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A delayed SEIQR epidemic model with pulse vaccination and the quarantine measure☆

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ABSTRACT

A delayed SEIQR epidemic model with pulse vaccination and the quarantine measure is investigated. Using the discrete dynamical system determined by the stroboscopic map, we obtain the exact periodic disease-free solution. Using the comparison method, we prove that the disease-free periodic solution is globally attractive when the basic reproductive number ($\Re^*$) is less than unity, and that the disease is permanent when another basic reproductive number ($\Re_*$) is greater than unity. In other words, the disease will be extinct if the pulse vaccination rate is larger than a critical value θ^* and the disease will be uniformly persistent if the vaccination rate is less than another critical value θ_*. Our results indicate that a longer latent period of the disease or a larger pulse vaccination rate will lead to the eradication of the disease, and whether the disease will be extinct or not is independent of the removal rate from the quarantined group. Furthermore, a larger fraction of susceptibles should be vaccinated against the disease unless the quarantine measure is taken. Finally, we find that the number of the infected decreases as the quarantine measure is taken. We carry out numerical simulations to verify our results.

1. Introduction

The health and socioeconomic risks posed by severe and sudden epidemics of infectious disease like West Nile Virus, or the assessed impact of a potential influenza pandemic, or measles pre- and post-eradication outbreaks, are compelling scientists to design and implement more effective control and preparedness programs. One intervention procedure to control the spread of infectious diseases is to isolate some infectors, in order to reduce transmission of the infection to susceptibles. Isolation may have been the first infection control method, since biblical passages refer to the ostracism of lepers, and plague sufferers were often isolated. The word quarantine originally corresponded to a period of forty days, which is the length of time that arriving ships suspected of plague infection were constrained from intercourse with the shore in Mediterranean ports during the 15–19th centuries [1]. Over the centuries, quarantine has been used to reduce the transmission of human diseases such as leprosy, plague, cholera, typhus, yellow fever, smallpox, diphtheria, tuberculosis, measles, mumps, ebola and lassa fever. Quarantine has also been used for preventing animal diseases such as rinderpest, foot and mouth, psittacosis, Newcastle disease and rabies.

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Epidemiology is the study of the spread of diseases in space and time, aiming at tracing factors that give rise to their occurrence. In recent years, epidemic mathematical models of ordinary differential equations have been studied by a number of scholars [2–9]. Most of the works in the research literature assume that the latent period of diseases is negligible. That is, once infected, each susceptible individual (in the class S) instantaneously becomes infectious (in the class I) and later recovers (in the class R) with a permanent or temporary acquired immunity. Today, we usually call these compartmental models by SIR or SIRS epidemic models. In the SIQR models for infections that confer permanent immunity, susceptibles become infected and then some infected individuals stay in the class I while they are infectious and then move to the removed class R upon recovery. Other infected individuals are transferred into the quarantine class Q while they are infectious, and then move into the removed class R. The models here have a variable total population size, because they have recruitment into the susceptible class by births or immigration and they have both natural and disease-related deaths [10]. However, many infective diseases such as tuberculosis, measles, AIDS and SARS have an incubation period. The disease will incubate inside the host for a period of time before the host becomes infectious. A susceptible individual first goes through a latent period (often called the exposed or in the class E) before being infectious. The mathematical models obtained by the compartmental approach are said to be SEIR or SEIRS epidemic models, respectively, depending on whether the acquired immunity is permanent or not. Global stability results on SEI and SEIR epidemic models without delays are discussed in many papers [3,11,12].

Infectious diseases have tremendous influence on human life. Every year, millions of people die of various infectious diseases. Controlling infectious diseases has been an increasingly complex issue in recent years. Pulse vaccination is an effective method in attempts to control infectious diseases. Pulse vaccination, the repeated application of vaccine over a defined age range, is gaining prominence as a strategy for the elimination or eradication of childhood viral infectious such as measles and polio. Pulse vaccination was considered in many papers [13–15], and has gained in prominence as a result of its highly successfully application in the control of poliomyelitis and measles throughout Central and South America [16]. But to our knowledge there have been no results on the pulse vaccination epidemic models with time delay and the quarantine class. The proposed work in this paper on delay impulsive systems is new in the literature. The related stability problems are interesting and challenging.

The organization of this paper is as follows. In the next section, we propose the SEIQR epidemic model with pulse vaccination and time delay. To prove our main results, we also give some preliminaries. In Section 3, using the discrete dynamical system determined by the stroboscopic map, we establish a sufficient condition for the global attractivity of disease-free periodic solution. The sufficient condition for the permanence of the disease is obtained in Section 4, and we try to interpret our mathematical results in terms of their ecological implication by numerical examples.

2. Model formulation and preliminaries

Suppose that the population $N(t)$ is divided into five compartments with $N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$. Here, $S(t)$ is the number of individuals in the susceptible class. $E(t)$ is the number of individuals who are infected but not yet infectious. $I(t)$ is the number of individuals who are infectious but not quarantined. $Q(t)$ is the number of individuals who are quarantined, and $R(t)$ is the number of individuals with permanent immunity. This model is called an SEIQR model since one typical pathway is: $S \to E \to I \to Q \to R$. In this section, we propose a pulse vaccination strategy in the SEIQR epidemic model:

$$\begin{cases} \left.\begin{aligned} S'(t) &= A - \beta S(t)I(t) - dS(t), \\ E'(t) &= \beta S(t)I(t) - \beta e^{-d\omega}S(t-\omega)I(t-\omega) - dE(t), \\ I'(t) &= \beta e^{-d\omega}S(t-\omega)I(t-\omega) - (\gamma+\delta+d+\alpha_1)I(t), \\ Q'(t) &= \delta I(t) - (\varepsilon+d+\alpha_2)Q(t), \\ R'(t) &= \gamma I(t) + \varepsilon Q(t) - dR(t), \end{aligned}\right\} \quad t \neq k\tau, k \in Z^+, \\ \left.\begin{aligned} S(t^+) &= (1-\theta)S(t), \\ E(t^+) &= E(t), \\ I(t^+) &= I(t), \\ Q(t^+) &= Q(t), \\ R(t^+) &= R(t) + \theta S(t), \end{aligned}\right\} \quad t = k\tau, k \in Z^+. \end{cases} \tag{2.1}$$

