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Automorphic forms and cohomology theories on Shimura curves of small discriminant

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Abstract

We apply Lurie's theorem to produce spectra associated to 1-dimensional formal group laws on the Shimura curves of discriminants 6, 10, and 14. We compute rings of automorphic forms on these curves and the homotopy of the associated spectra. At $p = 3$, we find that the curve of discriminant 10 recovers much the same as the topological modular forms spectrum, and the curve of discriminant 14 gives rise to a model of a truncated Brown–Peterson spectrum as an E_∞ ring spectrum.

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Keywords: Ring spectrum; Shimura curve; Brown–Peterson spectrum

1. Introduction

The theory TMF of topological modular forms associates a cohomology theory to the canonical 1-dimensional formal group law on the (compactified) moduli of elliptic curves. The chromatic data of this spectrum is determined by the height stratification on the moduli of elliptic

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curves, and the homotopy groups can be computed by means of a small Hopf algebroid associated to Weierstrass curves [4].

With newly available machinery this can be generalized to similar moduli. Associated to a discriminant N which is a product of an even number of distinct primes, there is a Shimura curve that parametrizes abelian surfaces with action of a division algebra. These are similar in spirit to the moduli of abelian varieties studied previously in topology [7,6]. Shimura curves carry 1-dimensional p-divisible groups which are split summands of the p-divisible group of the associated abelian surface, and Lurie's theorem allows these to give rise to associated E_∞ ring spectra analogous to TMF. See Section 2.6 for details.

In addition, there are further quotients of these Shimura curves by certain Atkin–Lehner involutions w_d for $d|N$. Given a choice of square root of $\pm d$ in $\mathbb{Z}_p$, we can lift this involution to an involution on the associated p-divisible group, and produce spectra associated to quotients of the Shimura curve.

Our goal in this paper is a computational exploration of spectra associated to these curves for the three smallest valid discriminants: 6, 10, and 14. These computations rely on a computation of rings of automorphic forms on these curves over $\mathbb{Z}[1/N]$ in the absence of anything approaching a Weierstrass equation. The associated cohomology theories, like TMF, only detect information in chromatic layers 0 through 2. One might ultimately hope that a library of techniques at low chromatic levels will allow inroads to chromatic levels 3 and beyond.

At discriminant 6, we find that the ring of automorphic forms away from 6 is the ring

$$\mathbb{Z}[1/6, U, V, W]/\big(U^4 + 3V^6 + 3W^2\big).$$

See Section 3. This recovers a result of Baba–Granath in characteristic zero [2]. The homotopy of the associated spectrum consists of this ring together with a Serre dual portion that is concentrated in negative degrees and degree 3. We also exhibit a connective version, with the above ring as its ring of homotopy groups.

At discriminant 14, we obtain an application to Brown–Peterson spectra. Baker defined generalizations of Brown–Peterson and Johnson–Wilson spectra with homotopy groups that are quotients of BP_* by regular sequence of generators other than the Araki or Hazewinkel generators [3], and Strickland made use of similar constructions in studying products on MU-modules [23]. We show that there exists such a generalized truncated Brown–Peterson spectrum $\mathrm{BP}\langle 2\rangle'$ admits an E_∞ structure at the prime 3. Associated to a prime p, these ring spectra BP and $\mathrm{BP}\langle n\rangle'$ are complex orientable cohomology theories carrying formal group laws of particular interest. The question of whether BP admits an E_∞ ring structure has been an open question for over 30 years, and has become increasingly relevant in modern chromatic homotopy theory. We find that the connective cover of the cohomology theory associated to an Atkin–Lehner quotient at discriminant 10 is a model for $(\mathrm{BP}\langle 2\rangle')^\wedge_3$. To our knowledge, this is the first known example of a generalized $\mathrm{BP}\langle n\rangle$ having an E_∞ ring structure for $n > 1$.

At discriminant 10, we obtain an example with torsion in its homotopy groups. We examine the homotopy of the spectrum associated to the Shimura curve, and specifically at $p = 3$ we give a complete computation together with the unique lifts of the Atkin–Lehner involutions. The invariants under the Atkin–Lehner involutions in the cohomology are isomorphic to the cohomology of the 3-local moduli of elliptic curves, and the associated homotopy fixed point spectrum has the same homotopy groups as the localization $\mathrm{TMF}_{(3)}$ of the spectrum of topological modular forms. This isomorphism does not appear to arise due to geometric considerations and the reason for it is, to be brief, unclear.

The main technique allowing these computations to be carried out is the theory of complex multiplication. One first determines data for the associated orbifold over $\mathbb{C}$, for example by simply taking tables of this data [1], or making use of the Eichler–Selberg trace formula. One can find defining equations for the curve over $\mathbb{Q}$ using complex multiplication points and level structures. Our particular examples were worked out by Kurihara [13] and Elkies [11]. We can then determine uniformizing equations over $\mathbb{Z}[1/N]$ by excluding the possibility that certain complex-multiplication points can have common reductions at most primes. This intersection theory on the associated arithmetic surface was examined in detail by Kudla, Rapoport, and Yang [12] and reduces in our case to computations of Hilbert symbols. The rings of automorphic forms can then be explicitly determined away from 2 in terms of meromorphic sections of the cotangent bundle. In order to compute the Atkin–Lehner operators, we apply the Eichler–Selberg trace formula. Finally, we move from automorphic forms to homotopy theory via the Adams–Novikov spectral sequence.

We now sketch the structure of the paper. In Section 2 we review background material on quaternion algebras, Shimura curves, Atkin–Lehner involutions, points with complex multiplication, and automorphic forms. In Section 2.6 we describe how Shimura curves give rise to spectra via Lurie's theorem, and how their homotopy groups are related to rings of automorphic forms. In Section 3 we find defining equations for the Shimura curve of discriminant 6 over $\mathbb{Z}[1/6]$ in detail by the sequence of steps described above, as well as the ring of automorphic forms and the homotopy of various associated spectra. In Section 4 we carry out the same program for an Atkin–Lehner quotient of the Shimura curve of discriminant 14, culminating in the construction of an E_∞-ring structure on a generalized truncated Brown–Peterson spectrum. In Section 5 we apply the same methods to the curve of discriminant 10. The existence of elliptic points of order 3, and hence 3-primary torsion information, complicates this computation and necessitates working equivariantly with smooth covers induced by level structures. In Sections 5.7, 5.8, and 5.9 we use this equivariant data to compute the homotopy groups of the resulting spectrum via homotopy-fixed-point spectral sequences.

2. Background

In this section we begin with a brief review of basic material on Shimura curves. Few proofs will be included, as much of this is standard material in number theory.

For a prime p and a prime power $q = p^k$, we write $\mathbb{F}_q$ for the field with q elements, $\mathbb{Z}_q$ for its ring of Witt vectors, and $\mathbb{Q}_q$ for the associated field of fractions. The elements ω and i are fixed primitive 3rd and 4th roots of unity in $\mathbb{C}$.

2.1. Quaternion algebras

A quaternion algebra over $\mathbb{Q}$ is a 4-dimensional simple algebra D over $\mathbb{Q}$ whose center is $\mathbb{Q}$. For each prime p, the tensor product $D \otimes \mathbb{Q}_p$ is either isomorphic to $M_2\mathbb{Q}_p$ or a unique division algebra

$$D_p = \{a + bS \mid a, b \in \mathbb{Q}_{p^2}\}, \tag{2.1}$$

with multiplication determined by the multiplication in $\mathbb{Q}_{p^2}$ and the relations $S^2 = -p$, $aS = Sa^\sigma$. (Here a^σ is the Galois conjugate of a.) We refer to p as either *split* or *ramified* accordingly. Similarly, the ring $D \otimes \mathbb{R}$ is either isomorphic to $M_2\mathbb{R}$ or the ring $\mathbb{H}$ of quaternions. We correspondingly refer to D as split at ∞ (*indefinite*) or ramified at ∞ (*definite*).

We have a bilinear Hilbert symbol $(-,-)_p : (\mathbb{Q}_p^\times)^2 \to \{\pm 1\}$. The element $(a,b)_p$ is 1 or -1 according to whether the algebra generated over $\mathbb{Q}_p$ by elements i and j with relations $i^2 = a$, $j^2 = b$, $ij = -ji$ is ramified or not.

Lemma 2.2. *(See [20].) Write $a = p^{\nu_p(a)}u$ and $b = p^{\nu_p(b)}v$. If p is odd, we have*

$$(a,b)_p = \left\{\frac{-1}{p}\right\}^{\nu_p(a)\nu_p(b)} \left\{\frac{\bar{u}}{p}\right\}^{\nu_p(b)} \left\{\frac{\bar{v}}{p}\right\}^{\nu_p(a)},$$

with $\bar{x}$ being the mod-p reduction. At $p = 2$, we have

$$(a,b)_2 = (-1)^{\frac{(u-1)(v-1)}{8}(2+(u+1)\nu_2(a)+(v+1)\nu_2(b))}.$$

At $p = \infty$, we have $(a,b)_\infty$ equal to -1 if and only if a and b are both negative.

Here $\{\frac{x}{p}\}$ is the Legendre symbol.

Theorem 2.3 *(Quadratic reciprocity). The set of ramified primes of a quaternion algebra D over $\mathbb{Q}$ is finite and has even cardinality. For any such set of primes, there exists a unique quaternion algebra of this type.*

We refer to the product of the finite, ramified primes as the discriminant of D. D is a division algebra if and only of this discriminant is not 1.

The left action of any element x on D has a characteristic polynomial of the form $(x^2 - \mathrm{Tr}(x)x + \mathrm{N}(x))^2$, and $x^2 - \mathrm{Tr}(x) + \mathrm{N}(x)$ is zero in D. We define $\mathrm{Tr}(x)$ and $\mathrm{N}(x)$ to be the *reduced trace* and *reduced norm* of x. This trace and norm are additive and multiplicative respectively. There is a canonical involution $x \mapsto x^\iota = \mathrm{Tr}(x) - x$ on D.

An *order* in D is a subring of D which is a lattice, and the notion of a *maximal order* is clear. The reduced trace and reduced norm of an element x in an order Λ must be integers. Two orders Λ, Λ' are *conjugate* if there is an element $x \in D$ such that $x\Lambda x^{-1} = \Lambda'$.

Theorem 2.4. *(See [10].) If D is indefinite, any two maximal orders Λ and Λ' are conjugate.*

Theorem 2.4 generalizes the Noether–Skolem theorem that any automorphism of D is inner. It is false for definite quaternion algebras.

Proposition 2.5. *An order is maximal if and only if the completions $\Lambda \otimes \mathbb{Z}_p$ are maximal orders of $D \otimes \mathbb{Q}_p$ for all primes p.*

At split primes, this is equivalent to $\Lambda \otimes \mathbb{Z}_p$ being some conjugate of $M_2(\mathbb{Z}_p)$. At ramified primes, there is a *unique* maximal order of the division algebra of Eq. (2.1) given by

$$\Lambda_p = \{a + bS \mid a, b \in \mathbb{Z}_{p^2}\}. \tag{2.6}$$

Let F be a quadratic extension of $\mathbb{Q}$. Then we say the field F *splits* D if any of the following equivalent statements hold:

- There is a subring of D isomorphic to F.
- $D \otimes F \cong M_2(F)$.
- No prime ramified in D is split in F.

2.2. Shimura curves

Fix an indefinite quaternion algebra D over $\mathbb{Q}$ with discriminant $N \neq 1$ and a maximal order $\Lambda \subset D$. We can choose an embedding $\tau : D \hookrightarrow M_2(\mathbb{R})$, and get a composite map $\Lambda^\times \to \mathrm{PGL}_2(\mathbb{R})$. The latter group acts on the space $\mathbb{C} \setminus \mathbb{R}$, which is the union $\mathcal{H} \cup \overline{\mathcal{H}}$ of the upper and lower half-planes. The stabilizer of $\mathcal{H}$ is the subgroup $\Gamma = \Lambda^{\mathrm{N}=1}$ of norm-1 elements in Λ, which is a cocompact Fuchsian group.

Definition 2.7. The complex Shimura curve of discriminant N is the compact Riemann surface $\mathcal{X}_{\mathbb{C}}^D = \mathcal{H}/\Gamma$.

The terminology is, to some degree, inappropriate, as it should properly be regarded as a stack or orbifold. When the division algebra is understood we simply write $\mathcal{X}_{\mathbb{C}}$.

This object has a moduli-theoretic interpretation. The embedding τ embeds Λ as a lattice in a 4-dimensional real vector space $M_2(\mathbb{R})$, and there is an associated quotient torus $\mathbb{T}$ with a natural map $\Lambda \to \mathrm{End}(\mathbb{T})$. The space $\mathcal{X}_{\mathbb{C}}$ is the quotient of the space of compatible complex structures by the Λ-linear automorphism group, and parametrizes 2-dimensional abelian varieties with an action of Λ. Such objects are called *fake elliptic curves*.

