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Expansion of the Weber Function D_ν for Small Order, with an Application*

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In this paper we investigate the Weber function D_ν for small index ν and obtain explicitly the first-order asymptotic behavior. We then apply this result to obtain an expression for the first moment of the first passage distribution for a stationary, Gauss–Markov (Ornstein–Uhlenbeck) process.

I. AN ASYMPTOTIC EXPANSION FOR THE WEBER FUNCTION

The Weber function $D_\nu(z)$ satisfies the (parabolic cylinder) differential equation

$$D_\nu''(z) + \left[\nu + \frac{1}{2} - \frac{z^2}{4}\right] D_\nu(z) = 0 \tag{1}$$

with the initial conditions

$$D_\nu(0) = \pi^{\frac{1}{2}} \frac{2^{\nu/2}}{\Gamma\left(\frac{1-\nu}{2}\right)}, \qquad D_\nu'(0) = -\pi^{\frac{1}{2}} \frac{2^{(\nu+1)/2}}{\Gamma\left(-\frac{\nu}{2}\right)},$$

where the prime denotes differentiation with respect to the argument z.

It is shown in [1] that $D_\nu(z)$ is an analytic function of both z and ν. In particular, there is an asymptotic expansion in ν of the form

$$D_\nu(z) = \sum_{n=0}^{\infty} y_n(z)\, \nu^n. \tag{2}$$

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We obtain the coefficients y_n by a perturbation approach to the differential equation (1).

We denote by $\mathbf{L}$ the differential operator

$$\mathbf{L} = \frac{d^2}{dz^2} + \frac{1}{2} - \frac{z^2}{4}$$

and expand the initial conditions as power series in ν:

$$D_\nu(0) = \sum_{n=0}^{\infty} a_n \nu^n, \qquad D_\nu'(0) = \sum_{n=0}^{\infty} b_n \nu^n.$$

Direct computation (by differentiation, for example) gives

$$\begin{aligned} a_0 &= 1, & b_0 &= 0, \\ a_1 &= -\frac{1}{2}(\gamma + \log 2), & b_1 &= \sqrt{\frac{\pi}{2}}, \\ &\cdots, & &\cdots, \end{aligned} \tag{3}$$

where γ is Euler's constant. By standard perturbation methods (see for example, [2]), we obtain the following differential recurrence relations for the y_n, $n \geqslant 0$:

$$\mathbf{L}y_n(z) = -y_{n-1}(z), \qquad y_n(0) = a_n, \qquad y_n'(0) = b_n \tag{4}$$

with

$$y_{-1}(z) = 0.$$

The solution of this system of equations is conveniently expressed in terms of the homogeneous Green's function for the operator $\mathbf{L}$ given by

$$g(z, \zeta) = \begin{cases} 0, & z \leqslant \zeta, \\ i[D_{-1}(i\zeta)\, D_0(z) - D_0(\zeta)\, D_{-1}(iz)], & z > \zeta. \end{cases} \tag{5}$$

The solution to (4) then has the form

$$y_n(z) = \int_0^\infty g(z, \zeta)\, y_{n-1}(\zeta)\, d\zeta + Y_n(z), \tag{6}$$

where the function $Y_n(z)$ satisfies the homogeneous problem

$$\mathbf{L}Y_n(z) = 0, \qquad Y_n(0) = a_n, \qquad Y_n'(0) = b_n.$$

It is easily checked that

$$Y_n(z) = \left(a_n - \sqrt{\frac{\pi}{2}}\, ib_n\right) D_0(z) + ib_n D_{-1}(iz). \tag{7}$$

Combining (6) and (7) we obtain

$$y_0(z) = D_0(z) = e^{-z^2/4}. \tag{8}$$

Similarly,

$$\begin{aligned} y_1(z) &= iD_0(z) \int_0^z D_0(\zeta)\, D_{-1}(i\zeta)\, d\zeta - iD_{-1}(iz) \int_0^z D_0^2(\zeta)\, d\zeta \\ &\quad - \frac{1}{2}(\gamma + \log 2 + i\pi)\, D_0(z) + i\sqrt{\frac{\pi}{2}}\, D_{-1}(iz). \end{aligned} \tag{9}$$