The following assumptions for model (2.1) are made:

(1) Parameters A, d and β are positive constants, and γ, δ, ϵ, ω, α_1 and α_2 are non-negative constants.
(2) The constant A is the recruitment rate of susceptibles corresponding to births and immigration.
(3) The parameter d is the per capita natural mortality rate, and α_1 and α_2 represent the extra disease-related death rate constants in class I and Q, respectively,
(4) The parameter δ is the rate constant for individuals leaving the infectious compartment I for the quarantine compartment Q. The Parameters γ and ε are the removal rate constants from group I and Q, respectively.
(5) Time delay ω is the latent period.

(6) The pulse vaccination is applied every τ years and θ $(0 \leq \theta \leq 1)$ denotes the proportion of those vaccinations successfully.

The total variable population size is given by the differential equation

$$N'(t) = A - dN(t) - \alpha_1 I(t) - \alpha_2 Q(t), \tag{2.2}$$

which is derived by adding the equations in model (2.1). Thus, the total population size may vary in time. From (2.2), we have

$$A - (d + \alpha_1 + \alpha_2)N(t) \leq N'(t) \leq A - dN(t).$$

It follows that

$$\frac{A}{d + \alpha_1 + \alpha_2} \leq \lim_{t\to\infty} \inf N(t) \leq \lim_{t\to\infty} \sup N(t) \leq \frac{A}{d}.$$

So we focus our attention only on the following equivalent model of model (2.1)

$$\begin{cases} \left.\begin{aligned} S'(t) &= A - \beta S(t)I(t) - dS(t), \\ I'(t) &= \beta e^{-d\omega} S(t-\omega)I(t-\omega) - (\gamma + \delta + d + \alpha_1)I(t), \\ Q'(t) &= \delta I(t) - (\varepsilon + d + \alpha_2)Q(t), \\ R'(t) &= \gamma I(t) + \varepsilon Q(t) - dR(t), \\ N'(t) &= A - dN(t) - \alpha_1 I(t) - \alpha_2 Q(t), \end{aligned}\right\} & t \neq k\tau, k \in Z^+, \\ \left.\begin{aligned} S(t^+) &= (1-\theta)S(t), \\ I(t^+) &= I(t), \\ Q(t^+) &= Q(t), \\ R(t^+) &= R(t) + \theta S(t), \\ N(t^+) &= N(t), \end{aligned}\right\} & t = k\tau, k \in Z^+. \end{cases} \tag{2.3}$$

The initial condition of (2.3) is given as

$$S(\theta) = \phi_1(\theta), I(\theta) = \phi_2(\theta), Q(\theta) = \phi_3(\theta), R(\theta) = \phi_4(\theta), N(\theta) = \phi_5(\theta), -\omega \leq \theta \leq 0. \tag{2.4}$$

Here $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)^T \in PC_+$ and PC_+ is the space of all piecewise functions $\phi : [-\omega, 0] \to \Re^5_+$ with points of discontinuity at $-k\tau (k \in Z^+)$ of the first kind and which are continuous from the left, i.e., $\phi(-k\tau - 0) = \phi(-k\tau)$, where

$$\Re^5_+ = \{(x_1, x_2, x_3, x_4, x_5) \in \Re^5 | x_i \geq 0, i = 1, 2, 3, 4, 5\}.$$

By biological meaning, we further assume that $\phi_i(0) > 0$ for $i = 1, 2, 3, 4, 5$. The meaningful domain of system (2.3) is

$$\Omega = \left\{(S, I, Q, R, N) \in \Re^5_+ | 0 \leq S + I + Q + R \leq \frac{A}{d}, N \leq \frac{A}{d}\right\},$$

and it is easy to prove that Ω is a positive invariant set.

To prove our main results we give the following lemmas.

Lemma 2.1 ([17]). *Consider the following impulsive differential equation*

$$\begin{cases} u'(t) = a - bu(t), & t \neq k\tau, \\ u(t^+) = (1-\theta)u(t), & t = k\tau, \end{cases} \tag{2.5}$$

where $a > 0, b > 0, 0 < \theta < 1$. Then there exists a unique positive periodic solution of system (2.5) given by

$$\tilde{u}(t) = \frac{a}{b} + \left(u^* - \frac{a}{b}\right) e^{-b(t-k\tau)}, \quad k\tau < t \leq (k+1)\tau,$$

which is globally asymptotically stable, where $u^* = \frac{a}{b}\frac{(1-\theta)(1-e^{-b\tau})}{1-(1-\theta)e^{-b\tau}}$.

We will use a basic result from Theorem 3.2.1 in [18] to obtain the following lemma.