The complex Shimura curve can be lifted to an algebraically defined moduli object. As in [14], we consider the moduli of 2-dimensional projective abelian schemes A equipped with an action $\Lambda \to \mathrm{End}(A)$. We impose a constraint on the tangent space $T_e(A)$ at the identity: for any such $A/\mathrm{Spec}(R)$ and a map $R \to R'$ such that $\Lambda \otimes R' \cong M_2(R')$, the two summands of the $M_2(R')$-module $R' \otimes_R T_e(A)$ are locally free of rank 1.

Theorem 2.8. *(See [8, Section 14].) There exists a smooth, proper Deligne–Mumford stack $\mathcal{X}^D$ over* $\mathrm{Spec}(\mathbb{Z}[1/N])$ *parametrizing* 2*-dimensional abelian schemes with Λ-action. The complex points form the curve $\mathcal{X}_{\mathbb{C}}^D$.*

Remark 2.9. We note that related PEL moduli problems typically include the data of an equivalence class of polarization $\lambda : A \to A^\vee$ satisfying $x^\vee\lambda = \lambda x^\iota$ for $x \in \Lambda$. In the fake-elliptic case, there is a unique such polarization whose kernel is prime to the discriminant, up to rescaling by $\mathbb{Z}[1/N]$.

Theorem 2.10. *(See [22].) The curve $\mathcal{X}$ has no real points.*

The object $\mathcal{X}$ has a model that is 1-dimensional and proper over $\mathrm{Spec}(\mathbb{Z})$ and smooth over $\mathrm{Spec}(\mathbb{Z}[1/N])$, but has singular fibers at primes dividing the discriminant. We may view this either as a curve over $\mathrm{Spec}(\mathbb{Z}[1/N])$ or an arithmetic surface.

2.3. Atkin–Lehner involutions

An abelian scheme A over a base scheme S has a subgroup scheme of n-torsion points $A[n]$; if n is invertible in S, the map $A[n] \to S$ is étale and the fibers over geometric points are isomorphic to $(\mathbb{Z}/n)^{2\dim(A)}$. We suppose that A is 2-dimensional and has an action of Λ, so that we inherit an action of the ring Λ/n on $A[n]$.

Let p be a prime dividing N. There is a two-sided ideal I of Λ such that $I^2 = (p)$; in the notation of Eq. (2.6), it is the intersection of Λ with the ideal of Λ_p generated by S. The scheme $A[p]$ has a subgroup scheme H_p of I-torsion, and any choice of generator π of $I_{(p)}$ gives an exact sequence

$$0 \to H_p \to A[p] \xrightarrow{\pi} H_p \to 0.$$

Hence the I-torsion is a subgroup scheme of rank p^2.

For $d|N$, the sum of these over p dividing d is the unique Λ-invariant subgroup scheme $H_d \subset A$ of rank d^2, and there is a natural quotient map $A \to A/H_d$. It is of degree d^2, Λ-linear, and these properties characterize A/H_d uniquely up to isomorphism under A. This gives a natural transformation

$$w_d : A \mapsto (A/H_d)$$

on this moduli of fake elliptic curves, and the composite w_d^2 sends A to $A/A[d]$, which is canonically isomorphic to A via the multiplication-by-d map $m_d : A/A[d] \to A$.

Proposition 2.11. *The map w_d gives rise to an involution on the moduli object called an Atkin–Lehner involution. These involutions satisfy $w_d w_{d'} \simeq w_{dd'}$ if d and d' are relatively prime.*

The Atkin–Lehner involutions form a group isomorphic to $(\mathbb{Z}/2)^k$, with a basis given by involutions w_p for primes p dividing N. There are associated quotient objects of $\mathcal{X}$ by any subgroup of Atkin–Lehner involutions, and we write $\mathcal{X}^*$ for the full quotient.

Theorem 2.12. *(See [16].) For any group G of Atkin–Lehner involutions, the coarse moduli object underlying $\mathcal{X}/G$ is smooth.*

Remark 2.13. The involution w_d on $\mathcal{X}$ does not lift to an involution on the universal abelian scheme $\mathcal{A}$, nor on the associated p-divisible group $\mathcal{A}[p^\infty]$. The natural isogeny $A \to A/H_d$ of abelian varieties with Λ-action gives rise to a natural Λ-linear isomorphism of p-divisible groups $t'_d : \mathcal{A}[p^\infty] \to (w_d)^* \mathcal{A}[p^\infty]$ over $\mathcal{X}$. However, the natural diagram

$$\begin{array}{ccccc} A & \xrightarrow{w_d} & A/H_d & \xrightarrow{w_d} & A/A[d] \\ & \searrow^{[d]} & & & \downarrow{\scriptstyle \sim\ m_d} \\ & & & & A \end{array}$$

gives rise to a composite Λ-linear isomorphism of p-divisible groups on $\mathcal{X}$ as follows:

$$\begin{array}{ccccc} \mathcal{A}[p^\infty] & \xrightarrow{t'_d} & (w_d)^*\mathcal{A}[p^\infty] & \xrightarrow{t'_d} & (w_d^2)^*\mathcal{A}[p^\infty] \\ & & & \searrow^{d} & \downarrow{\scriptstyle m_d} \\ & & & & \mathcal{A}[p^\infty] \end{array}$$

This shows that under the canonical isomorphism of $(w_d^2)^*\mathcal{A}$ with $\mathcal{A}$, the map $(t'_d)^2$ acts as the multiplication-by-d map on the p-divisible group rather than the identity. Therefore, the universal p-divisible group does not descend naturally to $\mathcal{X}/\langle w_d\rangle$. However, if $x \in \mathbb{Z}_p^\times$ satisfies $x^2 = \pm d$, we can lift the involution w_d of $\mathcal{X}$ to a Λ-linear involution $t_d = x^{-1}t'_d$ of the p-divisible group and produce a Λ-linear p-divisible group on the quotient stack.

2.4. *Complex multiplication in characteristic* 0

Fix an algebraically closed field k. A k-point of $\mathcal{X}$ consists of a pair (A, ϕ) of an abelian surface over $\mathrm{Spec}(k)$ and an action $\phi : \Lambda \to \mathrm{End}(A)$. We write $\mathrm{End}_\Lambda(A)$ for the ring of Λ-linear endomorphisms. The ring $\mathrm{End}(A)$ is always a finitely generated torsion-free abelian group, and hence so is $\mathrm{End}_\Lambda(A)$.

Proposition 2.14. *(See [17, p. 202].) If k has characteristic zero, then either* $\mathrm{End}_\Lambda(A) \cong \mathbb{Z}$ *or* $\mathrm{End}_\Lambda(A) \cong \mathcal{O}$ *for an order $\mathcal{O}$ in a quadratic imaginary field F that splits D. In the latter case, we say A has complex multiplication by F (or $\mathcal{O}$).*

Complex multiplication points over $\mathbb{C}$ have explicit constructions. Fix a quadratic imaginary subfield F of $\mathbb{C}$. Suppose we have an embedding $F \to D^{op}$, and let $\mathcal{O}$ be the order $F \cap \Lambda^{op}$. The ring Λ^{op} acts Λ-linearly on the torus $\mathbb{T} = \Lambda \otimes S^1$ via right multiplication. There are precisely two (complex-conjugate) complex structures compatible with this action, but only one compatible with the right action of $F \otimes \mathbb{R} = \mathbb{C}$. This embedding therefore determines a unique point on the Shimura curve with $\mathcal{O}$-multiplication. The corresponding abelian surface A splits up to isogeny as a product of two elliptic curves with complex multiplication by F.

Lemma 2.15. *Suppose A represents a point on $\mathcal{X}_\mathbb{C}$ fixed by an Atkin–Lehner involution w_d. Then A has complex multiplication by $\mathbb{Q}(\sqrt{-d})$ if $d > 2$, and by $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$ if $d = 2$.*

Proof. The statement $(w_d)^*A \cong A$ is equivalent to the existence of a Λ-linear isogeny $x : A \to A$ whose characteristic polynomial is of the form $x^2 + tx + d$, because the kernel of such an isogeny must be H_d. This element generates the quadratic field $\mathbb{Q}(\sqrt{t^2 - 4d})$. This must be an imaginary quadratic field splitting D, and so must not split at primes dividing d. A case-by-case check gives the statement of the lemma. □

2.5. *Automorphic forms*

Suppose p does not divide N, so that we may fix an isomorphism $\Lambda \otimes \mathbb{Z}_p \cong M_2(\mathbb{Z}_p)$; this is equivalent to choosing a nontrivial idempotent $e \in \Lambda \otimes \mathbb{Z}_p$. For any fake elliptic curve A with

Λ-action the p-divisible group $A[p^\infty]$ breaks into a direct sum $e \cdot A[p^\infty] \oplus (1-e) \cdot A[p^\infty]$ of two canonically isomorphic p-divisible groups of dimension 1 and height 2, and the formal component of $A[p^\infty]$ carries a 1-dimensional formal group law. Similarly, $e \cdot \mathrm{Lie}(A)$ is a 1-dimensional summand of the Lie algebra of A.

The Shimura curve $\mathcal{X}$ over $\mathbb{Z}_p$ therefore has two naturally defined line bundles. The first is the cotangent bundle κ over $\mathrm{Spec}(\mathbb{Z}[1/N])$. The second is the split summand $\omega = e \cdot z^* \Omega_{A/\mathcal{X}}$ of the pullback of the vertical cotangent bundle along the zero section $z: \mathcal{X} \to \mathcal{A}$ of the universal abelian surface. The sections of this bundle can be identified with invariant 1-forms on a summand of the formal group of $\mathcal{A}$. The Kodaira–Spencer theory takes the following form.

Proposition 2.16. *There are natural isomorphisms $\kappa \cong \omega^2 \cong z^*(\bigwedge^2 \Omega_{A/\mathcal{X}})$.*

Proof. This identification of the tangent space is based on the deformation theory of abelian schemes. See [18] for an introduction to the basic theory. For space reasons we will only sketch some details.

Given a point on the Shimura curve represented by A over k, the relative tangent space over $\mathbb{Z}[1/N]$ of the moduli of abelian varieties at A is in bijective correspondence with the set of lifts of A to $k[\epsilon]/(\epsilon^2)$. This set of deformations is isomorphic to the group

$$H^1(A, TA) \cong H^1(A, \mathcal{O}) \otimes T_e(A) \cong T_e\big(A^\vee\big) \otimes T_e(A).$$

Here TA is the relative tangent bundle of A, which is naturally isomorphic to the tensor product of the trivial bundle and the tangent space at the identity $T_e(A)$, and $A^\vee$ is the dual abelian variety. The polarization mentioned in Remark 2.9 makes $H^1(A, \mathcal{O}) \cong T_e(A^\vee)$ isomorphic to $T_e(A)$, with the right Λ-action induced by the canonical involution ι on Λ. The set of deformations that admit lifts of the Λ-action are those elements equalizing the endomorphisms $1 \otimes x$ and $x^\vee \otimes 1$ of $T_e(A^\vee) \otimes T_e(A)$ for all $x \in \Lambda$. The cotangent space of $\mathcal{X}$ at A, which is dual to this equalizer, therefore admits a description as a tensor product:

$$\Omega_{A^\vee/R} \otimes_\Lambda \Omega_{A/k} \cong (\Omega_{A/k})^t \otimes_{M_2(Z_p)} \Omega_{A/k}.$$

The tangent space of A being free of rank 2 implies that this is isomorphic to $\bigwedge^2 \Omega_{A/k}$. □

We view κ and ω^2 as canonically identified. An *automorphic form* of weight k on $\mathcal{X}$ is a section of $\omega^{\otimes k}$, and one of even weight is a section of $\kappa^{\otimes k/2}$.

The isomorphism $\kappa \cong \omega^2$ is only preserved by isomorphisms of abelian schemes. For instance, the natural map $A \mapsto A/A[n]$ gives rise to the identity self-map on the moduli via the isomorphism $m_n: A/A[n] \to A$, and acts trivially on the cotangent bundle, but it acts nontrivially on the vertical cotangent bundle $\Omega_{A/\mathcal{X}}$ of the universal abelian scheme $\mathcal{A}$.

The canonical negation map $[-1]: \mathcal{A} \to \mathcal{A}$ acts by negation on ω, and hence we have the following.

Lemma 2.17. *For k odd, the cohomology groups*

$$H^i\big(\mathcal{X}, \omega^k\big) \otimes \mathbb{Z}[1/2]$$

are zero.

2.6. Application to topology

As in Section 2.5, we fix a nontrivial idempotent e in $\Lambda \otimes \mathbb{Z}_p$. Given a point A of the moduli $\mathcal{X}$ over a p-complete ring, Serre–Tate theory shows that the deformation theory of A is the same as the deformation theory of $e \cdot A[p^\infty]$; see [7, Section 7.3] for the basic argument. We may then apply Lurie's theorem.

Theorem 2.18. *There is a lift of the structure sheaf $\mathcal{O}$ on the étale site of $\mathcal{X}_p^\wedge$ to a sheaf $\mathcal{O}^{der}$ of locally weakly even periodic E_∞ ring spectra, equipped with an isomorphism between the formal group data of the resulting spectrum and the formal part of the p-divisible group. Associated to an étale map $U \to \mathcal{X}_p^\wedge$, the Adams–Novikov spectral sequence for the homotopy of $\Gamma(U, \mathcal{O}^{der})$ takes the form*

$$H^s\big(U, \omega^{\otimes t}\big) \to \pi_{2t-s}\Gamma\big(U, \mathcal{O}^{der}\big).$$

In particular, when $U = \mathcal{X}_p^\wedge$ we have an associated global section object which might be described as a "fake-elliptic" cohomology theory.

