Computing instanton numbers of curve singularities

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Abstract

We present an algorithm for computing instanton numbers of curve singularities. A comparison is made between these and some other invariants of curve singularities. The algorithm is implemented in Macaulay2, and can be downloaded from http://www.math.nmsu.edu/~iswanson/instanton.m2 or from http://emmy.nmsu.edu/~gasparim/m2code.
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We start with any polynomial $p(x, y)$ defining a plane curve with singularity at $0 \in \mathbb{C}^2$. Let $\pi \colon \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$ denote the blow-up of $\mathbb{C}^2$ at the origin and let j be a positive integer. The data (j, p) determines a holomorphic bundle $E(j, p)$ on $\widetilde{\mathbb{C}^2}$ with splitting type j and extension class p. We algorithmically compute numerical invariants of the bundle $E(j, p)$ and use them as invariants of the curve.

Tables giving examples of these new invariants of the curve together with some classical invariants are given in Section 1. A specially interesting example appears in Table 4, where we give two inequivalent singularities that have all the same classical invariants, but are distinguished by instanton numbers.

The holomorphic bundle $E(j, p)$ and its numerical invariants have interpretation in mathematical physics as instantons on $\widetilde{\mathbb{C}^2}$ and numerical invariants of the instantons.

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Here we mention briefly some properties of these invariants. More details are given in Gasparim (2002). Instantons are well known to have a topological invariant called the charge, which in the case of compact surfaces corresponds to a second Chern number of a vector bundle. For the case of bundles on a blown-up surface, the local Chern number around the exceptional divisor decomposes as a sum of two numerical invariants (denoted w and h), which are local analytic invariants of the bundle. In Ballico and Gasparim (2002) it is shown that the pair of invariants (w, h) is strictly finer than the Chern numbers in the following sense. The pair (w, h) provides the coarsest stratification of moduli of instantons on the blown-up plane for which the strata are Hausdorff. In contrast, the stratification by topological charge does not provide Hausdorff strata (Ballico and Gasparim, 2002, Theorem 4.1). Hence, the pair (w, h) gives strong numerical invariants, detecting more than topological information.

The idea of using these invariants for curves is natural given that, as a first step toward resolving the singularity of the plane curve defined by p, one blows up the plane, thus arriving at $\widetilde{\mathbb{C}^2}$, the base space of the bundles we construct.

Firstly, let us explain how to pass from curves on $\mathbb{C}^2$ to vector bundles on $\widetilde{\mathbb{C}^2}$. In what follows we fix, once and for all, the following coordinate charts. Let (x, y) be the coordinates on $\mathbb{C}^2$. Write $\widetilde{\mathbb{C}^2} = U \cup V$ where

$$U = \{(z, u)\} \cong \mathbb{C}^2 \cong \{(\xi, v)\} = V$$

with $(\xi, v) = (z^{-1}, zu)$ in $U \cap V$. Note that in these coordinates, the blow-up map $\pi \colon \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$ gives $x = u$, $y = zu$. We denote by ℓ the exceptional divisor.

Once these charts are fixed, there is a canonical choice of transition matrix for bundles on $\widetilde{\mathbb{C}^2}$. If E is a holomorphic rank 2 bundle on $\widetilde{\mathbb{C}^2}$ with $c_1(E) = 0$ then there exists an integer j, called the *splitting type* of E, such that $E|_\ell \cong \mathcal{O}(j) \oplus \mathcal{O}(-j)$. By Gasparim (1998, Theorem 2.1) E has a canonical transition matrix of the form

$$\begin{pmatrix} z^j & q \\ 0 & z^{-j} \end{pmatrix} \tag{1}$$

from U to V, where

$$q := \sum_{i=1}^{2j-2} \sum_{l=i-j+1}^{j-1} q_{il} z^l u^i$$

is a polynomial in z, z^{-1} and u.

In this paper, we start with a polynomial $p \in \mathbb{C}[x, y]$ and for each positive integer j we associate with the pair (j, p) a holomorphic bundle $E(j, p)$ on $\widetilde{\mathbb{C}^2}$ obtained as follows.