Lemma 2.2. *Consider the following equation*

$$x'(t) = ax(t-\omega) - bx(t),$$

where $a > 0, b > 0, \omega > 0$; $x(t) > 0$ for $t \in [-\omega, 0]$, we have

(1) *if $a < b$, then $\lim_{t\to+\infty} x(t) = 0$,*
(2) *if $a > b$, then $\lim_{t\to+\infty} x(t) = +\infty$.*

3. Global attractivity of a disease-free periodic solution

Firstly, we demonstrate the existence of the disease-free periodic solution of system (2.3). When $I(t) \equiv 0$ for all $t \geq 0$, the Eq. (2.3) can be written as follows

$$\begin{cases} \left.\begin{aligned} S'(t) &= A - dS(t), \\ Q'(t) &= -(\varepsilon + d + \alpha_2)Q(t), \\ R'(t) &= \varepsilon Q(t) - dR(t), \\ N'(t) &= A - dN(t) - \alpha_2 Q(t), \end{aligned}\right\} \quad t \neq k\tau, k \in Z^+, \\ \left.\begin{aligned} S(t^+) &= (1-\theta)S(t), \\ Q(t^+) &= Q(t), \\ R(t^+) &= R(t) + \theta S(t), \\ N(t^+) &= N(t), \end{aligned}\right\} \quad t = k\tau, k \in Z^+. \end{cases} \tag{3.1}$$

If $I(t) \equiv 0$, from the fourth and ninth equations of model (2.1), we easily obtain that $\lim_{t\to\infty} Q(t) = 0$, and it follows from the second and seventh equations of model (2.1) that $\lim_{t\to\infty} E(t) = 0$. Further, it follows from the fourth and last equations of system (3.1) that $\lim_{t\to\infty} N(t) = \frac{A}{d}$. Therefore, we have the following limit system of system (3.1)

$$R(t) = \frac{A}{d} - S(t), \tag{3.2}$$

and

$$\begin{cases} S'(t) = A - dS(t), & t \neq k\tau, \\ S(t^+) = (1-\theta)S(t), & t = k\tau. \end{cases} \tag{3.3}$$

According to Lemma 2.1, we know that the unique periodic solution of system (3.3) given by

$$S^*(t) = \frac{A}{d}\left(1 - \frac{\theta \mathrm{e}^{-d(t-k\tau)}}{1-(1-\theta)\mathrm{e}^{-d\tau}}\right), \quad k\tau < t \leq (k+1)\tau, \tag{3.4}$$

is globally asymptotically stable. By (3.2), we obtain a periodic solution $(S^*(t), 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ of system (3.1). Hence, system (2.3) has a disease-free periodic solution $(S^*(t), 0, 0, \frac{A}{d} - S^*(t), \frac{A}{d})$.

In this section that follows we determine the global attractivity condition of the disease-free periodic solution $(S^*(t), 0, 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ of system (2.3).

Define the basic reproduction number

$$\Re^* = \frac{A\beta \mathrm{e}^{-d\omega}(1-\mathrm{e}^{-d\tau})}{d(\gamma+\delta+d+\alpha_1)(1-(1-\theta)\mathrm{e}^{-d\tau})}.$$

Hence we have the following theorem.

Theorem 3.1. *The disease-free periodic solution $(S^*(t), 0, 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ of system (2.3) is globally attractive provided that $\Re^* < 1$.*

Proof. Since $\Re^* < 1$, we can choose $\varepsilon_0 > 0$ sufficiently small such that

$$\beta \mathrm{e}^{-d\omega}\left(\frac{A}{d}\frac{1-\mathrm{e}^{-d\tau}}{1-(1-\theta)\mathrm{e}^{-d\tau}} + \varepsilon_0\right) < \gamma + \delta + d + \alpha_1. \tag{3.5}$$

It follows from the first equation of system (2.3) that $S'(t) \leq A - dS(t)$. Then we consider the comparison system with pulse

$$\begin{cases} x'(t) = A - dx(t), & t \neq k\tau, \\ x(t^+) = (1-\theta)x(t), & t = k\tau. \end{cases} \tag{3.6}$$

In view of Lemma 2.1, we know that the unique periodic solution of system (3.6)

$$x^*(t) = \frac{A}{d}\left(1 - \frac{\theta \mathrm{e}^{-d(t-k\tau)}}{1-(1-\theta)\mathrm{e}^{-d\tau}}\right), \quad k\tau < t \leq (k+1)\tau,$$

is globally asymptotically stable.

Let $(S(t), I(t), Q(t), R(t), N(t))$ be the solution of system (2.3) with initial values (2.4) and $S(0^+) = S_0 > 0$, and $x(t)$ be the solution of system (3.6) with initial value $x(0^+) = S_0$. By the comparison theorem in impulsive differential equation [19], there exists an integer $k_1 > 0$ such that

$$S(t) \le x(t) < x^*(t) + \varepsilon_0, \quad k\tau < t \le (k+1)\tau, k > k_1.$$

That is

$$S(t) < S^*(t) + \varepsilon_0 \le \frac{A}{d}\frac{1-\mathrm{e}^{-d\tau}}{1-(1-\theta)\mathrm{e}^{-d\tau}} + \varepsilon_0 \doteq S^M, \quad k\tau < t \le (k+1)\tau, k > k_1, \tag{3.7}$$

where $S^*(t)$ is defined in (3.4). Further, from the second equation of system (2.3), we know that (3.7) implies

$$I'(t) \le \beta \mathrm{e}^{-d\omega} S^M I(t-\omega) - (\gamma + \delta + d + \alpha_1) I(t)$$

for all $t \ge k\tau + \omega$ and $k > k_1$.