Definition 2.19. We define the *p-complete fake topological modular forms spectrum* associated to the quaternion algebra D to be the p-complete E_∞ ring spectrum

$$\mathrm{TAF}_p^D = \Gamma\big(\mathcal{X}_p^\wedge, \mathcal{O}^{der}\big).$$

We define the *complete fake topological modular forms spectrum* associated to the quaternion algebra D to be the profinitely-complete E_∞ ring spectrum

$$\prod_{p \nmid N} \mathrm{TAF}_p^D.$$

If some subgroup of the Atkin–Lehner operators w_d on the p-divisible group are given choices of rescaling to involutions at p (see Remark 2.13), we obtain an action of a finite group $(\mathbb{Z}/2)^r$ on this spectrum by E_∞ ring maps, with associated homotopy fixed point objects equivalent to global section objects on the quotient stack.

We may attempt to define "p-local" and "global" objects by lifting the natural arithmetic squares

$$\begin{array}{ccc} \mathcal{X}_p^\wedge \otimes \mathbb{Q} & \longrightarrow & \mathcal{X}_p^\wedge \\ \downarrow & & \downarrow \\ \mathcal{X} \otimes \mathbb{Q} & \longrightarrow & \mathcal{X}_{(p)} \end{array} \qquad \begin{array}{ccc} \prod \mathcal{X}_p^\wedge \otimes \mathbb{Q} & \longrightarrow & \prod \mathcal{X}_p^\wedge \\ \downarrow & & \downarrow \\ \mathcal{X} \otimes \mathbb{Q} & \longrightarrow & \mathcal{X} \end{array}$$

to diagrams of E_∞ ring spectra on global sections, perhaps after a connective cover. The spectrum associated to $\mathcal{X} \otimes \mathbb{Z}_p$ is constructed by Lurie's theorem, whereas it may be possible to construct the associated rational object by methods of multiplicative rational homotopy theory. We will discuss such a global section object in a specific case in Section 3.7.

Remark 2.20. It seems conceivable that a direct construction analogous to that for topological modular forms [5] might be possible, bypassing the need to invoke Lurie's theorem. However, this construction is unlikely to be simpler than a construction of TMF with level structure.

Additionally, given more data about the structure of the modular curve over $\mathbb{Q}$, it seems plausible that the methods of [5, Section 9] might provide a direct lift of the structure sheaf of $\mathcal{X}$ to E_∞ ring spectra. However, as the rings of rational automorphic forms have more complicated structure than those in the modular case this requires further investigation.

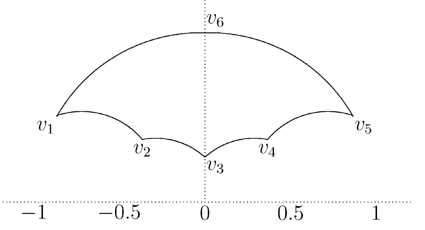
3. Discriminant 6

In this section, D denotes the quaternion algebra of discriminant $N = 6$ and $\mathcal{X} = \mathcal{X}^D$ the associated Shimura curve. Our goal in this section is to compute the cohomology

$$H^s\big(\mathcal{X}, \kappa^{\otimes t}\big),$$

together with the action of the Atkin–Lehner involutions. Write $\mathcal{X} \to X$ for the map to the underlying coarse moduli.

A fundamental domain for the action of the group of norm-1 elements $\Gamma = \Lambda^{N=1}$ on the upper half-plane is pictured as follows. (See [1].)



The edges meeting at v_2 are identified, as are the edges meeting at v_4 and those meeting at v_6. The resulting surface has 2 elliptic points of order 2, coming from v_6 and $\{v_1, v_3, v_5\}$, and 2 elliptic points of order 3, coming from v_2 and v_4. The underlying surface has genus 0 and hyperbolic volume $\frac{2\pi}{3}$.

This fundamental domain leads to a presentation of the group Γ:

$$\Gamma = \big\langle \gamma_{v_2}, \gamma_{v_4}, \gamma_{v_6} \bigm| \gamma_{v_2}^3 = \gamma_{v_4}^3 = \gamma_{v_6}^2 = \big(\gamma_{v_2}^{-1}\gamma_{v_6}\gamma_{v_4}\big)^2 = 1\big\rangle.$$

Here γ_v generates the stabilizer of the vertex v. In particular, Γ has a quotient map to $\mathbb{Z}/6$ that sends γ_{v_2} and γ_{v_4} to 2 and γ_{v_6} to 3; this is the maximal abelian quotient of Γ. Let K be the kernel of this map. This corresponds to a Galois cover $\mathcal{X}' \to \mathcal{X}$ of the Shimura curve $\mathcal{X}$ by a smooth curve, with Galois group cyclic of order 6. The Riemann–Hurwitz formula implies that $\mathcal{X}'$ is of genus 2. This cover can be obtained by imposing level structures at the primes 2 and 3.

3.1. Points with complex multiplication

We would like to now examine some specific points with complex multiplication on this curve. We need to first state some of Shimura's main results on complex multiplication [21] and establish some notation.

Fix a quadratic imaginary field F with ring of integers $\mathcal{O}_F$. Let $I(F)$ be the symmetric monoidal category of fractional ideals of $\mathcal{O}_F$, i.e. finitely generated $\mathcal{O}_F$-submodules of F with monoidal structure given by product of ideals. The morphisms consist of multiplication by scalars of F. We write $\mathrm{Cl}(F)$ for this *class groupoid* of isomorphism classes in $I(F)$, with cardinality $\#\{\mathrm{Cl}(F)\}$ and mass $|\mathrm{Cl}(F)| = \#\{\mathrm{Cl}(F)\}/|\mathcal{O}_F^\times|$. There is an associated Hilbert class field H_F which is a finite unramified extension of F with abelian Galois group canonically identified with $\mathrm{Cl}(F)$.

The Shimura curve $\mathcal{X}_\mathbb{C}$ has a finite set of points associated to abelian surfaces with endomorphism ring $\mathcal{O}_F$. The following theorem (in different language) is due to Shimura.

Theorem 3.1. *(See [21, 3.2, 3.5].) Let A be a point with complex multiplication by $\mathcal{O}_F$. Then A is defined over the Hilbert class field H_F. If S is the groupoid of points with $\mathcal{O}_F$-multiplication that are isogenous to A, then S is a complete set of Galois conjugates of A over F, and there is a natural equivalence of S with the groupoid $I(F)$ of ideal classes. This equivalence takes the action of the class group* $\mathrm{Cl}(F)$ *to the Galois action.*

In the specific case of discriminant 6, the curve $\mathcal{X}_\mathbb{Q}$ has two elliptic points of order 2 that have complex multiplication by the Gaussian integers $\mathbb{Z}[i]$ and are the unique fixed points by the involution w_2. They must be defined over the class field $\mathbb{Q}(i)$, and since the curve $\mathcal{X}$ has no real points they must be Galois conjugate over $\mathbb{Q}$. The involutions w_3 and w_6 interchange them. We denote these points by P_1 and P_2.

Similarly, there are two elliptic points of order 3 with complex multiplication by $\mathbb{Z}[\omega]$ that are Galois conjugate, defined over $\mathbb{Q}(\omega)$, and have stabilizers w_3. We denote these points by Q_1 and Q_2.

Finally, there are two ordinary points fixed by the involution w_6 with complex multiplication by $\mathbb{Z}[\sqrt{-6}]$ that are Galois conjugate. If K is the field of definition, then K must be quadratic imaginary and $K(\sqrt{-6})$ must be the Hilbert class field $\mathbb{Q}(\sqrt{-6}, \omega)$, which has only 3 quadratic subextensions generated by $\sqrt{-6}$, ω, and $\sqrt{2}$. Therefore, these CM-points must be defined over $\mathbb{Q}(\omega)$. We denote these points by R_1 and R_2.

3.2. Reduction mod p and divisor intersection

An abelian surface A in characteristic zero with complex multiplication has reductions at finite primes. In this section, we examine when two points with complex multiplication can have a common reduction at a prime p not dividing the discriminant N.

Suppose K is an extension field of $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$. As $\mathcal{X}$ is a proper Deligne–Mumford stack, any map $\mathrm{Spec}(K) \to \mathcal{X}$ extends to a unique map $\mathrm{Spec}(\mathcal{O}_K) \to \mathcal{X}$, representing canonical isomorphism classes of reductions mod-p of an abelian surface over K. The associated image of $\mathrm{Spec}(\mathcal{O}_K)$ can be viewed as a horizontal divisor on the arithmetic surface $\mathcal{X}$ over $\mathcal{O}_K$. If the point has complex multiplication by a quadratic imaginary field, then so do all points on this divisor.

We wish to obtain criteria to check when two such CM divisors can have nonempty intersection over $\mathrm{Spec}(\mathbb{Z}[1/N])$. Our treatment is based on the intersection theory developed in [12]. However, our needs for the theory are more modest, as we only want to exclude such intersections.

Proposition 3.2. *(See [14, 5.2].) Suppose k is an algebraically closed field of characteristic $p > 0$, $p \nmid N$. The endomorphism ring of a point (A, ϕ) of the Shimura curve $\mathcal{X}$ over k must be one of the following three types.*

- $\mathrm{End}_\Lambda(A) = \mathbb{Z}$.
- $\mathrm{End}_\Lambda(A) = \mathcal{O}$ *for an order in a quadratic imaginary field F.*
- $\mathrm{End}_\Lambda(A)$ *is a maximal order in D', where D' is the unique "switched" quaternion algebra ramified precisely at ∞, the characteristic p, and the primes dividing N.*

We refer to the third type of surface as *supersingular*, and in this case A is isogenous to a product of two supersingular elliptic curves. Over a finite field, the endomorphism ring of (A, ϕ) is always larger than $\mathbb{Z}$.

Proposition 3.3. *Suppose A and A' are CM-points in characteristic zero, with endomorphism rings generated by $x \in \mathrm{End}_\Lambda(A)$ and $y \in \mathrm{End}_\Lambda(A')$, with d_x and d_y the discriminants of the characteristic polynomials of x and y. Assume these generate non-isomorphic field extensions of $\mathbb{Q}$. If the divisors associated to A and A' intersect at a point in characteristic p, then there exists an integer m with*

$$0 > \Delta \stackrel{\mathrm{def}}{=} \bigl(2m + \mathrm{Tr}(x)\,\mathrm{Tr}(y)\bigr)^2 - d_x d_y$$

such that the Hilbert symbols

$$(\Delta, d_y)_q = (d_x, \Delta)_q$$

are nontrivial precisely when $q = \infty$ or $q \mid pN$.

Proof. As the divisors associated to A and A' intersect, they must have a common reduction $\bar{A} \cong \bar{A}'$ in characteristic p with endomorphism ring containing both x and y. This forces the object $\bar{A}$ to be supersingular.

Let $\Lambda' = \mathrm{End}_\Lambda(\bar{A})$ be the associated maximal order in the division algebra D'. The subset $L = \{u + vx + wy \mid u, v, w \in \mathbb{Z}\}$ has a ternary quadratic form induced by the reduced norm, taking integral values:

$$\begin{aligned}(u + vx + wy) &\mapsto -\mathrm{N}_{D'}(u + vx + wy)\\ &= -u^2 - \mathrm{N}(x)v^2 - \mathrm{N}(y)w^2 - \mathrm{Tr}(x)uv - \mathrm{Tr}(y)uw + m \cdot vw.\end{aligned}$$

Here $m \in \mathbb{Z}$ is an unknown integer. On trace-zero elements, this quadratic form takes an element to its square. Let $W \subset L \otimes \mathbb{Q}$ be the subset of trace-zero elements (having

$2u = -v\,\mathrm{Tr}(x) - w\,\mathrm{Tr}(y)$); the quadratic form restricts on W to the form

$$(v, w) \mapsto \frac{d_x}{4} \cdot v^2 + \frac{d_y}{4} \cdot w^2 + \left(\frac{\mathrm{Tr}(x)\,\mathrm{Tr}(y)}{2} + m\right) \cdot vw.$$

This quadratic form determines the division algebra D' as follows. We can choose two elements α and β in $W \subset D'$ diagonalizing the form:

$$v'\alpha + w'\beta \mapsto d_1 v'^2 + d_2 w'^2.$$

In particular, we can complete the square in such a way as to obtain the diagonalized form with $(d_1, d_2) = (d_x/4, -\Delta/4d_x)$, which is equivalent to (d_x, Δ), or symmetrically complete in the other variable.

The elements α and β then satisfy $\alpha^2 = d_1$, $\beta^2 = d_2$, and $\alpha^2 + \beta^2 = (\alpha + \beta)^2$, so they anti-commute. The Hilbert symbols $(d_1, d_2)_p$ must then be nontrivial precisely at the places where D' is ramified. □

As m ranges over integer values, Proposition 3.3 leaves a finite number of cases that can be checked by hand. For example, if D is a division algebra ramified at a pair of distinct primes $2 < p < q$, then no divisor associated to a fixed point of the involution w_p intersects any divisor associated to a fixed point of w_q.