Definition 0.1. Given a polynomial $p(x, y)$ defined over $\mathbb{C}[x, y]$ we first consider the polynomial $p(u, zu) = \sum_i \sum_l p_{il} z^l u^i$ obtained from p by making $x = u$ and $y = zu$. If furthermore a positive integer j is chosen, we define $\bar{p} = \sum_i \sum_l \bar{p}_{il} z^l u^i$ by the rule

$$\bar{p}_{il} = \begin{cases} p_{il} & \text{if } 1 \leq i \leq 2j - 2 \text{ and } 0 \leq l \leq j - 1; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We then set $E := E(j, \bar{p})$ to be the bundle given in canonical coordinates (as above) by the transition matrix

$$T = \begin{pmatrix} z^j & \bar{p} \\ 0 & z^{-j} \end{pmatrix}. \tag{3}$$

We have therefore associated with each pair (j, p) formed by an integer and a polynomial $p(x, y)$ a unique vector bundle $E(j, \bar{p})$ over $\widetilde{\mathbb{C}^2}$. We now define instanton invariants w and h for these vector bundles.

Definition 0.2. Given a bundle E over $\widetilde{\mathbb{C}^2}$ we define the sheaf Q by the exact sequence

$$0 \to \pi_* E \to (\pi_* E)^{\vee\vee} \to Q \to 0$$

and set

$$w(E) := l(Q), \qquad h(E) := l(R^1 \pi_* E).$$

In instanton terminology, w is called the width of the instanton and h is called the height of the instanton. The charge of the instanton is $c = w + h$. These are clearly analytic invariants of the bundle. To use them as invariants of the singularity, and to perform the calculations, we choose charts for the bundle and a representative p of the singularity. Such choice works particularly well in the case of quasi-homogeneous singularities, where preferred representatives are known to exist, cf. Saito (1971) or Arnold (1981, page 95).

If a different representative p' of the germ of the singularity is chosen, then by classical theory it is known that there exists a holomorphic change of coordinates taking p to p'. The same change of coordinates has to be made to take the bundle $E(j, \bar{p})$ to a bundle $E'(j, p')$. E and E' are isomorphic bundles and therefore have the same invariants. In this sense the numbers can be considered as analytic invariants for germs of curve singularities.

In this paper we prove that the computation of h and w can be automated. We implemented our algorithm in the symbolic computer algebra package Macaulay2 (due to Grayson and Stillman (1996)). Having implemented algorithms to compute both invariants, we obtained a large amount of examples, which lead us to a simple formula for h.

Theorem 3.3. *Let m denote the largest power of u dividing $\bar{p}$. If E is the bundle defined by data $(j, \bar{p})$, then $h(E) = \binom{j}{2} - \binom{j-m}{2}$.*

The algorithm we present here has as input data the polynomial $p(x, y)$ and the integer j, and as output the values of the instanton height $h(E(j, \bar{p}))$ and width $w(E(j, \bar{p}))$. Our implementation on Macaulay2 is for p with rational coefficients.

1. Some tables

Before giving the algorithm, we illustrate some instanton numbers by tabulating them together with some classical invariants. Tables 1 and 2 give examples of instanton numbers of monomials.

Table 3 illustrates that for a fixed polynomial, we may get different invariants as we vary the splitting type.

Table 1

Polynomial	Splitting type 2		
	w	h	Charge
x	1	1	2
y	1	1	2
xy	2	1	3

Table 2

Polynomial	Splitting type 3		
	w	h	Charge
x	1	2	3
y	1	2	3
x^2	3	3	6
xy	2	3	5
y^2	3	3	6
$x^2 y$	4	3	7
xy^2	4	3	7
$x^2 y^2$	5	3	8

Table 3

Polynomial	Splitting type 7			Splitting type 8		
	w	h	Charge	w	h	Charge
$x^2 - y^2$	2	11	13	3	13	16
$x^2 - y^3$	3	11	14	3	13	16
$x^2 - y^5$	3	11	14	3	13	16
$x^3 - y^3$	4	15	19	4	18	22
$x^3 - y^4$	6	15	21	6	18	24
$x^3 - y^5$	6	15	21	6	18	24

Our interest in instanton numbers was partially fueled by the fact that in some cases, instanton numbers give finer information than the classical invariants. We considered the invariants: multiplicity m, $\delta_P = \dim (\widetilde{\mathcal{O}}/\mathcal{O})$, $\mu =$ Milnor number, and $\tau =$ Tjurina number. Note that the Milnor and Tjurina numbers are defined only for isolated singularities, but instanton numbers are well defined for non-isolated singularities as well.

In Table 4 we give an example where instanton numbers distinguish the singularities, but other invariants do not. Note that the first polynomial is irreducible, whereas the second is reducible in the local ring, cf. Hartshorne (1977, ex. I 5.14), so they define inequivalent singularities. We believe that the fact that the classical invariants m, δ_P, μ, and τ do not distinguish some inequivalent singularities is evidence that finer invariants are useful.