This expression can be simplified considerably by using the relation

$$\int_0^z D_0^2(\zeta)\, d\zeta = \sqrt{\frac{\pi}{2}} - e^{-z^2/4}\, D_{-1}(z)$$

and observing that since

$$\frac{d}{dz}\left[D_{-1}(z)\, D_{-1}(iz) + \int_0^z D_0(\zeta)\, D_{-1}(i\zeta)\, d\zeta\right] = -\, iD_0(iz)\, D_{-1}(z),$$

we have

$$D_{-1}(z)\, D_{-1}(iz) + \int_0^z D_0(\zeta)\, D_{-1}(i\zeta)\, d\zeta = \frac{\pi}{2} - i \int_0^z D_0(i\zeta)\, D_{-1}(\zeta)\, d\zeta.$$

Equation (9) for y_1 may then be rewritten as

$$y_1(z) = e^{-z^2/4} \left\{\frac{\gamma + \log 2}{2} - \int_0^z D_0(it)\, D_{-1}(t)\, dt\right\}. \tag{10}$$

In terms of Rosser's G function [3]

$$G(z, p) = \int_0^z e^{-p^2 y^2} \int_0^y e^{-x^2}\, dx\, dy,$$

y_1 can alternatively be expressed as

$$y_1(z) = e^{-z^2/4} \left\{\frac{\gamma + \log 2}{2} - i\sqrt{2}\, \mathrm{erf}\left(-\frac{iz}{\sqrt{2}}\right) + 2G(z, i)\right\}. \tag{11}$$

Efficient computational formulas for G are known. For $0 \leqslant z \leqslant 6$ see [3, (16–7)]. For $z \geqslant 6$ ten significant figures may be obtained from the approximation

$$G(z, i) \approx \frac{\sqrt{\pi}}{2} \int_0^z e^{t^2}\, dt = \frac{-\pi i}{4} \operatorname{erf}(iz).$$

Since $G(z, i)$ cannot be expressed in terms of elementary functions, it does not seem possible to obtain y_2 in simple form.

Combining (2), (8), and (11) we have the first order asymptotic development

$$D_\nu(z) = e^{-z^2/4} + e^{-z^2/4} \left\{ \frac{\gamma + \log 2}{2} + i\sqrt{2} \operatorname{erf}\left(-\frac{iz}{\sqrt{2}}\right) + 2G(z, i) \right\} \nu + O(\nu^2). \tag{12}$$

II. Application to the Ornstein–Uhlenbeck Process

We now apply the results of the first section to find the first moment of the first passage distribution of the special diffusion process $\xi(t)$ which is stationary, Markovian and Gaussian with mean 0 and covariance

$$E[\xi(t_1)\, \xi(t_2)] = \frac{\sigma^2}{2\rho} e^{-\rho|t_2 - t_1|}.$$

This process describes the motion of a harmonically bound Brownian particle drawn to the origin by a force whose magnitude is proportional to its displacement with proportionality constant $\rho > 0$. We consider only the case $\sigma^2/2 = 1$ and $\rho = 1$ since the general case can be obtained by a simple scale change. The transition density $p(t, x, y)$ of this process satisfies the backward type Kolmogorov equation

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial x^2} - \rho x \frac{\partial p}{\partial x}. \tag{12}$$