We consider the following comparison equation

$$y'(t) = \beta \mathrm{e}^{-d\omega} S^M y(t-\omega) - (\gamma + \delta + d + \alpha_1) y(t). \tag{3.8}$$

From (3.5), we have $\beta \mathrm{e}^{-d\omega} S^M < \gamma + \delta + d + \alpha_1$. By Lemma 2.2, it is easily obtained that $\lim_{t\to\infty} y(t) = 0$. By the comparison theorem and non-negativity of $I(t)$, we get

$$\lim_{t\to\infty} I(t) = 0. \tag{3.9}$$

Therefore, for any sufficiently small $\varepsilon_1 \in (0, 1)$, there exists an integer $k_2 > k_1$ such that $I(t) < \varepsilon_1$ for all $t > k_2\tau$.

From the first equation of system (2.3), we have

$$S'(t) \ge A - (\beta\varepsilon_1 + d) S(t), \quad \text{for } t > k_2\tau.$$

Consider the comparison impulsive differential equation for $t > k_2\tau$ and $k > k_2$,

$$\begin{cases} z'(t) = A - (\beta\varepsilon_1 + d) z(t), & t \ne k\tau, \\ z(t^+) = (1-\theta) z(t), & t = k\tau. \end{cases} \tag{3.10}$$

By Lemma 2.1, we have the unique periodic solution of system (3.10) given by

$$z^*(t) = \frac{A}{\beta\varepsilon_1 + d}\left(1 - \frac{\theta \mathrm{e}^{-(\beta\varepsilon_1+d)(t-k\tau)}}{1-(1-\theta)\mathrm{e}^{-(\beta\varepsilon_1+d)\tau}}\right), \quad k\tau < t \le (k+1)\tau,$$

which is globally asymptotically stable.

Let $(S(t), I(t), Q(t), R(t), N(t))$ be the solution of system (2.3) with initial values (2.4) and $S(0^+) = S^0 > 0$, and $z(t)$ be the solution of system (3.10) with initial value $z(0^+) = S^0$. In view of the comparison theorem in impulsive differential equation, there exists an integer $k_3 > k_2$ such that

$$S(t) > z^*(t) - \varepsilon_1, \quad k\tau < t \le (k+1)\tau, k > k_3. \tag{3.11}$$

Because ε_0 and ε_1 are sufficiently small, it follows from (3.7) and (3.11) that

$$\lim_{t\to\infty} S(t) = S^*(t). \tag{3.12}$$

When $t > k_3\tau$, from the second and seventh equations of system (2.1) we have

$$E'(t) \le \beta\frac{A}{d}\varepsilon_1 - dE(t).$$

This implies that there exists an integer $k_4 > k_3$ such that

$$E(t) \le \left(1 + \frac{\beta A}{d^2}\right)\varepsilon_1, \tag{3.13}$$

for all $t \ge k_4\tau$. When $t > k_4\tau$, from the third and eighth equations of system (2.3), we obtain

$$Q'(t) \le \delta\varepsilon_1 - (\varepsilon + d + \alpha_2) Q(t).$$

This implies that there exists an integer $k_5 > k_4$ such that

$$Q(t) \le \left(\frac{\delta}{\varepsilon + d + \alpha_2} + 1\right)\varepsilon_1, \tag{3.14}$$

for all $t \ge k_5\tau$.

From the fifth equation of system (2.3), we have

$$N'(t) \geq A - dN(t) - \alpha_1\varepsilon_1 - \alpha_2\left(\frac{\delta}{\varepsilon + d + \alpha_2} + 1\right)\varepsilon_1, \quad t > k_5\tau. \tag{3.15}$$

Consider the following comparison system

$$v'(t) = A - \alpha_1\varepsilon_1 - \alpha_2\left(\frac{\delta}{\varepsilon + d + \alpha_2} + 1\right)\varepsilon_1 - dv(t), \quad t > k_5\tau.$$

It is easy to obtain $\lim_{t\to\infty} v(t) = \frac{A-\alpha_1\varepsilon_1-\alpha_2(\frac{\delta}{\varepsilon+d+\alpha_2}+1)\varepsilon_1}{d}$. By comparison theorem, there is an integer $k_6 > k_5$ such that

$$N(t) \geq \frac{A - \alpha_1\varepsilon_1 - \alpha_2\left(\frac{\delta}{\varepsilon+d+\alpha_2} + 1\right)\varepsilon_1}{d} - \varepsilon_1, \tag{3.16}$$

for all $t > k_6\tau$. Because ε_1 can be arbitrarily small and $\limsup_{t\to\infty} N(t) \leq \frac{A}{d}$, it follows from (3.13), (3.14) and (3.16) that

$$\lim_{t\to\infty} E(t) = 0, \qquad \lim_{t\to\infty} Q(t) = 0, \qquad \lim_{t\to\infty} N(t) = \frac{A}{d}. \tag{3.17}$$

Finally, it follows from (3.9), (3.12) and (3.17) that the disease-free periodic solution $(S^*(t), 0, 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ of system (2.3) is globally attractive. The proof is completed. □

Set

$$\tau_* = \frac{1}{d}\ln\left[1 + \frac{d(\gamma+\delta+d+\alpha_1)\theta}{A\beta e^{-d\omega} - d(\gamma+\delta+d+\alpha_1)}\right],$$

$$\theta^* = 1 - \frac{d(\gamma+\delta+d+\alpha_1)e^{d\tau} - A\beta e^{-d\omega}(e^{d\tau}-1)}{d(\gamma+\delta+d+\alpha_1)}$$

and

$$\omega^* = \frac{1}{d}\ln\left[\frac{A\beta(1-e^{-d\tau})}{d(\gamma+\delta+d+\alpha_1)(1-(1-\theta)e^{-d\tau})}\right].$$

According to Theorem 3.1 we can easily obtain the following results.

Corollary 3.1. *The disease-free periodic solution* $(S^*(t), 0, 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ *of system* (2.3) *is globally attractive provided that* $\theta > \theta^*$ *or* $\omega > \omega^*$.