We now apply this to the curve of discriminant 6. In the previous section we constructed points P_i over $\mathbb{Q}(i)$ with endomorphism ring $\mathbb{Z}[i]$, Q_i over $\mathbb{Q}(\omega)$ with endomorphism ring $\mathbb{Z}[\omega]$, and R_i over $\mathbb{Q}(\omega)$ with endomorphism ring $\mathbb{Z}[\sqrt{-6}\,]$. The discriminants of these rings are -4, -3, and -24.

Proposition 3.4. *The divisors $P_1 + P_2$, $Q_1 + Q_2$, and $R_1 + R_2$ are pairwise nonintersecting on $\mathcal{X}$.*

Proof. Proposition 3.3 implies that the divisors associated to P_i and Q_j can only intersect in characteristic p if there exists an integer m such that the Hilbert symbols $(-4, 4m^2 - 12)_q$ are nontrivial precisely for $q \in \{2, 3, p, \infty\}$.

- When $m^2 = 0$, $(-4, -12)_q \neq 1$ iff $q \in \{3, \infty\}$.
- When $m^2 = 1$, $(-4, -8)_q \neq 1$ iff $q \in \{2, \infty\}$.
- When $|m| > 1$, $(-4, 4m^2 - 12)_\infty = 1$.

Therefore, no pair P_i and Q_j can intersect over $\mathrm{Spec}(\mathbb{Z}[1/6])$.

Similarly, the intersections $P_i \cap R_k$ are governed by Hilbert symbols of the form $(-4, 4m^2 - 96)_q$. In order for this to be ramified at 3, we must have the second term divisible by 3, which reduces to the cases $(-4, -96)_q$ and $(-4, -60)_q$. These are both nontrivial only at 3 and ∞.

Finally, the intersections $Q_j \cap R_k$ are governed by Hilbert symbols of the form $(-3, 4m^2 - 72)_q$. If m is not divisible by 3, the Hilbert symbol at 3 is $\{\frac{(2m)^2}{3}\} = 1$. We can then reduce to the cases $(-3, -72)_q$ and $(-3, -36)_q$, neither of which can be nontrivial at any finite primes greater than 3. □

3.3. Defining equations for the Shimura curve

In this section we obtain defining equations for the underlying coarse moduli X of the Shimura curve $\mathcal{X}$ over $\mathbb{Z}[1/6]$. Our approach is based on that of Kurihara [13]. These equations are well known, at least over $\mathbb{Q}$, but we will illustrate the method.

In general, a general smooth proper curve $\mathcal{C}$ of genus zero over a Dedekind domain R may not be isomorphic to $\mathbb{P}^1_R$ for several reasons.

First, a general curve may fail to be geometrically connected, or equivalently the ring of constant functions may be larger than R. In our case, the curve $\mathcal{X}$ is connected in characteristic zero, and hence connected over $\mathbb{Z}[1/6]$.

Second, a general curve may fail to have any points over the field of fractions K. A geometrically connected curve of genus zero over $\operatorname{Spec}(K)$ is determined by a non-degenerate quadratic form in three variables over K that has a solution if and only if the curve is isomorphic to $\mathbb{P}^1_K$. For example, X itself has no real points, and so is not isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$.

However, the Galois conjugate points $P_j \in X_{\mathbb{Q}}$ are interchanged by w_2 and w_6, and hence have a common image P, defined over $\mathbb{Q}$, in X/w_2 and X/w_6. Similarly, the Galois conjugate points Q_j have a common image Q defined over $\mathbb{Q}$ in X/w_3 and X/w_6, and R_j have a common image R in X/w_2 and X/w_3. The curve $X^* = X/\langle w_2, w_3\rangle$ has points P, Q, and R over $\mathbb{Q}$. Therefore, these quotient curves are all isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$.

Finally, if a general curve $\mathcal{C}$ of genus zero has a $\operatorname{Spec}(K)$-point, we can fix the associated divisor and call it ∞. Let $G = \mathcal{O} \rtimes \mathcal{O}^*$ be the group of automorphisms of $\mathbb{P}^1$ preserving ∞, or equivalently the group of automorphisms of $\mathbb{A}^1$. The object $\mathcal{C}$ is then a form of $\mathbb{P}^1$ classified by an element in the Zariski cohomology group $H^1_{Zar}(\operatorname{Spec}(R), G)$. In general we have that $H^1_{Zar}(\operatorname{Spec}(R), \mathcal{O})$ is trivial, and so such a form is classified by an element of $H^1_{Zar}(\operatorname{Spec}(R), \mathcal{O}^*)$, which is isomorphic to the ideal class group of R. In particular:

Lemma 3.5. *If R has trivial class group, any geometrically connected curve of genus zero over $\mathbb{P}^1_R$ having a K-point is isomorphic to $\mathbb{P}^1_R$.*

Given two such curves $\mathcal{C}$ and $\mathcal{C}'$, suppose P_i and P'_i $(i \in \{1, 2, 3\})$ are $\operatorname{Spec}(K)$*-points on $\mathcal{C}$ and $\mathcal{C}'$ whose associated* $\operatorname{Spec}(R)$*-points are nonintersecting divisors. There exists a unique isomorphism $\mathcal{C} \to \mathcal{C}'$ over R taking P_i to P'_i.*

At each residue field k of R, the associated map $\mathcal{C}_k \to \mathcal{C}'_k$ is the uniquely determined linear fractional transformation moving the reduction of P_i to that of P'_i.

As $\mathbb{Z}[1/6]$ has trivial class group, Lemma 3.5 implies that the four curves X/w_2, X/w_3, X/w_6, and X^* are noncanonically isomorphic to $\mathbb{P}^1_{\mathbb{Z}[1/6]}$. The divisors fixed by the involutions w_2, w_3, and w_6 are nonintersecting over $\operatorname{Spec}(\mathbb{Z}[1/6])$ by Proposition 3.4, allowing explicit uniformizations to be constructed.

We can choose a defining coordinate $z : X^* \to \mathbb{P}^1$ such that $z(P) = 0$, $z(Q) = \infty$, $z(R) = 1$.

As the divisors $\{P_1, Q, R\}$ are nonintersecting on X/w_3 over $\mathbb{Z}[1/6, \omega]$ and $\{\sqrt{-3}, 0, \infty\}$ are distinct on $\mathbb{P}^1$ over $\mathbb{Z}[1/6, \omega]$, Lemma 3.5 implies that there is a unique coordinate $x : X/w_3 \to \mathbb{P}^1$ defined over $\mathbb{Z}[1/6, \omega]$ such that $x(P_1) = \sqrt{-3}$, $x(Q) = 0$, and $x(R) = \infty$. The involutions w_2 and w_6 fix Q and R, and hence must send x to $-x$. This is invariant under the Galois action and thus lifts to an isomorphism of coarse moduli over $\mathbb{Z}[1/6]$. We have $1 + (3/x^2) = z$.

Similarly, as $\{P, Q_1, R\}$ are nonintersecting on X/w_2 over $\mathbb{Z}[1/6, i]$, there is a unique coordinate y such that $y(P) = 0$, $y(Q_1) = i$, and $y(R) = \infty$, invariant under the Galois action and thus defined over $\mathbb{Z}[1/6]$. We have $1 + 1/(y^2 + 1) = z$. The involutions w_3 and w_6 send y to $-y$.

We then have

$$1 + 3/x^2 = 1 - 1/\bigl(y^2 + 1\bigr), \quad \text{or} \quad x^2 + 3y^2 + 3 = 0.$$

The vanishing locus of x consists of the points Q_i, and the vanishing locus of y consists of the points P_i. The Atkin–Lehner involutions act by $w_2x = -x$, $w_2y = y$ and $w_3x = x$, $w_3y = -y$.

The coordinates x and y generate the pullback curve X, and we thus have the description of X as the smooth projective curve with homogeneous equation

$$X^2 + 3Y^2 + 3Z^2 = 0.$$

Equations equivalent to this were given in [13,11].

3.4. The ring of automorphic forms

In this section we use the defining equations for X to compute the ring of automorphic forms on $\mathcal{X}$.

The graded ring $\bigoplus H^0(\mathcal{X}, \omega^k)$ of automorphic forms can be understood via the coarse moduli object away from primes dividing the order of elliptic points.

Lemma 3.6. *Suppose $\mathcal{X}$ has a finite number of elliptic points p_i of order n_i relatively prime to M. Then pullback of forms induces an isomorphism*

$$H^0\Bigl(X, \kappa_X^t\Bigl(\sum \lfloor t(n_i - 1)/n_i \rfloor p_i\Bigr)\Bigr) \otimes \mathbb{Z}[1/M] \to H^0\bigl(\mathcal{X}, \kappa_{\mathcal{X}}^t\bigr) \otimes \mathbb{Z}[1/M].$$

In other words, a section of the t-fold power of the canonical bundle κ of $\mathcal{X}$ is equivalent to a section of κ^t on the coarse moduli that has poles only along the divisors p_i of degree less than or equal to $\frac{t(n_i-1)}{n_i}$. This can be established by pulling back a meromorphic 1-form ω on X to an equivariant 1-form on $\mathcal{X}$, and examining the zeros and poles. Any other equivariant meromorphic section differs by multiplication by a meromorphic function on X.

For discriminant 6, then, the group of automorphic forms of weight $k = 2t$ on $\mathcal{X}$ is identified with the set of meromorphic sections of κ^t on the curve $\{X^2 + 3Y^2 + 3Z^2 = 0\}$ in $\mathbb{P}^2_{\mathbb{Z}}[1/6]$ that only have poles of order at most $\lfloor t/2 \rfloor$ along the divisor $X = 0$ and at most $\lfloor 2t/3 \rfloor$ along $Y = 0$.

In particular, we have the coordinates $x = X/Z$ and $y = Y/Z$. The form dx has simple zeros along $Y = 0$ and double poles along $Z = 0$. We have the following forms:

$$U = \frac{dx^3}{xy^5}, \qquad V = \frac{dx^2}{xy^3}, \qquad W = \frac{dx^6}{x^3y^{10}}.$$

These have weights 6, 4, and 12 respectively. They generate all possible automorphic forms, and are subject only to the relation

$$U^4 + 3V^6 + 3W^2 = 0.$$

3.5. Atkin–Lehner involutions and the Eichler–Selberg trace formula

In this section we compute the action of the Atkin–Lehner involutions on the ring of automorphic forms on $\mathcal{X}$. The main tool in this is the Eichler–Selberg trace formula, which is similar in application to the Riemann–Roch formula or the Lefschetz fixed-point formula.

Associated to a union of double cosets $\Gamma\alpha\Gamma$ in Λ, there is a Hecke operator on the ring of automorphic forms. In the particular case where this union consists of the set of elements of reduced norm $\pm n$, the associated Hecke operator is denoted by $T(n)$. If d divides N, then the Atkin–Lehner operator w_d is the Hecke operator $T(d)$.

We state the Eichler–Selberg trace formula in our case, as taken from Miyake with minor corrections [15, Section 6]. For our purposes, we will not require level structure, nor terms involving contributions from cusps that occur only in the modular case.

Theorem 3.7. *(See [15, Theorem 6.8.4].) The trace of the Hecke operator $T(n)$ on the vector space of complex automorphic forms of even weight $k \geqslant 2$ is given by*

$$\operatorname{Tr}\bigl(T(n)\bigr) = \delta_{2,k} + \epsilon_n \frac{k-1}{12} n^{k/2-1} \prod_{p|N} (p-1) - \sum_t \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta} \prod_{p|N} \left(1 - \left\{\frac{d}{p}\right\}\right) \sum_{\mathcal{O}} \bigl|\mathrm{Cl}(\mathcal{O})\bigr|.$$

Here t ranges over integers such that $d = t^2 - 4n$ is negative, α and β are roots of $X^2 + tX + n$, and $\mathcal{O}$ ranges over all rings of integral elements $\mathcal{O} \supset \mathbb{Z}[\alpha]$ of the field $\mathbb{Q}(\sqrt{d})$. The element $\delta_{2,k}$ is 1 if $k = 2$ and 0 if $k > 2$, and ϵ_n is 0 unless n is a square.

Remark 3.8. For the unique order $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_F$ of index f of a quadratic imaginary field $\mathbb{Q}(\sqrt{d})$, the mass of the ring class groupoid $\mathrm{Cl}(\mathcal{O})$ is given by

$$\bigl|\mathrm{Cl}(\mathcal{O})\bigr| = \bigl|\mathrm{Cl}(\mathcal{O}_F)\bigr| \cdot f \prod_{p|f} \left(1 - p^{-1}\left\{\frac{d}{p}\right\}\right).$$

Remark 3.9. A word of caution about normalizations is in order.