Table 4

Polynomial	m	δ_P	μ	τ	Splitting type 3		
					w	h	Charge
$x^3 - x^2 y + y^3$	3	3	4	4	4	3	7
$x^3 - x^2 y^2 + y^3$	3	3	4	4	5	3	8

Table 5

Polynomial	m	δ_P	μ	τ	Splitting type 4		
					w	h	Charge
$x^2 - y^7$	2	3	6	6	3	5	8
$x^3 - y^4$	3	3	6	6	6	6	12

Table 6

Polynomial	m	δ_P	μ	τ	Splitting type 4		
					w	h	Charge
$x^4 - xy^5$	4	9	17	17	10	6	16
$x^4 - x^2 y^3 - x^2 y^5 - y^8$	4	9	17	15	8	6	14

Table 7

Polynomial	m	δ_P	μ	τ	Splitting type 8		
					w	h	Charge
$x^3 + x^2 y^3 + y^9 + xy^7$	3	9	15	16	6	18	24
$x^3 y + x^2 y^3 + xy^6 + y^7$	4	9	14	15	7	22	29
$x^4 + x^2 y^3 + y^6$	4	8	15	15	10	22	32
$(x^2 + y^3)^2 + xy^4$	4	7	12	13	9	22	31
$(x^2 + y^3)^2 + xy^3$	4	6	9	9	6	22	28

In Tables 5–7 we give examples of inequivalent singularities which are distinguished by instanton numbers, and also by one other classical invariant.

Table 6 comes from Hartshorne (1977, ex. V 3.8) on page 395. However, in the statement of this exercise, the first polynomial contains an incorrect exponent: it is written as "$x^4 y - y^4$" but it should be "$x^5 y - y^4$".

Table 7 comes from the list of bimodular singularities given by Arnold in Arnold (1981, page 159).

Table 8 below comes from the list of exceptional families of unimodal singularities in Arnold (1981, page 95).

Table 8

Polynomial	Splitting type 7		
	w	h	Charge
$x^3 + y^7 + xy^5$	6	15	21
$x^3 + y^8 + xy^6$	6	15	21
$x^3 y + xy^4 + x^2 y^3$	7	18	25
$x^3 + xy^5 + y^8$	6	15	21
$x^3 y + y^5 + xy^4$	7	18	25
$x^3 y + y^6 + xy^5$	7	18	25
$x^4 + xy^4 + y^6$	9	18	25

2. Computing the instanton width

In this section we present an algorithm that takes as input the pair (j, p) and computes the instanton width $w = l(Q)$, where Q is the skyscraper sheaf as given in Definition 0.2. Since Q is supported at zero, $l(Q)$ is the dimension of the stalk at zero. Let M be the completion of the stalk $\pi_* E$ at zero, that is, $M := (\pi_* E)_0^\wedge$. Then the length of Q equals the dimension of the cokernel of the canonical map from M to its double dual. If we can compute M, we can also compute Q via the following lemma:

Lemma 2.1. *Let R be a commutative Noetherian ring and A an $n \times m$ matrix with entries in R. Let M be the R-module such that $R^m \xrightarrow{A} R^n \to M \to 0$ is an exact sequence. Let N be the kernel of the transpose of A. Then N is a submodule of R^n, say generated by t elements. Let B be the $n \times t$ matrix whose columns are the given generators of N. Let C be the matrix such that $R^k \xrightarrow{C} R^t \to N \to 0$ is exact. Let $\_^\vee = \operatorname{Hom}_R(\_, R)$. Then*

 (i) *$M^{\vee\vee}$ is isomorphic to the kernel of the transpose of C,*
 (ii) *the image of the canonical map $M \to M^{\vee\vee}$ is isomorphic to the module generated by the columns of the transpose of B,*
(iii) *$M^{\vee\vee}/M$ is isomorphic to (kernel C^T)/(image B^T).*

Proof. By left-exactness of the Hom functor, $M^\vee = \operatorname{Hom}_R(M, R)$ is the kernel of the map defined by the transpose of A. Thus $N = M^\vee$. Similarly, by the definition of C, $M^{\vee\vee}$ is the kernel of the transpose of C. Let $f_1, \ldots, f_t$ generate $M^\vee \subseteq R^n$. The natural map $M \to M^{\vee\vee}$ takes m to φ_m, where for each $f \in M^\vee$, $\varphi_m(f) = f(m)$. Thus, in coordinates, the image of $m \in M$ in $M^{\vee\vee} \subseteq R^t$ is the vector $(f_1(m), \ldots, f_t(m))$, therefore the image of M in $M^{\vee\vee}$ is generated by the columns of the transpose of B. $\quad\square$