Corollary 3.2. (i) *If* $A\beta e^{-d\omega} \leq d(\gamma+\delta+d+\alpha_1)$, *then the disease-free periodic solution* $(S^*(t), 0, 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ *of system* (2.3) *is globally attractive;*

(ii) *If* $A\beta e^{-d\omega} > d(\gamma+\delta+d+\alpha_1)$, *then the disease-free periodic solution* $(S^*(t), 0, 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ *of system* (2.3) *is globally attractive provided that* $\tau < \tau_*$.

Remark 3.1. Theorem 3.1 determines the global attractivity of the disease-free periodic solution $(S^*(t), 0, 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ of system (2.3) in Ω for the case $\Re^* < 1$. Its epidemiology implication is that the infective population vanishes i.e. the disease dies out. Corollary 3.1 implies that the disease will disappear if the pulse vaccination rate is lager than the threshold value θ^* or the length of latent period exceeds ω^*.

4. Permanence

In this section, we say the disease becomes endemic if the infectious population persists above a certain positive level for a long period.

Definition 4.1. In system (2.3), the disease is said to be permanent if there is a positive constant q such that $\liminf_{t\to\infty} I(t) > q$ for any positive solution $(S(t), I(t), Q(t), R(t), N(t))$ of system (2.3) with initial condition (2.4).

Set

$$\Re_* = \frac{A\beta e^{-d\omega}(1-\theta)(1-e^{-d\tau})}{d(\gamma+\delta+d+\alpha_1)(1-(1-\theta)e^{-d\tau})} = (1-\theta)\Re^*, \tag{4.1}$$

and

$$I^* = \frac{d}{\beta}(\Re_* - 1). \tag{4.2}$$

Theorem 4.1. *If $\Re_* > 1$, then the disease is permanent in model (2.3).*

Proof. Let us consider the following continuous function

$$W(t) = I(t) + \beta e^{-d\omega} \int_{t-\omega}^{t} S(s)I(s)ds. \tag{4.3}$$

The derivative of $W(t)$ along solutions of system (2.3) is

$$W'(t) = [\beta e^{-d\omega} S(t) - (\gamma + \delta + d + \alpha_1)]I(t). \tag{4.4}$$

Since $\Re_* > 1$, we easily see that $I^* > 0$, and there exists a sufficiently small $\varepsilon_2 > 0$ such that

$$\frac{\beta e^{-d\omega}}{\gamma + \delta + d + \alpha_1}\xi > 1, \tag{4.5}$$

where

$$\xi = \frac{A(1-\theta)(1 - e^{-(\beta I^* + d)\tau})}{(\beta I^* + d)(1 - (1-\theta)e^{-(\beta I^* + d)\tau})} - \varepsilon_2.$$

We claim that it is impossible that $I(t) \le I^*$ for all $t \ge t_0$ (t_0 is a certain non-negative constant). Suppose the contrary, then as $t \ge t_0$,

$$S'(t) \ge A - (\beta I^* + d)S(t).$$

Consider the following comparison impulsive system for all $t \ge t_0$,

$$\begin{cases} u'(t) = A - (\beta I^* + d)u(t), & t \ne k\tau, \\ u(t^+) = (1-\theta)u(t), & t = k\tau. \end{cases} \tag{4.6}$$

According to Lemma 2.1, we obtain that

$$u^*(t) = \frac{A}{\beta I^* + d} + \left(\bar{u} - \frac{A}{\beta I^* + d}\right) e^{-(\beta I^* + d)(t - k\tau)}, \quad k\tau < t \le (k+1)\tau,$$

is the unique globally asymptotically stable positive periodic solution of system (4.6). Here

$$\bar{u} = \frac{A}{\beta I^* + d} \frac{(1-\theta)(1 - e^{-(\beta I^* + d)\tau})}{1 - (1-\theta)e^{-(\beta I^* + d)\tau}}.$$

Thus, there exists a $T^* > 0$ satisfying

$$S(t) > u^*(t) - \varepsilon_2 \ge \bar{u} - \varepsilon_2 \doteq \xi, \tag{4.7}$$

for all $t \ge t_0 + T^* \doteq t_1$. By (4.4) and (4.7), we have

$$W'(t) \ge (\gamma + \delta + d + \alpha_1)\left(\frac{\beta e^{-d\omega}}{\gamma + \delta + d + \alpha_1}\xi - 1\right) I(t), \quad t \ge t_1. \tag{4.8}$$

Set

$$I_l = \min_{t \in [t_1, t_1 + \omega]} I(t).$$

We will show that $I(t) \ge I_l$ for all $t \ge t_1$. If it is not true, then there exists a $T_0 \ge 0$ such that $I(t) \ge I_l$ for $t_1 \le t \le t_1 + \omega + T_0$, $I(t_1 + \omega + T_0) = I_l$ and $I'(t_1 + \omega + T_0) \le 0$. However, the second equation of system (2.3) implies that

$$\begin{aligned} I'(t_1 + \omega + T_0) &= \beta e^{-d\omega} S(t_1 + T_0)I(t_1 + T_0) - (\gamma + \delta + d + \alpha_1)I(t_1 + \omega + T_0) \\ &\ge [\beta e^{-d\omega}\xi - (\gamma + \delta + d + \alpha_1)]I_l > 0. \end{aligned} \tag{4.9}$$

This is a contradiction. Hence, $I(t) \ge I_l$ for all $t \ge t_1$. Consequently, for all $t \ge t_1$, we have that

$$W'(t) \ge (\gamma + \delta + d + \alpha_1)\left(\frac{\beta e^{-d\omega}}{\gamma + \delta + d + \alpha_1}\xi - 1\right) I_l > 0.$$

Since $W(t)$ is continuous on $[0, +\infty)$ and these points at which $W(t)$ is not derivable are at most countable, then $W(t) \to +\infty$ as $t \to +\infty$. This is a contrary to the boundedness of $W(t)$. Hence, the claim is proved. For the claim, we will discuss the following two possibilities.