There are multiple possible choices of normalization for the action of the Hecke operators depending on perspective. In the formula of Theorem 3.7, the Hecke operators $T(n)$ are normalized so that an element $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M_2(\mathbb{R})$ acts on a form of weight k by

$$f(z) \mapsto \frac{\det(A)^{k-1}}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right).$$

In particular, the double coset generated by a scalar $n \in \mathbb{Z}$ acts trivially on forms of weight 2 and rescales functions. This formula is computationally simpler and closely related to zeta functions, but is not suitable for our purposes. The normalization we require is

$$f(z) \mapsto \left(\frac{\det(A)}{cz+d}\right)^k f\left(\frac{az+b}{cz+d}\right),$$

which is compatible with the description of automorphic forms as sections of a tensor power of the line bundle of invariant differentials on the universal abelian scheme. In particular, this normalization makes the Atkin–Lehner operators w_d into ring homomorphisms.

As an immediate application of the trace formula, we find the following.

Corollary 3.10. *If $n > 3$, the self-map w_n on the group of automorphic forms of even weight $k > 0$ has trace*

$$d \cdot \delta_{2,k} + (-n)^{k/2} \prod_{p|N} \left(1 - \left\{\frac{-n}{p}\right\}\right) \sum_{\mathcal{O}} \left|\mathrm{Cl}(\mathcal{O})\right|.$$

The traces of w_2 and w_3 do not admit as compact a formulation.

The Eichler–Selberg trace formula on the curve of discriminant 6 implies that the Atkin–Lehner involutions on the spaces of automorphic forms have the following traces.

κ^t	0	1	2	3	4	5	6	7	8	9	10	11
id	1	0	1	1	1	1	3	1	3	3	3	3
w_2	1	0	-2^2	2^3	2^4	-2^5	-2^6	2^7	2^8	-2^9	-2^{10}	2^{11}
w_3	1	0	-3^2	-3^3	3^4	3^5	3^6	-3^7	-3^8	-3^9	3^{10}	3^{11}
w_6	1	0	6^2	-6^3	6^4	-6^5	6^6	-6^7	6^8	-6^9	6^{10}	-6^{11}

At various primes, as in Remark 2.13, we can extend the universal p-divisible group to quotients of $\mathcal{X}$ and obtain further invariant subrings. However, these depend on choices.

Example 3.11. If 2 and 3 have roots in $\mathbb{Z}_p$ (i.e. $p \equiv \pm 1 \bmod 24$), we can lift these involutions w_d of $\mathcal{X}$ to involutions t_d on the ring of automorphic forms. The rescaling $t_2 = (\sqrt{2})^{-1} w_2$ on forms of weight $2t$ has trace equal to the trace of w_2 divided by $(\sqrt{2})^{2t}$, and similarly for t_3 and t_6. These being ring homomorphisms, it suffices to determine their effects on the generators. In particular, when $t = 2$ we find that the form V is fixed only by t_6, when $t = 3$ we find that the form U is fixed only by t_2, and when $t = 6$, knowing that U^2 is fixed and V^3 is fixed only by t_6 we find that W is fixed only by t_3. (The multiples of the form W are the only ones having only zeros at the points R_i interchanged by the Atkin–Lehner operators, and hence W can only be rescaled.)

The ring of invariants under the resulting action of $\mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by U^2, V^2, and UVW, as the element W^2 becomes redundant. These generators are subject only to the relation

$$\left(U^2\right)^3 V^2 + 3U^2 V^8 + 3(UVW)^2 = 0.$$

Example 3.12. As complementary examples, at $p = 5$ we have square roots of ± 6, and hence two lifts of the involution w_6 to the ring of automorphic forms. The involution $(\sqrt{6})^{-1} w_6$ fixes V and W, negating U, and leaves the invariant subring

$$\mathbb{Z}_5\left[U^2, V, W\right] / \left(U^2\right)^2 + 3V^6 + 3W^2.$$

The involution $(\sqrt{-6})^{-1} w_6$ fixes U and V, negating W, and leaves instead the subring

$$\mathbb{Z}_5[U, V].$$

3.6. Higher cohomology

Since the orders of the elliptic points are units, the only higher cohomology groups are determined by a Serre duality isomorphism

$$H^1(\mathcal{X}, \kappa^t) \cong H^0(\mathcal{X}, \kappa^{1-t}).$$

Explicitly, the open subsets $\mathcal{X} \setminus \{P_j\}$ and $\mathcal{X} \setminus \{Q_j\}$ give an affine cover of $\mathcal{X}$ and produce a Mayer–Vietoris exact sequence on cohomology. Taking invariants under the action of the Atkin–Lehner involutions, we can also obtain exact sequences for the higher cohomology of quotient curves.

Letting R denote the graded ring of automorphic forms

$$R = \mathbb{Z}[1/6, U, V, W]/(U^4 + 3V^6 + 3W^2),$$

we have a Mayer–Vietoris exact sequence as follows.

$$0 \to R \to U^{-1}R \oplus V^{-1}R \to (UV)^{-1}R \to \bigoplus_t H^1(\mathcal{X}, \kappa^t) \to 0.$$

In particular, we find that the element $W(UV)^{-1}$ gives rise to a class in $H^1(\mathcal{X}, \kappa)$ such that multiplication induces duality between H^0 and H^1.

3.7. The associated spectrum

For a prime p, the Adams–Novikov spectral sequence for the associated spectrum TAF_p^D is concentrated in filtrations 0 and 1, which determine each other by duality. The spectral sequence therefore collapses at E_2.

Theorem 3.13. *In positive degrees,*

$$\pi_* \mathrm{TAF}_p^D \cong \mathbb{Z}_p[U, V, W]/(U^4 + 3V^6 + 3W^2) \oplus \mathbb{Z}_p\{D\},$$

where $|U| = 12$, $|V| = 8$, $|W| = 24$, and D is a Serre duality class in degree 3 such that multiplication induces a perfect pairing

$$\pi_t \mathrm{TAF}_p^D \otimes \pi_{3-t} \mathrm{TAF}_p^D \to \mathbb{Z}_p.$$

We remark that we can construct a connective E_∞ ring spectrum taf_p^D as the pullback of a diagram of Postnikov sections.

$$\begin{array}{ccc} \mathrm{taf}_p^D & \longrightarrow & \mathrm{TAF}_p^D[0..\infty] \\ \downarrow & & \downarrow \\ \mathbb{S}_p[0..3] & \longrightarrow & \mathrm{TAF}_p^D[0..3] \end{array}$$

(The lower-left corner is an Eilenberg–Mac Lane spectrum, as $\mathbb{S}_p$ has no nontrivial homotopy groups below degree 7. We denote it this way to show that the lower map exists as a map of Postnikov sections.) The lower map has cofiber $\Sigma^3 \mathbb{HZ}_p$ with third homotopy group generated by the Serre duality generator from $H^1(\mathcal{X}, \kappa)$. The homotopy of taf_p^D is then simply a subring of $\pi_* \mathrm{TAF}_p^D$ mapping isomorphically to the ring of automorphic forms on $\mathcal{X}_p^\wedge$. Other than aesthetic or computational reasons, there is currently no particular reason to prefer this connective cover.

We now discuss the construction of an integral global section object. We have a map of spectra $\prod_{p \nmid N} \mathrm{taf}_p^D \to (\prod_{p \nmid N} \mathrm{taf}_p^D)_{\mathbb{Q}}$ that, on homotopy groups, is the map from the ring of automorphic forms on $\mathcal{X}^\wedge$ to the ring of automorphic forms on the rationalization $\mathcal{X}^\wedge \otimes \mathbb{Q}$.

This latter ring contains the ring of automorphic forms on $\mathcal{X}_{\mathbb{Q}}$. There is a unique homotopy type $\mathrm{taf}_{\mathbb{Q}}^D$ of E_∞-ring spectrum with these homotopy groups. It can be defined as the following left-hand pushout diagram of free E_∞ algebras over the Eilenberg–Mac Lane spectrum $\mathbb{HQ}$, with the homotopy groups given at right:

$$
\begin{array}{ccc}
\mathbb{P}_{\mathbb{Q}}[S^{48}] & \longrightarrow & \mathbb{P}_{\mathbb{Q}}[S^{12} \vee S^8 \vee S^{24}] \\
\downarrow & & \downarrow \\
\mathbb{S}_{\mathbb{Q}} & \longrightarrow & \mathrm{taf}_{\mathbb{Q}}^D
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{Q}[R] & \longrightarrow & \mathbb{Q}[U, V, W] \\
\downarrow & & \downarrow \\
\mathbb{Q} & \longrightarrow & \mathbb{Q}[U, V, W]/(U^4 + 3V^6 + 3W^2)
\end{array}
$$

For any E_∞ algebra T, this homotopy pushout diagram gives rise to a Mayer–Vietoris sequence of homotopy classes of maps of E_∞-algebras out to T, as follows

$$\pi_{49}(T) \to \left[\mathrm{taf}_{\mathbb{Q}}^D, T\right]_{E_\infty} \to \pi_{12}(T) \times \pi_8(T) \times \pi_{24}(T) \to \pi_{48}(T).$$

In particular, as $\pi_{49}(\mathrm{taf}_{\mathbb{Q}}^D)$ and $\pi_{49}(\mathrm{taf}_p^D)$ are both zero, this shows any other E_∞ algebra with this ring of homotopy groups accepts a weak equivalence from $\mathrm{taf}_{\mathbb{Q}}^D$, and $(\prod \mathrm{taf}_p^D)_{\mathbb{Q}}$ accepts a unique homotopy class of map such that the image on homotopy is the subring of rational automorphic forms.

We may then define taf^D as the homotopy pullback in the resulting diagram:

$$
\begin{array}{ccc}
\mathrm{taf}^D & \longrightarrow & \prod_{p \nmid N} \mathrm{taf}_p^D \\
\downarrow & & \downarrow \\
\mathrm{taf}_{\mathbb{Q}}^D & \longrightarrow & (\prod_{p \nmid N} \mathrm{taf}_p^D)_{\mathbb{Q}}
\end{array}
$$

The spectrum taf^D is a connective E_∞-ring spectrum whose homotopy groups form the ring $\mathbb{Z}[1/6, U, V, W]/(U^4 + 3V^6 + 3W^2)$ of modular forms on $\mathcal{X}^D$.

From this, we can form a homotopy pullback diagram of localizations:

$$\begin{array}{ccc} \mathrm{TAF}^D & \longrightarrow & U^{-1}\,\mathrm{taf}^D \\ \downarrow & & \downarrow \\ V^{-1}\,\mathrm{taf}^D & \longrightarrow & (UV)^{-1}\,\mathrm{taf}^D \end{array}$$

The spectrum TAF^D has natural maps to TAF^D_p, each of which is a p-completion map.

4. Discriminant 14

In this section, D denotes the quaternion algebra of discriminant 14 and $\mathcal{X} = \mathcal{X}^D$ the associated Shimura curve. The complex curve $\mathcal{X}_{\mathbb{C}}$ is of genus 1 and has only two elliptic points, both of order 2. The computations are very similar to those in discriminant 6. We will explain some of the main details with the aim of understanding the full Atkin–Lehner quotient $\mathcal{X}^*$ at the prime 3.

The Atkin–Lehner operator w_2 has four fixed points. There are two Galois conjugate points with complex multiplication by $\mathbb{Z}[i]$ defined over $\mathbb{Q}(i)$, and there are also two Galois conjugate points with complex multiplication by $\mathbb{Z}[\sqrt{-2}\,]$ defined over $\mathbb{Q}(\sqrt{-2}\,)$.

The operator w_7 has no fixed points, as $\mathbb{Q}(\sqrt{-7}\,)$ does not split this division algebra.

The operator w_{14} has four fixed points, each having complex multiplication by $\mathbb{Z}[\sqrt{-14}\,]$ and all Galois conjugate.

Straightforward application of Proposition 3.3 shows that the divisors associated to points with complex multiplication by i, $\sqrt{-2}$, and $\sqrt{-14}$ cannot intersect.

Elkies [11, 5.1] constructs a coordinate t on the curve $\mathcal{X}^*$ over $\mathbb{Q}$ taking the points with complex multiplication by i, $\sqrt{-2}$, and $\sqrt{-14}$ to ∞, 0, and the roots of $16t^2 + 13t + 8$ respectively. These are distinct away from the prime 2 and hence determine an integral coordinate on $\mathcal{X}^*$. The images of the points fixed by Atkin–Lehner operators give 4 elliptic points, one of order 4 and three of order 2.

The Eichler–Selberg trace formula implies that the Atkin–Lehner operators have the following traces on forms of even weight $k = 2t > 2$ on $\mathcal{X}$.

- The trace of the identity (the dimension) is t if t is even and $t - 1$ if t is odd.
- The trace of w_2 is $(-1)^t 2^{t+1}$ if $t \equiv 0, 1 \bmod 4$ and 0 otherwise.
- The trace of w_7 is 0.
- The trace of w_{14} is $(-1)^t 2 \cdot 14^t$.

When $k = 2$, w_2 has trace -2, w_7 has trace 7, and w_{14} has trace -14.

We have $\{\frac{-2}{3}\} = \{\frac{7}{3}\} = \{\frac{-14}{3}\} = 1$, so at $p = 3$ we may renormalize these operators to involutions t_d whose fixed elements are automorphic forms on the quotient curve $\mathcal{X}^*$. These involutions have the following traces:

- The trace of t_2 is 2 if $t \equiv 0, 1 \bmod 4$ and 0 otherwise.
- The trace of t_7 is 0.
- The trace of t_{14} is 2.