Once Q is computed as in the previous lemma, its length can be computed as well. Thus it remains to compute the $\mathbb{C}[[x, y]]$-module M. By the Theorem on Formal Functions,

$$M \cong \varprojlim H^0(\ell_n, E|_{\ell_n})$$

as $\mathbb{C}[[x, y]]$-modules, where ℓ_n is the nth formal neighborhood of ℓ. We use the following lemma:

Lemma 2.2 (*Gasparim, 2000, Lemma 2.2*). Set $\bar{M} = H^0(\ell_{2j-2}, E|_{\ell_{2j-2}})$, and let $\rho \colon \bar{M} \hookrightarrow \bar{M}^{\vee\vee}$ be the natural inclusion of $\bar{M}$ into its bidual. Then $l(Q) = \dim \text{coker } \rho$.

This lemma greatly simplifies the calculations. Using this lemma, for the purpose of finding $l(Q)$, we may assume $M = \bar{M}$. Under this identification, and with our choice of transition matrix T in (3), it then follows that M is generated by all holomorphic vectors v over $\mathbb{C}[[z, u]]$ for which Tv is a holomorphic vector over $\mathbb{C}[[z^{-1}, zu]]$. Thus, we need to find all vectors $v = (a, b)$, given by $a = \sum_{i,l} a_{il} z^l u^i$, $b = \sum_{i,l} b_{il} z^l u^i$, where all i and l are non-negative, and a_{il}, b_{il} are (unknowns) in the field, for which Tv is holomorphic over $\mathbb{C}[[z^{-1}, zu]]$. This restriction yields relations among the unknowns a_{il}, b_{il}. To get all such relations, at the beginning we treat a_{il} and b_{il} as variables.

Lemma 2.3. Whenever $l > i + j$, then $b_{il} = 0$. Whenever $l > i$, then $a_{il} = 0$.

Proof. The second coordinate of Tv is $z^{-j} b = \sum_{i,l} b_{il} z^{l-j} u^i = \sum_{i,l} b_{il} z^{l-j-i} (zu)^i$. In order for this to be holomorphic in z^{-1} and zu, necessarily the coefficients b_{il} with $l > i + j$ must vanish.

The first entry of Tv is

$$z^j \sum_{i,l} a_{il} z^l u^i + \bar{p} \sum_{i_1, l_1} b_{i_1 l_1} u^{i_1} z^{l_1}$$

$$= \sum_{i,l} a_{il} z^{l-i+j} (zu)^i + \sum_{i_0=1}^{2j-2} \sum_{l_0=i_0-j+1}^{j-1} \bar{p}_{i_0 l_0} (zu)^{i_0} z^{l_0-i_0} \sum_{i_1, l_1} b_{i_1 l_1} z^{l_1-i_1} (zu)^{i_1}.$$

Let m be the minimum u-degree occurring in $\bar{p}$, i.e., the minimum i_0 such that $\bar{p}_{i_0 l_0} \neq 0$. Then for each $i < m$, the coefficient of u^i in the first entry of Tv is $\sum_l a_{il} (zu)^i z^{l-i+j}$. In order for this to be holomorphic in zu and z^{-1}, necessarily for all $l > i - j$, $a_{il} = 0$. Now consider the case $i \geq m$. The coefficient of $(zu)^i z^{l+j-i}$ in the first entry of Tv is

$$a_{il} + \sum_{i_0=1}^{2j-2} \sum_{l_0=i_0-j+1}^{j-1} \bar{p}_{i_0 l_0} b_{i-i_0, l+j-l_0},$$

where $b_{i',l'}$ is treated as zero whenever i' or l' is negative. By the established bounds, whenever $l > i - i_0 + l_0$, then $b_{i-i_0, l+j-l_0} = 0$. Since $\bar{p}$ is a subpolynomial of $p(u, zu)$, the only pairs (i_0, l_0) to consider are those with $l_0 \leq i_0$. Thus whenever $l > i - i_0 + l_0$, then $b_{i-i_0, l+j-l_0} = 0$. In particular, if $l > i$, then all $b_{i-i_0, l+j-l_0}$ are zero, so that the coefficient of $(zu)^i z^{l+j-i}$ in the first entry of Tv is a_{il}. But if Tv is to be holomorphic in zu, z^{-1}, then necessarily $a_{il} = 0$. $\square$

Thus for each i, we need only to consider finitely many unknowns a_{il}, b_{il} to construct the vectors v as in the set-up above, and find relations on these. Using Lemma 2.2, we may assume that $M = H^0(\ell_{2j-2}, E|_{\ell_{2j-2}})$. This means that we only need relations involving a_{il}, b_{il} for $i \leq 2j-2$. But the relations involving b_{il} with $i \leq 2j-2$ arising from $z^j a + \bar{p} b$ being holomorphic over $\mathbb{C}[[u, zu]]$ involve variables $a_{i'l}$ and $b_{i'l}$ with $i' \leq 2j-2+\deg_u \bar{p}$, so that