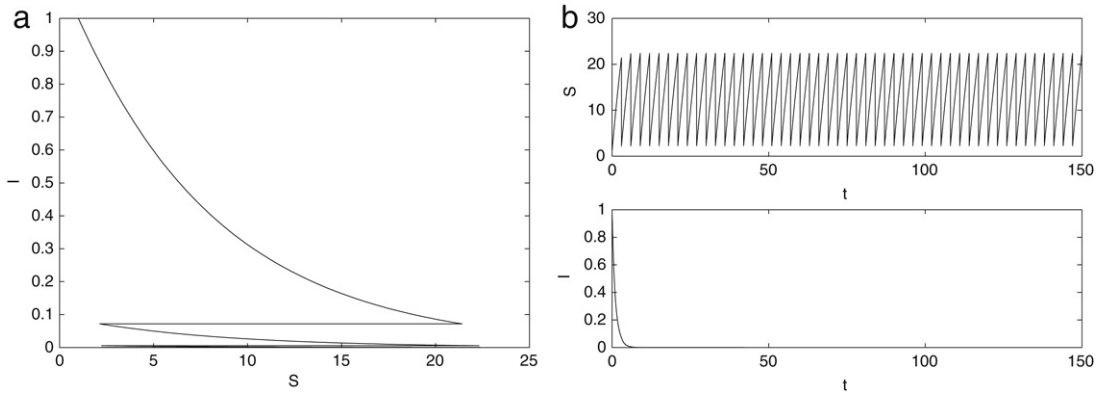

Fig. 1. This figure displays movement paths of S and I and the extinction of the disease with $A = 8$, $\beta = 0.01$, $d = 0.1$, $\alpha_1 = 0.2$, $\gamma = 0.4$, $\delta = 0.3$, $\omega = 0.4$, $\tau = 3$ and $\theta = 0.9$. (a) Phase portrait. (b) Time series. The variables $S(t)$ and $I(t)$ in system (2.1) are integrated 50 pulsing cycles with initial value $S(0) = 2$, $I(0) = 2$ and the last 20 are plotted in (a).

(i) $I(t) \geq I^*$ for all large t;
(ii) $I(t)$ oscillates about I^* for all large t.

Finally, we will show that $I(t) \geq I^* e^{-(\gamma+\delta+d+\alpha_1)(T^*+\omega)} \doteq q$ as t is large enough. Evidently, we only need consider the case (ii). Let t_1 and t_2 be large sufficiently and satisfy

$$I(t_1) = I(t_2) = I^*,$$
$$I(t) < I^* \quad \text{as } t \in (t_1, t_2).$$

If $t_2 - t_1 \leq T^* + \omega$, then $I'(t) \geq -(\gamma + \delta + d + \alpha_1)I(t)$ and $I(t_1) = I^*$ imply $I(t) \geq q$ for all $t \in [t_1, t_2]$. If $t_2 - t_1 > T^* + \omega$, then it is clear that $I(t) \geq q$ for all $t \in [t_1, t_1 + T^* + \omega]$. Thus, proceeding exactly as the proof of (4.7), we see that $S(t) > \xi$ for all $t \in [t_1 + T^*, t_2]$. Next, we will prove that it is still valid that $I(t) \geq q$ for all $t \in [t_1 + T^* + \omega, t_2]$. If it is not true, there is a $T_1 \geq 0$ such that $I(t) \geq q$ for all $t \in [t_1, t_1 + T^* + \omega + T_1]$, $I(t_1 + T^* + \omega + T_1) = q$ and $I'(t_1 + T^* + \omega + T_1) \leq 0$. Using the second equation of system (2.3), as $t = t_1 + T^* + \omega + T_1$, we further obtain

$$\begin{aligned} I'(t) &= \beta e^{-d\omega} S(t-\omega) I(t-\omega) - (\gamma + \delta + d + \alpha_1) I(t) \\ &\geq [\beta e^{-d\omega} \xi - (\gamma + \delta + d + \alpha_1)] q > 0. \end{aligned}$$

This is a contrary. So, $I(t) \geq q$ is valid for all $t \in [t_1, t_2]$. The proof of Theorem 4.1 is completed. □

Denote

$$\tau^* = \frac{1}{d} \ln \left[1 + \frac{d(\gamma + \delta + d + \alpha_1)\theta}{A\beta e^{-d\omega}(1-\theta) - d(\gamma + \delta + d + \alpha_1)} \right],$$
$$\theta_* = 1 - \frac{d(\gamma + \delta + d + \alpha_1) e^{d\tau}}{A\beta e^{-d\omega}(e^{d\tau} - 1) + d(\gamma + \delta + d + \alpha_1)},$$

and

$$\omega_* = \frac{1}{d} \ln \left[\frac{A\beta(1-\theta)(1 - e^{-d\tau})}{d(\gamma + \delta + d + \alpha_1)(1 - (1-\theta)e^{-d\tau})} \right].$$

Corollary 4.1. *If $\theta < \theta_*$ or $\omega < \omega_*$, then the disease is permanent.*

Corollary 4.2. *If $A\beta e^{-d\omega}(1-\theta) > d(\gamma + \delta + d + \alpha_1)$ and $\tau > \tau^*$, then the disease is permanent.*

Remark 4.1. By parameter values used in Fig. 1, it follows that $\Re^* = 0.2152 < 1$. Hence from Theorem 3.1, the disease will disappear (see Fig. 1). In the same way, by parameter values used in Fig. 2, it is obtained that $\Re_* = 1.3161 > 1$. According to Theorem 4.1, the disease will be permanent (see Fig. 2). These results validate our theoretical result.