When $k = 1$ or 2, these operators all have trace 1. Elementary character theory implies that the group of automorphic forms of weight $2t$ on $\mathcal{X}^*$ has dimension $1 + \lfloor t/4 \rfloor$, from which the following theorem results.

Theorem 4.1. *This ring of automorphic forms on $(\mathcal{X}^*)^{\wedge}_{(3)}$ has the form*

$$\mathbb{Z}_3[U, V].$$

The generator U is in weight 2*, having a simple zero at the divisors with complex multiplication by $\mathbb{Z}[i]$, and the second generator V in weight* 8 *is non-vanishing on these divisors.*

The homotopy spectral sequence associated to this quotient Shimura curve at 3 collapses at E_2, and the dual portion is concentrated in degrees -11 and below. The connective cover taf_3^D has homotopy concentrated in even degrees and is therefore complex orientable, via a ring spectrum map $\mathrm{BP} \to \mathrm{taf}_3^D$. The homotopy of taf_3^D is a polynomial algebra on the image of v_1 in π_4 (a unit times U that vanishes on the height 2 locus) and the image of v_2 in π_{16} (a combination of U^4 and a unit times V that is non-vanishing on the height 2 locus). The composite ring homomorphism

$$\mathbb{Z}_3[v_1, v_2] \hookrightarrow \mathrm{BP}_* \to \pi_* \mathrm{taf}_3^D$$

is therefore an isomorphism, and the ring $\pi_* \mathrm{taf}_3^D$ is obtained from $(\mathrm{BP}_*)^{\wedge}_3$ by killing a regular sequence $v_3', v_4', \ldots$ of polynomial generators.

Theorem 4.2. *There is a generalized truncated Brown–Peterson spectrum* $\mathrm{BP}\langle 2\rangle'$ *admitting an E_∞ ring structure at the prime* 3.

Proof. The spectrum $\mathrm{taf}_3^{D,*} = \Gamma((\mathcal{X}^*)^{\wedge}_3, \mathcal{O}^{der})$ is an E_∞-ring spectrum with the homotopy type of a 3-complete generalized Brown–Peterson spectrum. Similar to Section 3.7, we construct an E_∞ ring $\mathrm{taf}_{\mathbb{Q}}^{D,*}$ as the free $\mathbb{HQ}$ algebra $\mathbb{P}_{\mathbb{Q}}(S^4 \vee S^{16})$ on generators in degree 4 and 16, and the images of the generators of the rational ring of automorphic forms on $\mathcal{X}^*$ determine a unique homotopy class of map $\mathrm{taf}_{\mathbb{Q}}^{D,*} \to (\mathrm{taf}_3^{D,*})_{\mathbb{Q}}$. We then construct a homotopy pullback square of E_∞ ring spectra as follows:

$$\begin{array}{ccc} \mathrm{taf}_{(3)}^{D,*} & \longrightarrow & \mathrm{taf}_3^{D,*} \\ \downarrow & & \downarrow \\ \mathrm{taf}_{\mathbb{Q}}^{D,*} & \longrightarrow & (\mathrm{taf}_3^{D,*})_{\mathbb{Q}} \end{array}$$

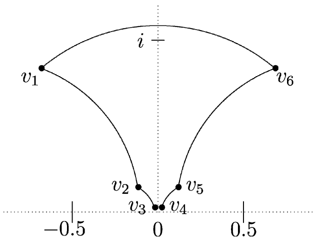
We have $\pi_* \mathrm{taf}^{D,*}(3) \cong \mathbb{Z}_{(3)}[U, V]$. The resulting spectrum is therefore complex orientable via a p-typical orientation, and the composite map

$$\mathbb{Z}_{(3)}[v_1, v_2] \hookrightarrow \mathrm{BP}_* \to \pi_* \mathrm{taf}^{D,*}$$

is an isomorphism. $\square$

5. Discriminant 10

In this section, D denotes the quaternion algebra of discriminant 10 and $\mathcal{X} = \mathcal{X}^D$ the associated Shimura curve. The fundamental domain appears as follows.



The edges meeting at v_2 are identified, as are the edges identified at v_5. The top and bottom edges are identified as well. The genus of the resulting curve is 0. There are four elliptic points of order 3 and none of order 2.

5.1. *Points with complex multiplication*

The involutions w_2, w_5, and w_{10} each have two fixed points on the curve $\mathcal{X}$, having complex multiplication by $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-5})$, and $\mathbb{Q}(\sqrt{-10})$ respectively. The associated Hilbert class fields are $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-5}, i)$, and $\mathbb{Q}(\sqrt{-10}, \sqrt{-2})$. The fixed points are therefore defined over $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(i)$, and $\mathbb{Q}(\sqrt{-2})$ due to Theorem 3.1.

The curve has four elliptic points of order 3, with complex multiplication by $\mathbb{Z}[\omega]$, that are acted on transitively by the Atkin–Lehner involutions. Their field of definition is $\mathbb{Q}(\omega)$ and they are Galois conjugate in pairs. Each divisor intersects its Galois conjugate at the prime 3.

Proposition 3.3 implies that the divisors associated to the fixed points of the Atkin–Lehner involutions are nonintersecting. The divisors associated to the elliptic points cannot intersect the fixed divisors of w_2 or w_5, and can intersect those of w_{10} only at the prime 3. Therefore, the involution w_{10} must act as Galois conjugation on the elliptic points.

5.2. *Level structures*

On the curve of discriminant 10, the elliptic points of order 3 create torsion phenomena and obstruct use of the previous methods for computation. In order to determine higher cohomology, it is necessary to impose level structure to remove the 3-primary automorphism groups.

Write $\mathcal{A} \to \mathcal{X}$ for the universal abelian surface over the Shimura curve $\mathcal{X}$. Associated to an ideal I of Λ satisfying $M\Lambda \subset I \subset \Lambda$ we can consider the closed subobjects $\mathcal{A}[I] \subset \mathcal{A}[M] \subset \mathcal{A}$ of I-torsion points and M-torsion points. Over $\mathrm{Spec}(\mathbb{Z}[1/NM])$, the maps $\mathcal{A}[I] \to \mathcal{A}[M] \to \mathcal{X}$ are étale. We have a natural decomposition by annihilating ideal:

$$\mathcal{A}[I] \cong \coprod_{\Lambda \supset J \supset I} \mathcal{A}\langle J\rangle.$$

In particular, if I is maximal then $\mathcal{A}\langle I\rangle$ is $\mathcal{A}[I]$ minus the zero section.

We will write $\mathcal{X}_1(I) = \mathcal{A}\langle I\rangle$ as an arithmetic surface over $\mathrm{Spec}(\mathbb{Z}[1/NM])$. It is a modular curve parametrizing abelian surfaces with a chosen primitive I-torsion point. If I is a two-sided ideal, the cover $\mathcal{X}_1(I) \to \mathcal{X}$ is Galois with Galois group $(\Lambda/I)^\times$. For sufficiently small I (such that no automorphism of an abelian surface can preserve such a point), the object $\mathcal{X}_1(I)$ is represented by a smooth curve.

In particular, for d dividing N there is an ideal I_d which is the kernel of the map $\Lambda \to \prod_{p|d} \mathbb{F}_{p^2}$. This ideal squares to (d), and so we write $\mathcal{X}_1(\sqrt{d})$ for the associated cover over $\mathbb{Z}[1/N]$ with Galois group $\prod_{p|d} \mathbb{F}_{p^2}^\times$.

For a general two-sided ideal I, the surface $\mathcal{X}_1(I)$ is generally not geometrically connected, or equivalently its ring of constant functions may be a finite extension of $\mathbb{Z}[1/NM]$.

Theorem 5.1. *(See [21, 3.2].) Let $(n) = I \cap \mathbb{Z}$. The ring of constant functions on the curve $\mathcal{X}(I)_{\mathbb{Q}}$ is the* narrow class field $\mathbb{Q}(\zeta_n)$ *generated by a primitive n'th root of unity.*

In the case of discriminant 10, to find a cover of the curve with no order-3 elliptic points we impose level structure at the prime 2. In this case, the choice of level structure is equivalent to the choice of a *full* level structure, so we will drop the subscript. Let $\mathcal{X}(\sqrt{2}) \to \mathcal{X}$ be the degree 3 Galois cover with Galois group $\mathbb{F}_4^\times$. The cover is a smooth curve of genus 2, and the map on underlying curves is ramified precisely over the four divisors with complex multiplication by $\mathbb{Z}[\omega]$. The ring of constant functions on $\mathcal{X}(\sqrt{2})_{\mathbb{Q}}$ is $\mathbb{Q}$ by Theorem 5.1.

Shimura's theorem on fields of definition of CM-points has a generalization to include level structures. Suppose $\mathfrak{a} \subset \mathcal{O}_F$ is an ideal, and let $I(F, \mathfrak{a})$ be the groupoid parametrizing fractional ideals I of F together with a chosen generator of $I/\mathfrak{a} \cong \mathcal{O}_F/\mathfrak{a}$. Write $\mathrm{Cl}(F, \mathfrak{a})$ for the associated *ray class group* under tensor product, with mass

$$\left|\mathrm{Cl}(F, \mathfrak{a})\right| = \left|\mathrm{Cl}(F)\right| \cdot \left|(\mathcal{O}/\mathfrak{a})^\times\right|$$

and cardinality

$$\#\left\{\mathrm{Cl}(F, \mathfrak{a})\right\} = \left|\mathrm{Cl}(F, \mathfrak{a})\right| \cdot \left|(1 + \mathfrak{a})^\times\right|.$$

There is an associated ray class field $H_{F,\mathfrak{a}}$, an extension of F generalizing the Hilbert class field, with Galois group $\mathrm{Cl}(F, \mathfrak{a})$.

Theorem 5.2. *(See [21, 3.2, 3.5].) Suppose we have an embedding $F \to D^{op}$ with corresponding abelian surface A acted on by $\mathcal{O}_F$, together with a two-sided ideal J of Λ and a level J structure on A. Let $(n) = J \cap \mathbb{Z}$ and $\mathfrak{a} = J \cap \mathcal{O}_F$. Then A is defined over the compositum $\mathbb{Q}(\zeta_n) \cdot H_{F,\mathfrak{a}}$ of the ray class field with the cyclotomic field, and the set of elements isogenous to A is isomorphic to the ray class groupoid $I(F, \mathfrak{a})$ as* $\mathrm{Cl}(F, \mathfrak{a})$*-sets.*

For the curve of discriminant 10, the points with complex multiplication by $\mathbb{Z}[\omega]$, $\mathbb{Z}[\sqrt{-2}]$, $\mathbb{Z}[\sqrt{-5}]$, and $\mathbb{Z}[\sqrt{-10}]$ must have associated ideals (2), $(\sqrt{-2})$, $(2, 1+\sqrt{-5})$, and $(2, \sqrt{-10})$ respectively, with ray class fields $\mathbb{Q}(\omega)$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-5}, i)$, and $\mathbb{Q}(\sqrt{-10}, \sqrt{-2})$. In particular, the order-3 elliptic points and the fixed points of w_2 have all lifts defined over the same base field.

5.3. Defining equations for the curve

Let Q_j denote the 4 elliptic points of $\mathcal{X}_{\mathbb{Q}}$, and P_j, P'_j, P''_j the points with complex multiplication by $\sqrt{-2}$, $\sqrt{-5}$, and $\sqrt{-10}$ respectively. These points have rational images Q, P, P', and P'' on $\mathcal{X}^*$. Elkies has shown [11, Section 4] that there exists a coordinate t on the underlying curve X^* taking the following values $t(Q)=0$, $t(P)=\infty$, $t(P')=2$, and $t(P'')=27$. As the divisors associated to P, P', and P'' are nonintersecting over $\mathbb{Z}[1/10]$, this extends to a unique isomorphism $X^* \to \mathbb{P}^1_{\mathbb{Z}[1/10]}$.

There is a coordinate y on X/w_5 such that $y(P)=0$, $y(P'')=\infty$, and without loss of generality we can scale y so that $y(P'_j)=\pm 5i$. Similarly, there exists a coordinate z on X/w_2 such that $z(P')=0$, $z(P'')=\infty$, and $z(P_j)=\pm 5\sqrt{-2}$. These coordinates satisfy $y^2=t-27$ and $z^2=4-2t$, and generate all functions on the curve.

We find, after rescaling, the coarse moduli X can defined by the homogeneous equation

$$2Y^2+Z^2+2W^2=0$$

over $\mathbb{Z}[1/10]$ [13].

5.4. Defining equations for the smooth cover

Any element of norm 5 in Λ has the trivial conjugation action on the quotient ring $\mathbb{F}_4$, and hence the involution w_5 commutes with the group of deck transformations of $\mathcal{X}(\sqrt{2})$. The quotient curve $X(\sqrt{2})/w_5$ has genus 0. The involutions w_2 and w_{10} anticommute with the group of deck transformations.