Lemma 2.4. *Let N be the sum of $2j - 2$ plus the u-degree of $\bar{p}$. Then M is generated by vectors (a, b) with $a = \sum_{i,l} a_{il} z^l u^i$, $b = \sum_{i,l} b_{il} z^l u^i$, where all i and l are non-negative, $a_{il}, b_{il} \in \mathbb{C}$, and*

(i) $a_{il} = b_{il} = 0$ *whenever* $i > N$.

(ii) $b_{il} = 0$ *whenever* $l > i + j$.

(iii) $a_{il} = 0$ *whenever* $l > i$. $\square$

The conditions on the a_{il} and b_{il} as in the lemma above are not the only ones that arise from the requirement that Tv be holomorphic in z^{-1}, zu. Finding all the conditions amounts to finding the generators of M.

Definition 2.5. All the remaining relations on the variables a_{il}, b_{il} arise from the condition that in the first entry of Tv, whenever $l > i$, then the coefficient of $u^i z^l$ must be 0. From now on, we refer to these coefficients as the **generating relations**.

Note that these coefficients are all linear forms in the ring $\mathbb{C}[a_{il}, b_{il}]$.

As in Gasparim (2000), we find these relations successively in the zeroth through Nth neighborhoods. Below follows our algorithm which computes the generating relations, arising from ensuring that the first coordinate of Tv is holomorphic in zu, z^{-1}. We write all algorithms in this paper in pseudocode, close to the Macaulay2 code that we implemented.

Algorithm "getrelations" to get the generating relations on the a_{il}, b_{il}
Input: non-negative integer N,
       fTv = first entry of Tv
       ring R, the polynomial ring in u, z, all a_{il}, b_{il}
Output: ideals nonfree and relations in R

```
    nonfree = zero ideal in R
    relations = zero ideal in R
    k = 0
    while (k ≤ N) (
       tempoly = truncation of fTv to terms of u-degree at most k
       while (tempoly != 0) (
          tempterm = leading term of tempoly
          i = u exponent of tempterm
          l = z exponent of tempterm
          partp = coefficient of uⁱzˡ in tempoly, linear form in aᵢₗ, bᵢₗ
          tempoly = tempoly - partp * uⁱzˡ
          fTv = fTv - partp * uⁱzˡ
          if (l > i) then (
             relations = relations + ideal(partp)
             nonfree = nonfree + ideal (leading variable in partp)
          )
       )
       k = k + 1
    )
    return nonfree, relations
)
```

The ideal of all the generating relations obtained in this way is called `relations`. The ideal `nonfree` contains the leading variable of each relation: in the sense of linear algebra, these leading variables are not free. We use the following ordering of the monomials.

Definition 2.6. We say that $a_{il} > a_{i'l'}$ if $i > i'$ or if $i = i'$ and $l > l'$, and similarly $b_{il} > b_{i'l'}$ if $i > i'$ or if $i = i'$ and $l > l'$, and furthermore that $a_{il} > b_{i'l'}$ for all i, l, i', l'.

Thus by the form of Tv, each relation contains at most one a_{il}, and each a_{il} appears at most once in a generating relation. Observe that if all the coefficients of $p(x, y)$ are in a subfield F of $\mathbb{C}$, then all the generating relations have coefficients in F. The algorithm `"getrelations"` above computes the relations among the given a_{il}, b_{il}, but some of these relations are "fake" in the following sense:

Example. Let $p(x, y) = x^2 - y^3, j = 3$. Then with notations as above, $N = 7$, $\bar{p} = u^2 - u^3 z^3$. For holomorphic $a = \sum a_{il} u^i z^l$, $b = \sum b_{il} u^i z^l$, the coefficient of $u^8 z^9$ in $z^j a + \bar{p} b$ is $0 = a_{86} + b_{69} - b_{56}$. However, if we restrict the first index of a_{il} to only vary from 0 to N, then `"getrelations"` gives the "fake" relation $0 = b_{69} - b_{56}$.