Remark 4.2. From Theorems 3.1 and 4.1, we know that the critical values $\Re^*$ and $\Re_*$ are independent of ε, which implies that whether the disease will be extinct or not is independent of the removal rate from the quarantined group.

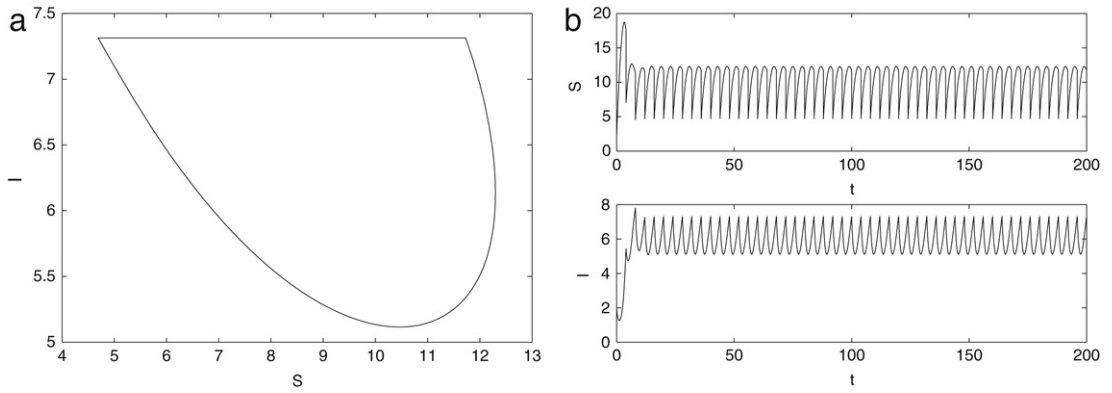

Fig. 2. This figure displays movement paths of S and I and the epidemic of the disease with $A = 10$, $\beta = 0.1$, $d = 0.2$, $\alpha_1 = 0.3$, $\gamma = 0.3$, $\delta = 0.2$, $\omega = 0.1$, $\tau = 4$ and $\theta = 0.6$. (a) Phase portrait. (b) Time series. The variables $S(t)$ and $I(t)$ in system (2.1) are integrated 50 pulsing cycles with initial value $S(0) = 2, I(0) = 2$ and the last 20 are plotted in (a).

5. An SEIR epidemic model with pulse vaccination and time delay

If there is no quarantine class Q, system (2.3) becomes

$$\begin{cases} \left.\begin{aligned} S'(t) &= A - \beta S(t)I(t) - dS(t), \\ I'(t) &= \beta \mathrm{e}^{-d\omega} S(t-\omega)I(t-\omega) - (\gamma + d + \alpha_1)I(t), \\ R'(t) &= \gamma I(t) - dR(t), \\ N'(t) &= A - dN(t) - \alpha_1 I(t), \end{aligned}\right\} & t \neq k\tau, k \in Z^+, \\ \left.\begin{aligned} S(t^+) &= (1-\theta)S(t), \\ I(t^+) &= I(t), \\ R(t^+) &= R(t) + \theta S(t), \\ N(t^+) &= N(t), \end{aligned}\right\} & t = k\tau, k \in Z^+. \end{cases} \tag{5.1}$$

Denote

$$\mathfrak{R}_0^* = \frac{A\beta \mathrm{e}^{-d\omega}(1 - \mathrm{e}^{-d\tau})}{d(\gamma + d + \alpha_1)(1 - (1-\theta)\mathrm{e}^{-d\tau})},$$

and

$$\mathfrak{R}_*^0 = \frac{A\beta \mathrm{e}^{-d\omega}(1-\theta)(1 - \mathrm{e}^{-d\tau})}{d(\gamma + d + \alpha_1)(1 - (1-\theta)\mathrm{e}^{-d\tau})}.$$

Then by Theorems 3.1 and 4.1, we get the corollary as follows.

Corollary 5.1. *The disease-free periodic solution* $(S^*(t), 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ *of system* (5.1) *is globally attractive provided that* $\mathfrak{R}_0^* < 1$,*where* $S^*(t)$ *is defined in* (3.4). *If* $\mathfrak{R}_*^0 > 1$, *then the disease is permanent in model* (5.1).

Denote

$$\tau_*^0 = \frac{1}{d} \ln\left[1 + \frac{d(\gamma + d + \alpha_1)\theta}{A\beta \mathrm{e}^{-d\omega} - d(\gamma + d + \alpha_1)}\right],$$

$$\theta_0^* = 1 - \frac{d(\gamma + d + \alpha_1)\mathrm{e}^{d\tau} - A\beta \mathrm{e}^{-d\omega}(\mathrm{e}^{d\tau} - 1)}{d(\gamma + d + \alpha_1)}$$

and

$$\omega_0^* = \frac{1}{d} \ln\left[\frac{A\beta(1 - \mathrm{e}^{-d\tau})}{d(\gamma + d + \alpha_1)(1 - (1-\theta)\mathrm{e}^{-d\tau})}\right].$$

Then by Corollaries 3.1 and 3.2, we get the corollaries as follows.

Corollary 5.2. *The disease-free periodic solution* $(S^*(t), 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ *of system* (5.1) *is globally attractive provided that* $\theta > \theta_0^*$ *or* $\omega > \omega_0^*$.

