JOURNAL OF ALGEBRA 76, 138–152 (1982)

Representation Theory of Graded Artin Algebras

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Communicated by N. Jacobson

Received September 20, 1980

INTRODUCTION

Having initiated the study of graded Artin algebras Λ in [9], here we initiate the study of their representation theory. Our point of view is to study graded Λ-modules, which we believe to be somewhat more tractable than ungraded ones, in order to obtain information about all Λ-modules.

This point of view leads to the introduction, in Section 1, of the full subcategory $\operatorname{mod}_\infty(\Lambda)$ of $\operatorname{mod}\Lambda$ consisting of the gradable objects of $\operatorname{mod}\Lambda$; that is, the finitely generated Λ-modules which support a gradation. In these terms, one of the major results of the paper asserts that $\operatorname{mod}_\infty(\Lambda)$ has finite representation type if and only if $\operatorname{mod}\Lambda$ has finite representation type.

Thus, in Section 3 we introduce a number $G = G(\Lambda)$ designed to measure the size of $\operatorname{mod}_\infty(\Lambda)$. If $G = \infty$, we show that Λ has infinite representation type. If $G < \infty$, we show that there is a graded Artin algebra Ω of a certain specified form such that Λ has infinite representation type precisely when Ω has infinite representation type. The Artin algebra Ω has, in particular, desirable diagrammatic properties; and these we will exploit elsewhere.

We speculate that when G is finite, every finitely generated Λ-module is gradable. Now, in case Λ has finite representation type, it is obvious that G is finite. We show, indeed, that for every gradation of a given Artin algebra of finite representation type, every module is gradable.

This result, and the others cited, are chiefly consequences of a result proved in Section 4: If a component of the Auslander–Reiten graph of a graded Artin algebra contains a gradable module, then the component

* Both authors gratefully acknowledge the partial support of the National Science Foundation.

0021-8693/82/050138–15$02.00/0

consists entirely of gradable modules. This, in turn, is the culmination of an analysis, performed in Section 3, of almost split sequences one end term of which is gradable.

Only $\mathbb{Z}$-graded rings and modules will be considered in this paper. Also, indecomposable modules will be assumed to be nonzero and finitely generated. Finally, we assume some familiarity with the paper [9] as well as with the representation theory of Artin algebras.

1. Preliminaries

We recall from [9] that a graded Artin algebra is an Artin algebra Λ together with a gradation for which Λ is a graded ring. We denote the category of graded finitely generated Λ-modules by $\operatorname{gr}\Lambda$. Objects of $\operatorname{gr}\Lambda$ are written $X = \bigoplus X_n$, signifying that the homogeneous elements of X of degree n are the nonzero elements of X_n. A fact of basic importance established in [9] is that $X_n = 0$ for $|n| \gg 0$. We remind the reader that a morphism in $\operatorname{gr}\Lambda$ is a Λ-morphism $\alpha: \bigoplus X_n \to \bigoplus Y_n$ such that $\alpha(X_n) \subseteq Y_n$ for all n.

We reserve the letter F for the forgetful functor $\operatorname{gr}\Lambda \to \operatorname{mod}\Lambda$, where $\operatorname{mod}\Lambda$ is our notation for the category of finitely generated Λ-modules. By [9, Theorem 3.2], we know that an object X of $\operatorname{gr}\Lambda$ is indecomposable if and only if FX is an indecomposable Λ-module. Also, if X and Y are indecomposable objects of $\operatorname{gr}\Lambda$, we know, by [9, Theorem 4.1], that $FX \simeq FY$ if and only if $X \simeq \sigma(i)Y$ for some i, where $\sigma(i)$ is the ith shift functor. (That is, $\sigma(i)$ is the automorphism of $\operatorname{gr}\Lambda$ defined by $(\sigma(i)X)_n = X_{n-i}$.)

We use the notation $\operatorname{mod}_\infty(\Lambda)$ for $F(\operatorname{gr}\Lambda)$, the full subcategory of $\operatorname{mod}\Lambda$ with objects isomorphic to Λ-modules of the form FX, X in $\operatorname{gr}\Lambda$. An equivalent formulation of [9, Theorem 3.2] is that $\operatorname{mod}_\infty(\Lambda)$ is closed under direct summands. Objects of $\operatorname{mod}_\infty(\Lambda)$ are said to be gradable.