Let ℓ denote a constant threshold level and τ the first passage time across the level ℓ, i.e.,

$$\tau = \inf\{t \geqslant 0 : \xi(t) > \ell\}.$$

Then it is shown in [4] that the Laplace transform for the distribution of τ is given by

$$R(\alpha, x) = \int_0^\infty e^{-\alpha t} P^x[\tau > t]\, dt = \begin{cases} \dfrac{1}{\alpha}\left[1 - \dfrac{e^{x^2/4} D_{-\alpha}(-x)}{e^{\ell^2/4} D_{-\alpha}(-\ell)}\right], & x < \ell, \\ 0, & x \geqslant \ell, \end{cases} \tag{13}$$

where x is the starting point of the process. Let $\mu_n(x, \ell)$ be the n-th moment of the first passage time distribution, i.e.,

$$\begin{aligned}\mu_n(x, \ell) &= \int_0^\infty t^n \, dP^x[\tau \leqslant t] \\ &= n \int_0^\infty t^{n-1} P^x[\tau > t] \, dt, \qquad n \geqslant 1.\end{aligned} \tag{14}$$

Then, in view of (13), we may write

$$\mu_n = (-1)^{n-1} n \lim_{\alpha \downarrow 0} \frac{\partial^{n-1} R}{\partial \alpha^{n-1}} .$$

In particular,

$$\mu_1(x, \ell) = \lim_{\alpha \downarrow 0} R(\alpha, x).$$

Applying the asymptotic expansion in (12) for $D_{-\alpha}$ we have

$$\mu_1(x, \ell) = \lim_{\alpha \downarrow 0} \frac{1}{\alpha} \left\{ \frac{e^{\ell^2/4}[y_0(-\ell) - \alpha y_1(-\ell) + O(\alpha^2)] - e^{x^2/4}[y_0(-x) - \alpha y_1(-x) + O(\alpha^2)]}{e^{\ell^2/4}[y_0(-\ell) + O(\alpha)]} \right\}$$

so that

$$\mu_1(x, \ell) = e^{x^2/4} y_1(-x) - e^{\ell^2/4} y_1(-\ell). \tag{15}$$

Explicitly, we have

$$\mu_1(x, \ell) = i\sqrt{2}\left[\operatorname{erf}\left(\frac{i\ell}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{ix}{\sqrt{2}}\right)\right] + 2[G(-x, i) - G(-\ell, i)]. \tag{16}$$

Successive differentiation shows that the expression in (16) can be written as the simple power series

$$\mu_1(x, \ell) = \sqrt{\pi} \sum_{n=1}^{\infty} \frac{\ell^n - x^n}{2^{n/2} n \Gamma\left(\frac{n+1}{2}\right)} . \tag{17}$$

It is of separate interest to note that this last series can be obtained directly from (13) using the integral representation of $D_{-\alpha}$ for $\alpha > 0$:

$$D_{-\alpha}(z) = \frac{e^{-z^2/4}}{\Gamma(\alpha)} \int_0^\infty e^{-zs - \frac{1}{2}s^2} s^{\alpha-1} \, ds.$$

Expanding the exponential e^{-xs} and interchanging the integration and summation we have

$$D_{-\alpha}(z) = \frac{e^{-z^2/4}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \left[\int_0^\infty e^{-\frac{1}{2}s^2} s^{\alpha+n-1}\, ds\right] \frac{(-x)^n}{n!}.$$

The interchange of operations is justified by monotone convergence for $x > 0$ and dominated convergence for $x \leqslant 0$. From these two representations it follows that

$$\int_0^\infty e^{-xs} e^{-\frac{1}{2}s^2} s^{\alpha-1}\, ds = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\, D_{-(\alpha+n)}(0)}{n!} (-x)^n$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{2^{(\alpha+n)/2}\Gamma\left(\frac{\alpha+n+1}{2}\right) n!} (-x)^n.$$

Substitution in (13) then gives

$$R(\alpha, x) = \frac{\displaystyle\sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)}{2^{(\alpha+n)/2}\Gamma\left(\frac{\alpha+n+1}{2}\right) n!} (\ell^n - x^n)}{\displaystyle\alpha \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{2^{(\alpha+n)/2}\Gamma\left(\frac{\alpha+n+1}{2}\right) n!} \ell^n},$$

from which the expression in (17) easily follows.