The points P_j with complex multiplication by $\sqrt{-2}$ each have three lifts to $\mathcal{X}(\sqrt{2})$ defined over $\mathbb{Q}(\sqrt{-2})$. The degree 5 isogeny $P_1 \to P_2$ canonically identifies the 2-torsion of P_1 with that of P_2, and so the involution w_5 lifts to an involution interchanging Galois conjugates. On the quotient curve, we therefore have that the rational point P of $\mathcal{X}/w_5$ has three rational points as lifts. As P has no automorphisms, the divisors of the associated lifts are nonintersecting.

Therefore, there exists a coordinate u on $X(\sqrt{-2})/w_5$ over $\mathbb{Z}[1/10]$ such that u takes the lifts of P to the points ± 1 and ∞. The order-3 group of deck transformations must therefore consist of the linear fractional transformations

$$u \mapsto \frac{3+u}{1-u} \mapsto \frac{u-3}{u+1}.$$

The fixed points, which are the lifts of the points Q_i, are the points satisfying $u^2+3=0$. The degree-3 function $\frac{u^3-9u}{u^2-1}$ is invariant under these deck transformations and hence descends to a coordinate on X/w_5. As it takes the value ∞ on P and $\pm 3\sqrt{-3}$ on Q_i, it must be one of the functions $\pm y$. By possibly replacing u with $-u$ we may assume

$$y=\frac{u^3-9u}{u^2-1}.$$

The function field is therefore generated by the coordinates u and z. Expressing y in terms of u, we find that these satisfy the relation

$$z^2\big(u^2-1\big)^2 = -2\big(u^2+1\big)\big(u^2+2u+5\big)\big(u^2-2u+5\big).$$

5.5. Automorphic forms

An automorphic form of weight $2t$ on $\mathcal{X}(\sqrt{2})$ is simply a holomorphic section of κ^t on $X(\sqrt{2})$. Any of these is of the form $f(u,z)\,du^t$, and to find generators one needs determine when such a function is holomorphic. The form du itself has simple zeros along the six divisors with $z=0$, and double poles along the two divisors with $u=\infty$.

The smooth cover $\mathcal{X}(\sqrt{2})$ has automorphic forms of weight 2 given by

$$U = \frac{u\,du}{z(u^2-1)}, \qquad V = \frac{du}{z(u^2-1)},$$

together with an automorphic form of weight 6 given by

$$W = \frac{du^3}{(u^2+1)(u^2+2u+5)(u^2-2u+5)}.$$

These generate the entire ring of forms. These satisfy the formula

$$W^2 = -2\big(U^2+V^2\big)\big(U^2+2UV+5V^2\big)\big(U^2-2UV+5V^2\big),$$

and the action of the deck transformation $\sigma : u \mapsto \frac{3+u}{1-u}$ is given as follows.

$$\begin{aligned} U &\mapsto (-U-3V)/2, \\ V &\mapsto (U-V)/2, \\ W &\mapsto W. \end{aligned}$$

We can then alternatively express the ring of automorphic forms on this smooth cover as having generators $\{A,B,C,W\} = \{2V, 2\sigma V, 2\sigma^2 V, W\}$, subject only to the relations $A+B+C=0$ and $W^2 = -(A^2+B^2)(B^2+C^2)(C^2+A^2)$.

We pause here to note the structure of the supersingular locus at 3. There are 2 supersingular points at 3 corresponding to the prime ideal generated by 3 and U, and V vanishes at neither point. Inverting V and completing with respect to the ideal $(3,U)$, we obtain the complete local ring

$$\mathbb{Z}_3[\![u]\!]\big[w, V^{\pm 1}\big]/\big(w^2+2\big(u^2+1\big)\big(u^2+2u+5\big)\big(u^2-2u+5\big)\big).$$

The generators are $u = U/V$ and $w = W/V^3$. The power series w^2 has a constant term which is, 3-adically, a unit and a square, and the binomial expansion allows us to extract a square root. Therefore, this ring is isomorphic to

$$\mathbb{Z}_3[\![u]\!]\big[V^{\pm 1}\big] \times \mathbb{Z}_3[\![u]\!]\big[V^{\pm 1}\big].$$

This is a product of two summands of the Lubin–Tate ring, each classifying deformations of one of the supersingular points.

5.6. Atkin–Lehner involutions

At the prime 3, both -2 and -5 have square roots. Therefore, the p-divisible group on $\mathcal{X}$ can be made to descend to the curves $\mathcal{X}^*$ and $\mathcal{X}(\sqrt{2})/w_5$. We write t_2 and t_5 for the associated involutions on the rings of automorphic forms.

The involution t_5 negates $w = W/V^3$ and fixes $u = U/V$, and also commutes with the action of the cyclic group of order 3. The lift t_5 must either fix all forms of weight 2 on $\mathcal{X}(\sqrt{2})$ and negate W, or negate all forms of weight 2 and fix W. The former must be the case: in the cohomology of $\mathcal{X}(\sqrt{2})/w_5$ the element U vanishing on the supersingular locus must survive to be a Hasse form v_1 in weight 2, as the element α_1 in the cohomology of the moduli of formal group laws has image zero.

Therefore, at 3 the curve $\mathcal{X}(\sqrt{2})/t_5$ has

$$\mathbb{Z}_3[U, V]$$

as its ring of automorphic forms.

The involution t_2 conjugate-commutes with the group of deck transformations, and choosing a lift gives rise to an action of the symmetric group Σ_3 on this ring. The fact that t_2 is non-trivial forces the representation on forms of weight 2 to be a $\mathbb{Z}[1/10]$-lattice in the irreducible 2-dimensional representation of Σ_3. There are two isomorphism classes of such lattices. Both have a single generator over the group ring $\mathbb{Z}[1/10, \Sigma_3]$, and are distinguished by whether their reduction mod 3 is generated by an element fixed by t_2 or negated by t_2. In the former case, $H^1(\mathcal{X}^*, \kappa) = 0$, whereas in the latter it is $\mathbb{Z}/3$. We will show in the following section that a non-zero element $\alpha_1 \in H^1(\mathcal{X}^*, \kappa)$ exists due to topological considerations.

Therefore, the ring of t_5-invariant automorphic forms can be expressed as a polynomial algebra

$$\mathbb{Z}_3[A, B, C]/(A + B + C)$$

on generators cyclically permuted by the group of deck transformations, and such that one lift of t_2 sends (A, B, C) to $(-A, -C, -B)$.

The ring of forms on $\mathcal{X}(\sqrt{2})$ invariant under the action of $\mathbb{Z}/3$ is the algebra

$$\mathbb{Z}_3[\sigma_4, \sigma_6, \sqrt{\Delta}, W]/\big(W^2 + 2\sigma_4^3 + \sigma_6^2, \sqrt{\Delta}^2 + 4\sigma_4^3 + 27\sigma_6^2\big).$$

Here σ_4, σ_6, and $\sqrt{\Delta}$ are standard invariant polynomials arising in the symmetric algebra on the reduced regular representation generated by A, B, and C. The subring of elements invariant under the involutions t_2 and t_5 is

$$\mathbb{Z}_3\big[\sigma_4, \sqrt{\Delta}, \sigma_6^2\big]/\big(\sqrt{\Delta}^2 + 4\sigma_4^3 + 27\sigma_6^2\big).$$

Abstractly, this ring is isomorphic to the ring of integral modular forms on the moduli of elliptic curves, with σ_6^2 playing the role of the discriminant.

5.7. Associated spectra

The $\mathbb{Z}/3$-Galois cover of $\mathcal{X}$ by $\mathcal{X}(\sqrt{2})$ gives rise to an expression of TAF_p^D as a homotopy fixed point spectrum of an object $\mathrm{TAF}_p^D(\sqrt{2})$ by an action of $\mathbb{Z}/3$. We will use the homotopy fixed point spectral sequence to compute the homotopy groups of TAF_p^D. The difference between this spectral sequence and the Adams–Novikov spectral sequence is negligible: the filtration of the portion arising via Serre duality is lowered by 1 in the homotopy fixed point spectral sequence.

The positive-degree homotopy groups of $\mathrm{TAF}_p^D(\sqrt{2})$ form the ring generated by the elements A, B, C, and W of Section 5.5, together with a Serre duality class D in degree 3 that makes the multiplication pairing perfect.

The main computational tool will be the Tate spectral sequence. The nilpotence theorem [9] guarantees that the Tate spectrum is contractible (the periodicity class β is nilpotent). Therefore, the Tate spectral sequence, formed by inverting the periodicity class, converges to 0. The Tate spectral sequence has the added advantage that the E_2-term can be written in terms of permanent cycles and a single non-permanent cycle.

Theorem 5.3. *The cohomology of $\mathbb{Z}/3$ with coefficients in the ring of automorphic forms on $\mathcal{X}(\sqrt{2})$ is given by*

$$E_2 = \mathbb{Z}_{(3)}[\sigma_4, \sigma_6, W, \sqrt{\Delta}\,][\beta] \otimes E(\alpha_1)/(3, \sigma_4, \sqrt{\Delta}\,) \cdot (\beta, \alpha_1),$$
$$W^2 + 2\sigma_4^3 + \sigma_6^2, \sqrt{\Delta}^2 + 4\sigma_4^3 + 27\sigma_6^2.$$

Here β is the generator of $H^2(\mathbb{Z}/3;\mathbb{Z})$ and α_1 is the element of H^1 coming from the reduced regular representation in degree 4.

The Atkin–Lehner operators act as follows. The involution t_5 fixes α_1, β, σ_4, σ_6, and $\sqrt{\Delta}$, but negates W. The involution t_2 fixes α_1, σ_4, and $\sqrt{\Delta}$, but negates β and σ_6.

This cohomology, 3-completed, forms the portion of the Adams–Novikov E_2-term for TAF_3^D generated by non-Serre dual classes.

Corollary 5.4. *For degree reasons, the first possible differentials are d_5 and d_9.*

The class α_1 is so named because it is the Hurewicz image of the generator of $\pi_3(S^0)$ at 3. This can be seen by $K(2)$-localization: the $K(2)$-localization of TAF_3^D is a wedge of four copies of $EO_2(\mathbb{Z}/3 \times \mathbb{Z}/2)$, corresponding to the four supersingular points on $\mathcal{X}$. The image of α_1 under the composite

$$\pi_3\big(S^0\big) \to \pi_3 \,\mathrm{TAF}_3^D \to \pi_3\big(EO_2(\mathbb{Z}/3 \times \mathbb{Z}/2)\big)$$

is the image of α_1 under the Hurewicz homomorphism for $EO_2(\mathbb{Z}/3 \times \mathbb{Z}/2)$, which is known to be non-zero. By considering the map of Adams–Novikov spectral sequences induced by the unit map $S^0 \to \mathrm{TAF}_3^D$ and the $K(2)$-localization map $\mathrm{TAF}_3^D \to L_{K(2)}\,\mathrm{TAF}_3^D$, we see that the Adams–Novikov filtration of the element representing α_1 can be at most 1, allowing us to explicitly determine the element. We note that (as required in the previous section) this also forces it to be fixed by the Atkin–Lehner involutions.

Proposition 5.5. *The element $\beta_1 = \langle\alpha_1, \alpha_1, \alpha_1\rangle$ is given by $\beta\sigma_6$.*

Proof. Given a torsion-free commutative ring R with G-action, a 1-cocycle $f : G \to R$ has a family of associated 1-cochains $f^k : g \mapsto f(g)^k$. These satisfy $\delta f^2 = -2(f \smile f)$ and $\delta f^3 = -3\langle f, -2f, f\rangle$. In particular, f^3 represents a cocycle mod 3 whose image under the Bockstein map is in $\langle f, 2f, f\rangle$.

In the case in question, the 1-cocycle $\alpha_1 \in H^1(\mathbb{Z}/3; \pi_4 \mathrm{TAF}_p^D(\sqrt{2}))$ is represented by the trace-zero element A, and the element A^3 represents, in $H^1(Z/3; \pi_4 \mathrm{TAF}_p^D(\sqrt{2})/3)$, the cube of the associated cocycle. The image under the Bockstein is $\beta(A^3 + B^3 + C^3)/3 = -\beta\sigma_6$. We then find $\beta\sigma_6 = \langle\alpha_1, \alpha_1, \alpha_1\rangle$ as desired. □

Remark 5.6. We can also detect β_1 using the $K(2)$-localization. There is only one class in the appropriate degree that is invariant under the Atkin–Lehner involutions, namely $\beta\sigma_6$. By an argument exactly like that for α_1, we conclude that this is β_1.

Corollary 5.7. *The class $\beta\sigma_6$ is a permanent cycle.*

Most classes on the zero-line in the Adams–Novikov spectral sequence are annihilated by multiplication by either α_1 or β – specifically, those classes that arise as elements in the image of the transfer. These play little role in the spectral sequence. However, there are classes which play important roles for the higher cohomology.

Lemma 5.8. *On classes of non-zero cohomological degree, multiplication by σ_6 is invertible.*

Proof. This follows easily from the cover of the moduli stack by affine spaces in which one of σ_4 or σ_6 is inverted. The Mayer–Vietoris sequence, together with the fact that α_1 and β are annihilated by σ_4, shows this result immediately. □

Corollary 5.9. *We can choose a different polynomial generator for the higher cohomology: $\beta_1 = \beta\sigma_6$.*

In particular, even though there is no class σ_6^{-1}, on classes of non-zero cohomological degree the use of such notation is unambiguous.