Thus, computation of the instanton width will have to account for and remove such "fake" relations. We do this as follows. Such relations only involve variables a_{il}, b_{il} with $i > 2j - 2$. Thus these variables are not allowed to be free variables in the sense of linear algebra. Using Lemma 2.2, the remaining free variables do give a generating set of M as follows. For each of the free variables, set that variable to 1 and all the others to 0 in (a, b). This produces a finite generating set of M as a module over $\mathbb{C}[[x, y]] = \mathbb{C}[[u, zu]]$ of elements whose entries are in $\mathbb{C}[[u, z]]$. As u is a non-zero divisor, M is isomorphic to $u^k M$ for arbitrary integer k. By Lemma 2.4, $u^j a$ and $u^j b$ are both polynomials in $u(= x)$ and $zu(= y)$, so that the generators of $u^j M$ can be written as pairs of polynomials in $\mathbb{C}[[x, y]]$.

Algorithm "polyconv" to convert $\mathbb{C}[u, z]$-polynomials to $\mathbb{C}[x, y]$-polynomials
Input: a polynomial f in u and z
Output: a "truncated" polynomial g(x, y) such that g(u, zu) = f',
        where f' is that part of f for which this can be done

```
g = 0 zero element of ℂ[u,z]
while (f != 0) (
    lf = leading term of f
    i := the u exponent of lf
    l := the z exponent of lf
    if (l ≤ i) then
        g = g + y^l * x^{i−l} * leading coefficient of lf
    f = f - lf
    )
return g
)
```

Algorithm "setvectors" to express generators of u^{N+j} M
            as vectors with entries in $\mathbb{C}[x, y]$
Input: polynomials u^j a, u^j b, named Apoly, Bpoly, respectively,
    lists changeables, allvars

Output: A, a presentation matrix for the $\mathbb{C}[[x, y]]$-module M

```
    Mxy =  zero submodule of C[x, y]²
    total = #changeables
    k = 0
    while (k < total) (
        tapoly = substitute in Apoly the kᵗʰ changeable variable to 1
        tbpoly = substitute in Bpoly the kᵗʰ changeable variable to 1
        tapoly = substitute in tapoly all other variables to 0
        tbpoly = substitute in tbpoly all other variables to 0
        A = convert tapoly into a polynomial in x and y (use polyconv)
        B = convert tbpoly into a polynomial in x and y (use polyconv)
        Mxy = Mxy + submodule of C[x,y]² generated by (A,B)
        k = k + 1
    )
    return a presentation of the C[x, y]-module Mxy
)
```

The output of the last routine is the presentation matrix of a $\mathbb{C}[x, y]$-module, which by faithful flatness of $\mathbb{C}[[x, y]]$ over $\mathbb{C}[x, y]$ is also the presentation matrix of the $\mathbb{C}[[x, y]]$-module M. Finally, tying it all together:

Algorithm to compute instanton width of the instanton with data $(\mathbf{j}, \mathbf{p})$
Input: a polynomial p in $\mathbb{C}[x, y]$ and a non-negative integer j
Output: the width of the instanton with data $(\mathbf{j}, \mathbf{p})$

```
    p̄ = p(u,zu) truncated to u-degree at most 2j-2
    N = 2j-2 + u-degree of p̄
    R = C[u,z,aᵢₗ, bᵢₗ], i < N+1, ordered as in Definition 2.6
    a =  Σ aᵢₗ uⁱ zˡ
    b =  Σ bᵢₗ uⁱ zˡ
    fTv = zʲ a + p̄ b
    compute relations and nonfree variables as in algorithm getrelations
    changeables = all aᵢₗ, bᵢₗ with i ≤ 2j − 2, which are not in nonfree
    a = a after applying all the relations
    b = b after applying all the relations
    A the presenting matrix of the C[x,y]-module M,
        output of setvectors(uʲ⁺ᴺapoly, uʲ⁺ᴺbpoly, changeables, allvars)
    Q = cokernel of the natural map M → M^∨∨, as in Lemma 2.1
    return (length of Q)
)
```

3. Computing the instanton height

In this section we compute the instanton height. Recall that the instanton height h is the length $l(R^1\pi_* E(j, \bar{p}))$. Another use of the Theorem on Formal Functions gives

$$(R^1\pi_* E)_0^\wedge = \varprojlim \ H^1(\ell_n, E|_{\ell_n}).$$

We use the following lemma:

Lemma 3.1 (*Gasparim, 2000, Lemma 2.3*). $l(R^1\pi_* E) = \dim_{\mathbb{C}} H^1(\ell_{2j-2}, E|_{\ell_{2j-2}})$.

This lemma greatly simplifies the calculations. We then proceed to compute the first Čech cohomology.