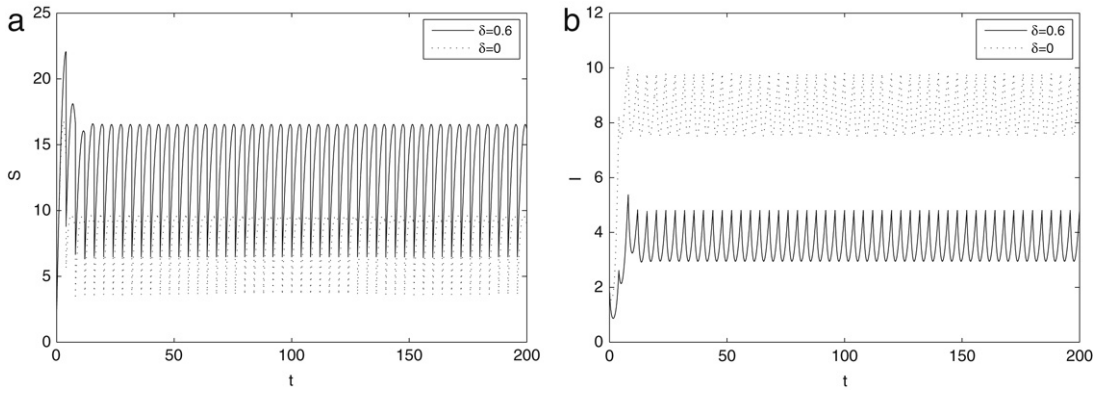

Fig. 3. This figure displays the effect of the quarantine on S and I **with** $A = 10$, $\beta = 0.1$, $d = 0.2$, $\alpha_1 = 0.3$, $\gamma = 0.3$, $\delta = 0.2$, $\omega = 0.1$, $\tau = 4$, $\theta = 0.6$. The solid line denotes time series of the variables when the quarantine measure is taken (the quarantine rate $\delta = 0.6$). The dotted line denotes time series of the variables when the quarantine measure is not taken (the quarantine rate $\delta = 0$). (a) Time series of the susceptible. (b) Time series of the infected.

Corollary 5.3. (i) *If* $A\beta e^{-d\omega} \le d(\gamma + d + \alpha_1)$, *then the disease-free periodic solution* $(S^*(t), 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ *of system* (5.1) *is globally attractive;*

(ii) *If* $A\beta e^{-d\omega} > d(\gamma + d + \alpha_1)$, *then the disease-free periodic solution* $(S^*(t), 0, \frac{A}{d} - S^*(t), \frac{A}{d})$ *of system* (5.1) *is globally attractive provided that* $\tau < \tau_*^0$.

Denote

$$\tau_0^* = \frac{1}{d} \ln\left[1 + \frac{d(\gamma + d + \alpha_1)\theta}{A\beta e^{-d\omega}(1-\theta) - d(\gamma + d + \alpha_1)}\right],$$

$$\theta_*^0 = 1 - \frac{d(\gamma + d + \alpha_1)e^{d\tau}}{A\beta e^{-d\omega}(e^{d\tau} - 1) + d(\gamma + d + \alpha_1)},$$

and

$$\omega_*^0 = \frac{1}{d} \ln\left[\frac{A\beta(1-\theta)(1 - e^{-d\tau})}{d(\gamma + d + \alpha_1)(1 - (1-\theta)e^{-d\tau})}\right].$$

Then by Corollaries 4.1 and 4.2, we get the corollaries as follows.

Corollary 5.4. *If* $\theta < \theta_*^0$ *or* $\omega < \omega_*^0$, *then the disease is permanent.*

Corollary 5.5. *If* $A\beta e^{-d\omega}(1-\theta) > d(\gamma + d + \alpha_1)$, *then the disease is permanent provided that* $\tau > \tau_0^*$.

Remark 5.1. Fig. 3 shows the effect of the quarantine on S and I. Obviously, the number of the infected decreases as the quarantine measure is taken.

6. Discussion

In this paper, we have studied the dynamical behavior of a delayed SEIQR epidemic model with pulse vaccination and the quarantine measure. We introduce two thresholds $\Re_*$ and $\Re^*$ (see Theorems 3.1 and 4.1) and further obtain that the disease will be extinct if $\Re^* < 1$, and the disease will be permanent if $\Re_* > 1$. (That is, after some period of time the disease will be endemic.) Corollaries 3.1 and 4.1 show that under the condition $\theta > \theta^*$ or $\omega > \omega^*$, the disease will fade out, and the disease will be uniformly persistent if $\theta < \theta_*$ or $\omega < \omega_*$. Our results indicate that a longer latent period of the disease or a larger pulse vaccination rate will lead to the eradication of a disease. On the other hand, by Corollaries 3.1 and 5.1, we know that the threshold value θ_0^* is larger than θ^*, which implies that a larger fraction of susceptibles should be vaccinated against the disease if the quarantine measure is not taken. Finally numerical results show that a decreased number of the infected is observed as the quarantine measure is taken, displaying the effect of quarantine measure in eradicating the disease.

We have only discussed two cases: (1) $\Re^* < 1$ (or $\theta > \theta^*$), (2) $\Re_* > 1$ (or $\theta < \theta_*$). Obviously, $\Re^* \ge \Re_*$. For $\Re_* \le 1 \le \Re^*$, the dynamical behaviors of model (2.3) have not been studied. For the pulse vaccination rate between θ_* and θ^*, the extinction and uniform persistence of the disease have not been obtained. These aspects will be considered elsewhere.

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