\big(S^1\big) \otimes \mathbb{R}[\Omega]; d_{\mathfrak{g},\delta}\big) \xrightarrow{\Omega\cdot} H^1\big(\Omega^\bullet\big(S^1\big) \otimes \mathbb{R}[\Omega]; d_{\mathfrak{g},\delta}\big),$$

$$C^\infty\big(S^1\big) \xrightarrow{\frac{\partial}{\partial\theta}} C^\infty\big(S^1\big).$$

It is easy to see now that the torsion submodule of $C^\infty(S^1)$ as a $\mathbb{R}[\Omega]$-module is infinitely generated (the functions $\sin(k\theta)$ belong to the torsion submodule). Therefore the cohomology $H^\bullet(\Omega^\bullet(S^1) \otimes \mathbb{R}[\Omega]; d_{\mathfrak{g},\delta})$ is infinitely generated as a $\mathbb{R}[\Omega]$-module and this prevents this cohomology to have a topological meaning.

Also note that the completed algebra $\Omega^\bullet(S^1) \otimes \mathbb{R}[[\Omega]]$ is isomorphic to the algebra

$$\Omega^\bullet\big(S^1\big) \otimes \mathbb{R}[\Omega] \otimes_{\mathbb{R}[\Omega]} \mathbb{R}\big[[\Omega]\big].$$

Then one may think that one can calculate the twisted equivariant cohomology by tensoring with $\otimes_{\mathbb{R}[\Omega]}\mathbb{R}[\![\Omega]\!]$ the cohomology of the uncompleted differential complex. This turns out to be false in general as one can check from the previous example: the twisted equivariant cohomology is $H^\bullet_{S^1}(S^1, d\theta\Omega) = \mathbb{R}$ meanwhile the cohomology of the uncompleted differential complex tensored with $\otimes_{\mathbb{R}[\Omega]}\mathbb{R}[\![\Omega]\!]$ is

$$H^\bullet\big(\Omega^\bullet\big(S^1\big) \otimes \mathbb{R}[\Omega]; d_{\mathfrak{g},\delta}\big) \otimes_{\mathbb{R}[\Omega]}\mathbb{R}[\![\Omega]\!] \cong \Omega^1\big(S^1\big) \otimes_{\mathbb{R}[\Omega]}\mathbb{R}[\![\Omega]\!]$$

which is an infinitely generated $\mathbb{R}[\![\Omega]\!]$-module.

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