Remark. We have two options to compute Čech cohomology. Given that our open cover of the base space is given by affine sets that are open in the analytic topology as well as in the Zariski topology, we have the option to compute either holomorphic Čech cohomology (by taking holomorphic cochains) or else algebraic Čech cohomology (by taking algebraic, i.e., polynomial cochains). Since ℓ is compact, so are its formal neighborhoods. By Serre's G.A.G.A., holomorphic bundles on a compact variety are algebraic, therefore $H^1_{\text{alg}}(\ell_n, E|_{\ell_n}) = H^1_{\text{hol}}(\ell_n, E|_{\ell_n})$. It follows that the holomorphic and the algebraic methods give the same answer.

We compute the instanton height using holomorphic Čech cohomology. The 1-cocycles consist of the vectors (a, b) which are holomorphic functions defined on the intersection $U \cap V$. Hence $a, b \in \mathbb{C}[[u, z, z^{-1}]]$. The coboundaries consist of the vectors of the form $v + T^{-1}v'$, where v is holomorphic in z, u (on U) and v' in z^{-1}, zu (on V). First of all we choose simple representatives for the cocycles:

Lemma 3.2. *Every 1-cocycle has a representative of the form*

$$\sum_{i=0}^{j-2} \sum_{l=i-j+1}^{-1} \binom{a_{il}}{0} z^l u^i,$$

with $a_{il} \in \mathbb{C}$. In particular, every 1-cochain represented by $\binom{a_{il}}{0} z^l u^i$ with $i, l \geq 0$ is a coboundary.

Proof. Let σ be a 1-cocycle and let $\sim$ denote cohomological equivalence. A power series representative for a 1-cochain has the form $\sigma = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \binom{a_{il}}{b_{il}} z^l u^i$, with $a_{il}, b_{il} \in \mathbb{C}$. The 1-cochain $s_1 = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \binom{a_{il}}{b_{il}} z^l u^i$ is holomorphic in U, hence is a coboundary. Hence

$$\sigma \sim \sigma - s_1 = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \binom{a_{il}}{b_{il}} z^l u^i.$$

After a change of coordinates

$$T\sigma = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \binom{z^j a_{il} + \bar{p} b_{il}}{z^{-j} b_{il}} z^l u^i,$$

but given that $s_2 = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \binom{0}{z^{-j} b_{il}} z^l u^i$ is holomorphic in V,

$$T\sigma \sim T\sigma - s_2 = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \binom{z^j a_{il} + \bar{p} b_{il}}{0} z^l u^i$$

and going back to the U-coordinate chart,

$$\sigma = T^{-1} T\sigma \sim \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \begin{pmatrix} a_{il} + z^{-j}\bar{p}\, b_{il} \\ 0 \end{pmatrix} z^l u^i.$$

But $\bar{p}$ contains only terms z^k for $k \le j-1$, therefore $z^{-j}\bar{p}$ contains only negative powers of z. Renaming the coefficients we may write $\sigma = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \binom{a_{il}'}{0} z^l u^i$ for some $a_{il}' \in \mathbb{C}$, and consequently $T\sigma = \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \binom{z^j a_{il}'}{0} z^l u^i$. Here each term $a_{il}' z^{j+l} u^i$ satisfying $j + l \le i$ is holomorphic in the V-chart. Subtracting these holomorphic terms we are left with an expression for a where the index l varies as $i - j + 1 \le l \le -1$. This in turn forces $i \le j - 2$, giving the claimed expression for the 1-cocycle. $\quad\square$

Thus we only have to consider cocycles of the form $(a, 0)$. Which of the cocycles $(a, 0)$ is a coboundary? In other words, which $(a, 0)$ equal to $(c, d) + T^{-1}(c', d')$, where c, d are holomorphic on U and c', d' on V, or even more simply, for what c and d holomorphic on U is $T(a + c, d)$ holomorphic on V? The second coordinate of $T(a + c, d)$ is $z^{-j}d$, and in order for that to be holomorphic, $d_{il} = 0$ whenever $l \ge i + j$. This is the only restriction on c and d obtained from the second coordinate. From the first coordinate of $T(a + c, d)$ we obtain the constraint that $z^j(a + c) + \bar{p}d$ be holomorphic on V, that is, holomorphic on coordinates z^{-1} and zu.