If $X = \bigoplus X_n$ is a nonzero object of $\operatorname{gr}\Lambda$, the lower bound, $\underline{\delta}X$, of X is defined by $X_{\underline{\delta}X} \neq 0$ and $X_n = 0$ for $n < \underline{\delta}X$. Similarly, the upper bound, $\bar{\delta}X$, of X is defined by $X_{\bar{\delta}X} \neq 0$ and $X_n = 0$ for $n > \bar{\delta}X$. We put $\underline{\delta}X = 0 = \bar{\delta}X$ if $X = 0$. If $-\infty \leqslant a \leqslant b \leqslant \infty$, the full subcategory of $\operatorname{gr}\Lambda$ whose nonzero objects X satisfy $\underline{\delta}X \geqslant a$ and $\bar{\delta}X \leqslant b$ is denoted by $\operatorname{gr}_a^b(\Lambda)$. We set $\operatorname{gr}_0^b(\Lambda) = \operatorname{gr}^b(\Lambda)$. Now, it can be shown that $\operatorname{gr}\Lambda$ is not equivalent to the category of finitely generated modules over any ring. However, by [9, Theorem 6.6], for a, b finite $\operatorname{gr}_a^b(\Lambda)$ is equivalent to the category of finitely generated modules over a graded Artin algebra.

In the same style as $\operatorname{mod}_\infty(\Lambda)$ was defined above, we define $\operatorname{mod}_d(\Lambda)$ to be $F(\operatorname{gr}^d(\Lambda))$. In fact, $\operatorname{mod}_\infty(\Lambda) = \bigcup_{0 \leqslant d < \infty} \operatorname{mod}_d(\Lambda)$. We note that $\operatorname{mod}_d(\Lambda)$ is closed under direct summands and, by [9, Lemma 6.5], $\operatorname{mod}_d(\Lambda) = F(\operatorname{gr}_a^b(\Lambda))$ whenever $b - a = d$.

Proposition 1.1. *If $d < \infty$, then the category $\operatorname{mod}_d(\Lambda)$ has finite* *representation type precisely when the category* $\mathrm{gr}^d(\Lambda)$ *has finite representation type. In particular, if* Λ *has finite representation type, then* $\mathrm{gr}^d(\Lambda)$ *has finite representation type for all finite* d.

Proof. If X is a nonzero object of $\mathrm{gr}^d(\Lambda)$ and d is finite, then $\sigma(i)X$ lies in $\mathrm{gr}^d(\Lambda)$ for only finitely many values of i. Thus, the result follows by [9, Theorem 3.2] allied with [9, Theorem 4.1]. ∎

If X and Y are in $\mathrm{gr}\,\Lambda$, nonzero elements of $F(\mathrm{Hom}_{\mathrm{gr}\,\Lambda}(X, \sigma(-i)Y))$ are called degree i Λ-morphisms. We denote the abelian group $F(\mathrm{Hom}_{\mathrm{gr}\,\Lambda}(X, \sigma(-i)Y))$ by $\mathrm{Hom}_\Lambda(X, Y)_i$ and the object $\mathrm{Hom}_\Lambda(FX, FY) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_\Lambda(X, Y)_i$ of $\mathrm{gr}\,\mathbb{Z}$ by $\mathrm{Hom}_\Lambda(X, Y)$.

The following lemma will be used frequently.

LEMMA 1.2. *Let* $0 \to X \to^\beta Y \to^\alpha Z \to 0$ *be an exact sequence in* $\mathrm{gr}\,\Lambda$. *Then this exact sequence splits if and only if the exact sequence* $0 \to FX \to^{F(\beta)} FY \to^{F(\alpha)} FZ \to 0$ *is split in* $\mathrm{mod}\,\Lambda$.