One may also easily verify (17) by direct substitution in the general differential recurrence relation for the moments obtained in [4]:

$$\frac{\partial^2 \mu_n}{\partial x^2} - x \frac{\partial \mu_n}{\partial x} = - n\mu_{n-1} \qquad \text{with} \qquad \mu_0 = 1, \quad \mu_n(\ell, \ell) = 0.$$

We remark that as a consequence of the finiteness of this first moment we have that for $x < \ell$

$$P^x[\xi(s) \leqslant \ell, 0 \leqslant s \leqslant t] = o\left(\frac{1}{t}\right) \qquad \text{as} \qquad t \to \infty.$$

This is in contrast to the case of a Brownian motion $X(t)$ with $E[X(t)] = 0$ and $\text{var}[X(t)] = \sigma^2 t$, where for $x < \ell$

$$P^x[X(s) \leqslant \ell, 0 \leqslant s \leqslant t] \sim \frac{\ell - x}{\sigma} \sqrt{\frac{2}{\pi t}} \qquad \text{as} \qquad t \to \infty$$

(see [5, p. 171]). The comparison is interesting in view of the fact that

$$\mathrm{var}[\xi(t)] = \frac{\sigma^2}{2\rho} \qquad \text{and} \qquad \mathrm{var}[X(t)] = \sigma^2 t,$$

and apparently reflects the propensity of Brownian motion to take extended excursions in both directions.

Figure 1 below indicates the behavior of μ_1 as a function of the threshold level ℓ for several parametric choices of the starting point x. It is interesting to note the marked departure of μ_1 from translational invariance, i.e., μ_1 is not a function only of the difference $\ell - x$. This behavior is of course the result of the central restoring force acting on the Brownian particle after which the stationary, Gauss–Markov process is modeled.

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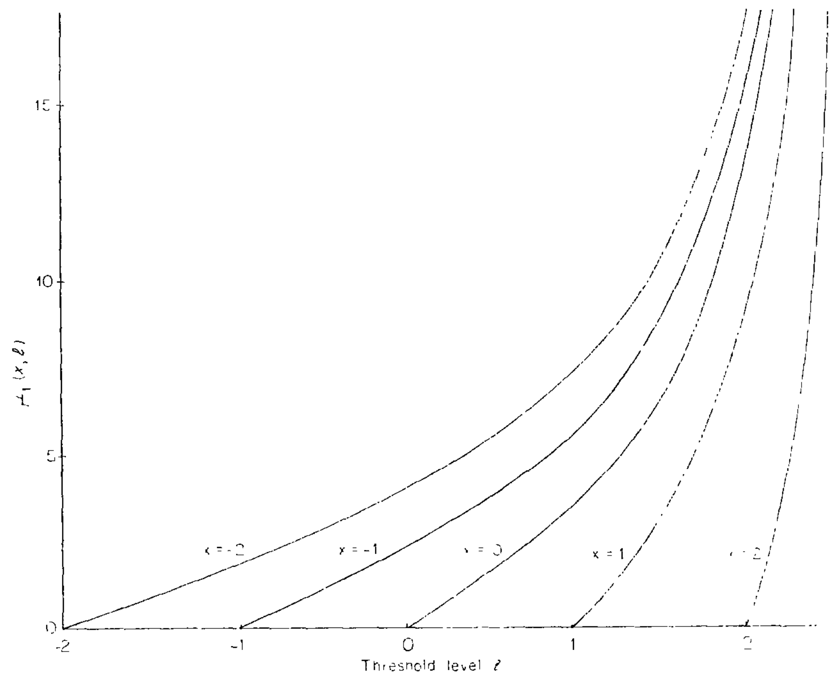

Fig. 1. Expected time to first passage across a fixed threshold for a stationary Gauss–Markov process. *Note*: x and ℓ expressed in units of $\sqrt{1/2\rho}\,\sigma$. μ_1 is expressed in units of $1/\rho$.

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