Lemma 3.3. *Let E be the bundle defined by data $(j, \bar{p})$, and m the smallest exponent of u appearing in $\bar{p}$. Then*

$$l(R^1\pi_* E) \ge \binom{j}{2} - \binom{j-m}{2}.$$

Proof. Let $\sigma = (a, 0)$ denote a 1-cocycle where $a = z^l u^i$ with $0 \le i \le m - 1$ and $i - j + 1 \le l \le -1$. We claim that σ represents a nonzero cohomology class. In fact, for σ to be a coboundary there must exist c and d, holomorphic in U, making the expression $z^j(a + c) + \bar{p}d$ holomorphic in V. However, $z^j a$ is not holomorphic in V. Moreover, by the choice of m, no term in $\bar{p}d$ cancels $z^j a$. Consequently, no choice of c and d solves the problem of holomorphicity on V. Hence $l(R^1\pi_* E)$ is at least the number of independent cocycles of the form $\sigma = (a, 0)$, where $a = z^l u^i$ with $0 \le i \le m - 1$ and $i - j + 1 \le l \le -1$. There are $\binom{j}{2} - \binom{j-m}{2}$ such terms. $\quad\square$

Theorem 3.4. *Let E be the bundle defined by data $(j, \bar{p})$, and m the smallest exponent of u appearing in $\bar{p}$. Then*

$$l(R^1\pi_* E) = \binom{j}{2} - \binom{j-m}{2}.$$

Proof. First note that, by Lemma 3.2, if $l \ge j$, then $(z^{l-j} u^i, 0)$ is a coboundary. Using the proof of Lemma 3.3, it suffices to show that if $j > l > i \ge m$, then $\sigma = (z^{l-j} u^i, 0)$ is a coboundary. Write $\bar{p} = \sum_{r=0}^{s} a_r z^{m-r} u^m + p'$, where $s \in \{0, \dots, m\}$ is some integer, a_r are constants, $a_s \ne 0$, and p' is a polynomial in u and zu each of whose terms has u-degree at least $m + 1$. Observe that $d = a_s^{-1} u^{i-m} z^{l-m+s}$ is holomorphic on U, since by

assumption $i \geq m$ and $l > m - s$. Therefore $\sigma \sim \sigma' = (z^{l-j}u^i, -d)$, where $\sim$ denotes cohomological equivalence. Changing coordinates we have $T\sigma' = (z^l u^i - \bar{p}d, -z^{-j}d)$. From $j > l > i \geq m \geq s$ we deduce that $z^{-j}d$ is holomorphic on V. We rewrite the first entry $T\sigma'$ as

$$z^l u^i - \bar{p}d = \sum_{r=0}^{s-1} a_s^{-1} a_r z^{l+s-r} u^i + a_s^{-1} u^{i-m} z^{l-m+s} p'.$$

By assumption, $l - j + s - r \leq s - r \leq s \leq m \leq i$, therefore the first sum on the right side of this expression is holomorphic on V. It follows that $T\sigma' \sim (a_s^{-1} u^{i-m} z^{l-m+s} p', 0)$.

Let $z^v u^r$ be an arbitrary term in p'. Then $r > m$ and $v \leq r$. If $i - m + r \geq l - m + s + v$, then the term $c = a_s^{-1} u^{i-m} z^{l-m+s} \cdot u^r z^v$ is holomorphic on V, therefore $T\sigma' \sim (a_s^{-1} u^{i-m} z^{l-m+s} p' - c, 0)$.

Removing all such terms c, we may now write $T\sigma' \sim (a_s^{-1} u^{i-m} z^{l-m+s} \bar{p}, 0)$, where $\bar{p}$ contains only terms in $z^v u^r$ such that $i - m + r < l - m + s + v$. Consequently $\sigma' \sim (z^{-j} a_s^{-1} u^{i-m} z^{l-m+s} \bar{p}, 0)$ and as $i - m + r > i$, by (reverse) induction on i and l, each term $(u^{i-m} z^{l-m+s} \cdot u^r z^v z^{-j}, 0)$ is a coboundary. Hence, σ' is a sum of coboundaries, and is itself a coboundary.　$\square$

Thus when starting with $p(x, y) \in \mathbb{C}[x, y]$, the computation of the instanton height is very fast: once m is determined, then the following routine `"iheight"` finishes the computation:

Algorithm "iheight" to compute instanton height
Input: a polynomial p and a non-negative integer j
Output: returns the length of R^1.

```
    p̄ = p(u,zu) truncated to u-degree at most 2j-2
        m = the largest power of u dividing p̄
        M = j*(j-1)/2;
        if (j > m+1) then M = M - (j-m)*(j-m-1)/2;
        return M
)
```

We implemented in Macaulay2 these algorithms for computing the instanton widths and heights. The computation of height of course only takes a negligible amount of time, and the computation of widths takes a few seconds. For example, `"iwidth(x^4-x*y^5,4)"` finishes in a Linux workstation in 17.07 s, and `"iwidth(x^4-x^2*y^3 -x^3*y^5-y^8,4)"` finishes in 32.02 s.

Acknowledgments

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