Corollary 5.10. *The Tate E_2-term is given by*

$$E_2 = \mathbb{F}_9\big[\beta_1^{\pm 1}, \sigma_6^{\pm 1}\big] \otimes E(\alpha_1).$$

Here the element $\sqrt{-1} \in \mathbb{F}_9$ is represented by the class W/σ_6, which must be a permanent cycle. The involution t_5 fixes α_1, β_1, and σ_6, but acts by conjugation on $\mathbb{F}_9$. The involution t_2 fixes α_1 and β_1, but negates σ_6 and acts by conjugation on $\mathbb{F}_9$.

5.8. The Tate spectral sequence

We suggest that the reader follow the arguments of this section with the aid of Fig. 1. In this picture, dots indicate a copy of $\mathbb{F}_9$, lines of slope $1/3$ correspond to α_1-multiplication, and the

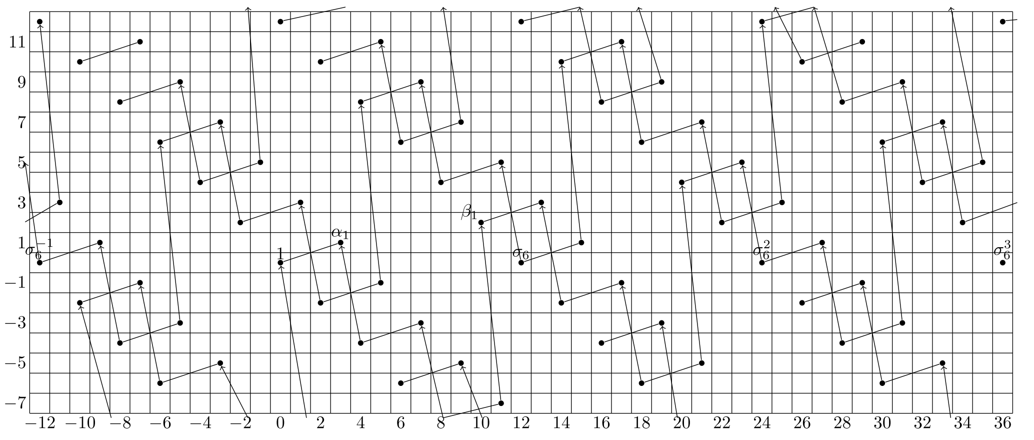


Fig. 1. The Tate E_2-term.

elements σ_6^i and β_1 are labeled. Both the d_5 and d_9-differentials are included, allowing the reader to readily see that these are essentially the only possibilities.

Though we use permanent cycles for our description of the higher cohomology, it is more straightforward to determine differentials originating from the periodicity class β.

Theorem 5.11. *There is a d_5-differential*

$$d_5(\beta) = \alpha_1\beta^3 = \alpha_1\beta_1^3\sigma_6^{-3}.$$

Proof. This is a relatively standard homotopy fixed point argument. The unit map $S^0 \to \mathrm{TAF}_3^D$ is a $\mathbb{Z}/3$-equivariant map. This therefore induces a map of filtered spectra on the homotopy fixed point objects:

$$D(B\mathbb{Z}/3_+) = \left(S^0\right)^{h\mathbb{Z}/3} \to \mathrm{TAF}_3^D.$$

This in turn gives rise to a map of homotopy fixed point spectral sequences, where the source spectral sequence coincides with the Atiyah–Hirzebruch spectral sequence for stable cohomotopy of $B\mathbb{Z}/3$. We can therefore identify classes in the source spectral sequence with homotopy classes and null-homotopies in the target spectral sequence.

If we denote by b_k the generator in $H^{2k}(\mathbb{Z}/3, \pi_0 S^0)$ on the E_2-page for the source spectral sequence, then by definition the image of b_k in the homotopy fixed point spectral sequence for TAF_3^D is β^k. Consideration of the cohomology of $B\mathbb{Z}/3$ shows that in the Atiyah–Hirzebruch spectral sequence b_1 supports a differential to α_1 times b_3. We conclude that in the homotopy fixed point spectral sequence for TAF_3^D, the class β supports a differential whose target is $\alpha_1\beta^3$. □

Corollary 5.12. *There is a d_5-differential of the form*

$$d_5(\sigma_6) = -\alpha_1\beta_1^2\sigma_6^{-1}.$$

Proof. Since $\beta\sigma_6$ is a permanent cycle, we must have this differential by the Leibniz rule. □

Remark 5.13. An analogous differential appears in the computation of the homotopy of tmf [4], and we can repeat the argument here as well. The Toda relation $\alpha_1\beta_1^3 = 0$, since it arises from the sphere, must be visible here. For degree reasons, the only possible way to achieve this is to have a differential on σ_6^2 hitting $\alpha_1\beta_1^2$, and this implies the aforementioned differential.

Remark 5.14. The differential on σ_6 also follows from geometric methods similar to those employed for β. In this case, instead of the unit map $S^0 \to \mathrm{TAF}_3^D$, we use a multiplicative norm map $S^{2\rho} \to \mathrm{TAF}_3^D(\sqrt{2})$, where ρ is the complex regular representation. The source homotopy fixed point spectral sequence is then the Atiyah–Hirzebruch spectral sequence for the Spanier–Whitehead dual of the Thom spectrum for a bundle over $B\mathbb{Z}/3$.

Since β_1 is an invertible permanent cycle, this leaves a relatively simple E_6-page.

Proposition 5.15. *As an algebra,*

$$E_6 = \mathbb{F}_9\left[\sigma_6^{\pm 3}, \beta_1^{\pm 1}\right] \otimes E\left([\alpha_1\sigma_6]\right).$$

A Toda style argument gives the remaining differential [19].

Proposition 5.16. *There is a d_9-differential of the form*

$$d_9\left(\alpha_1\sigma_6^4\right) = \left\langle \alpha_1, \alpha_1\beta_1^2, \alpha_1\beta_1^2\right\rangle = \beta_1^5.$$

Since β_1 is a unit, this gives a differential

$$d_9\left(\alpha_1\sigma_6^4\beta_1^{-5}\right) = 1,$$

so $E_{10} = 0$.

5.9. The homotopy fixed point spectral sequence

Having completed the Tate spectral sequence, the homotopy fixed point spectral sequence and the computation of the homotopy of TAF_3^D become much simpler. There is a natural map of spectral sequences from the homotopy fixed point spectral sequence to the Tate spectral sequence that controls much of the behavior. We first analyze the kernel of the map

$$H^*\left(\mathbb{Z}/3; H^*\left(\mathcal{X}(\sqrt{2})\right)\right) \to H^*_{Tate}\left(\mathbb{Z}/3; H^*\left(\mathcal{X}(\sqrt{2})\right)\right).$$

Lemma 5.17. *Everything in the ideal* $(3, \sigma_4, \sqrt{\Delta})$ *is a permanent cycle.*

Proof. These classes are all in the image of the transfer. □

This shows that the differentials and extensions take place in portions seen by the Tate spectral sequence, and we will henceforth work in the quotient of the homotopy fixed point spectral

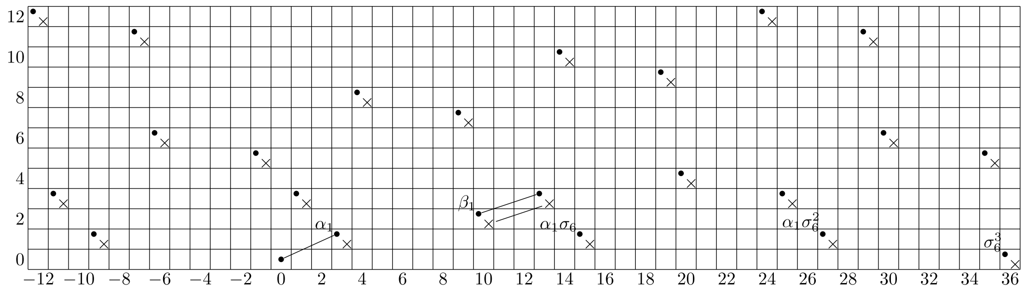


Fig. 2. The E_6-page of the homotopy fixed points.

sequence by this ideal. By naturality, the canonical map of spectral sequences is an inclusion through E_5, and the d_5-differential in the homotopy fixed point spectral sequence is the d_5-differential in the Tate spectral sequence.

Corollary 5.18.

$$d_5(\sigma_6) = -\alpha_1\beta^2\sigma_6, \qquad d_5(\beta) = \alpha_1\beta^3 \quad \textit{and} \quad d_5(W) = -\alpha_1\beta^2 W.$$

The classes βW and $W\sigma_6^2$, representing the classes $\sqrt{-1}\beta_1$ and $\sqrt{-1}\sigma_6^3$ in the Tate spectral sequence, are d_5-cycles. We depict the E_6-page of the homotopy fixed point spectral sequence in Fig. 2. In this picture, both dots and crosses represent a copy of $\mathbb{Z}/3$. The distinction reflects the action of the Atkin–Lehner involution t_5. Classes denoted by a dot are fixed by t_5, while those denoted by a cross are negated. This involution commutes with the differentials.

Sparseness of the spectral sequence guarantees that the next possible differential is a d_9, and this is again governed by the Tate spectral sequence.

Corollary 5.19. *There are d_9-differentials of the form*

$$d_9\big(\alpha_1\sigma_6 \cdot \sigma_6^3\big) = \beta_1^5 \quad \textit{and} \quad d_9\big(\alpha_1 W\sigma_6^3\big) = \beta_1^5 W\sigma_6^{-1}.$$

These produce a horizontal vanishing line of s-intercept 10, so there are no further differentials possible. In particular, all elements in the kernel of the map from the homotopy fixed point spectral sequence to the Tate spectral sequence are permanent cycles. The E_∞-page of the spectral sequence (again ignoring classes on the zero-line in the image of the transfer) is given in Fig. 3.

There are a number of exotic multiplicative extensions. These are all exactly analogous to those which arise with tmf.

Proposition 5.20. *There are exotic α_1-multiplications*

$$\alpha_1 \cdot \big[\alpha_1\sigma_6^2\big] = \beta_1^3 \quad \textit{and} \quad \alpha_1 \cdot [\alpha_1\sigma_6 W] = \beta_1^3 W\sigma_6^{-1}.$$

Proof. The class σ_6^2, via the d_5-differential, represents a null-homotopy of $\alpha_1\beta_1^2$. By definition, we therefore conclude that $[\alpha_1\sigma_6^2] = \langle\alpha_1, \alpha_1, \beta_1^2\rangle$. Standard shuffling results then show that

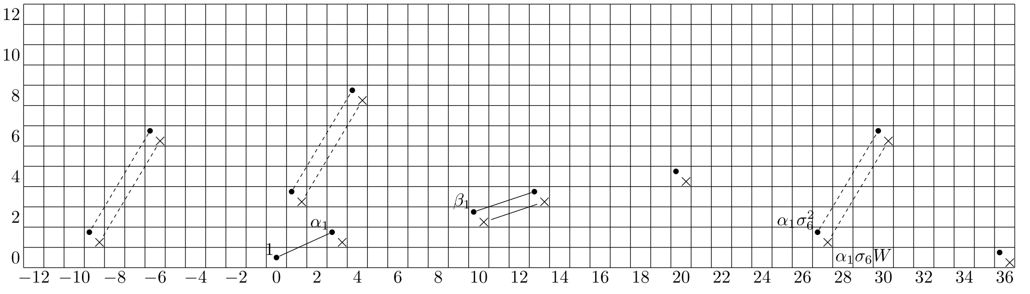


Fig. 3. The E_∞-page of the homotopy fixed points.

$$\alpha_1 \cdot \langle \alpha_1, \alpha_1, \beta^2 \rangle = \langle \alpha_1, \alpha_1, \alpha_1 \rangle \beta_1^2 = \beta_1^3,$$

giving the first result. The second is identical. □

In fact, these α_1 extensions, together with the fact that $\beta_1^2\alpha_1$ is zero, imply the d_9-differentials: σ_6^4 hits $\beta_1^2 \cdot \alpha_1\sigma_6^2$, so $\alpha_1\sigma_6^4$ hits $\alpha_1 \cdot \beta_1^2\alpha_1\sigma_6^2 = \beta_1^5$. For degree reasons, there are no other exotic multiplications, and there are no additive extensions, so the computation is concluded.

The homotopy fixed point spectrum under the action of t_2 and t_5 is much more surprising. Let taf_3^D be the connective cover of TAF_3^D.

Theorem 5.21. *As algebras,*

$$\pi_*\left(\mathrm{taf}_3^D\right)^{h\mathbb{Z}/2\times\mathbb{Z}/2} \cong \pi_* \mathrm{tmf}_3^\wedge .$$

Our argument is purely computational. The analysis of the action of t_2 and t_5 on the automorphic forms shows that the E_2-term for the homotopy fixed point spectral sequence for $\mathcal{X}^*$ is isomorphic to that of tmf [4]. Upon identification of the classes α_1 and β_1, the arguments used are formal and universal. This leads us to the conclusion that the homotopy rings are isomorphic, but for no obvious geometric reason.

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