Proof. Suppose that $0 \to FX \to^{F(\beta)} FY \to^{F(\alpha)} FZ \to 0$ splits. Then there exists $g \in \mathrm{Hom}_\Lambda(FY, FX)$ such that $gF(\beta) = 1_{FX}$. But, as we just saw, $g = \sum g_i$ for certain $g_i \in \mathrm{Hom}_\Lambda(Y, X)_i$. Thus, arguing by degrees, the only possibility is that $g_0 F(\beta) = 1_{FX}$. We have $F(1_X) = 1_{FX} = F(g_0')\,F(\beta) = F(g_0'\beta)$, where g_0' is the morphism in $\mathrm{gr}\,\Lambda$ with $F(g_0') = g_0$. But then, $g_0'\beta = 1_X$; so $0 \to X \to^\beta Y \to^\alpha Z \to 0$ splits in $\mathrm{gr}\,\Lambda$.

Conversely, since F is an additive functor, if $0 \to X \to^\beta Y \to^\alpha Z \to 0$ is split, then $0 \to FX \to^{F(\beta)} FY \to^{F(\alpha)} FZ \to 0$ splits. ∎

Since F is a faithful exact functor, every object of $\mathrm{gr}\,\Lambda$ has finite length. Let S be a simple object of $\mathrm{gr}\,\Lambda$. Then, since $(\mathrm{rad}\,\Lambda)FS$ is a homogeneous submodule of FS [9, Proposition 3.5], and $\mathrm{rad}\,\Lambda$ is nilpotent, $(\mathrm{rad}\,\Lambda)FS = 0$. It follows that FS is a simple Λ-module. From this we obtain the formula

$$l_{\mathrm{gr}\,\Lambda}(X) = l_\Lambda(FX)$$

valid for any X in $\mathrm{gr}\,\Lambda$, where the left-hand side indicates length in $\mathrm{gr}\,\Lambda$ and the right-hand side denotes length in $\mathrm{mod}\,\Lambda$. Also, since by the result of [9] just cited $\mathrm{soc}\,FX$ is a homogeneous submodule of FX, $F\,\mathrm{soc}\,X = \mathrm{soc}\,FX$, by Lemma 1.2. Similarly, if we define $\mathrm{rad}\,X$ to be the smallest subobject Y of X such that X/Y is semisimple, then $F\,\mathrm{rad}\,X = \mathrm{rad}\,FX$.

The observations made in the preceding paragraph will be used throughout the paper with no further elaboration. Next, we have

PROPOSITION 1.3. *If* Λ *is a graded Artin algebra, then the category* $\mathrm{gr}\,\Lambda$ *has projective covers. Moreover, if* $P \to^\alpha X$ *is a projective cover in* $\mathrm{gr}\,\Lambda$, *then* $FP \to^{F(\alpha)} FX$ *is a projective cover in* $\mathrm{mod}\,\Lambda$.

Proof. Let X be an object of gr Λ. By the usual argument, X has, apart from isomorphism, at most one projective cover in gr Λ. Thus, it suffices to show that X has a projective cover $P \to^{\alpha} X$ in gr Λ such that $FP \to^{F(\alpha)} FX$ is a projective cover in mod Λ. Indeed, since F reflects epimorphisms, it is enough to find a projective object P and a morphism α: $P \to X$ such that $F(\alpha)$: $FP \to FX$ is a projective cover. For this, let $X/\mathrm{rad}\, X = S_1 \oplus \cdots \oplus S_t$, where the S_i are simple objects. By [9, Proposition 5.8(i)], there are degree 0 primitive idempotents e_i of Λ such that $FS_i \simeq \Lambda e_i/\mathrm{rad}\, \Lambda e_i$. Hence, as we mentioned earlier in the section, there are integers a_i for which $S_i \simeq \sigma(a_i)(\Lambda e_i/\mathrm{rad}\, \Lambda e_i)$. But, by Lemma 1.2, $P = \bigoplus_{i=1}^{t} \sigma(a_i)\Lambda e_i$ is a projective object of gr Λ. Thus, for this object P, the obvious surjection $P \to X/\mathrm{rad}\, X$ lifts to a morphism α: $P \to X$ such that $F(\alpha)$: $FP \to FX$ is a projective cover. ∎

In [9, Proposition 2.5] we showed that if D: mod $\Lambda \to$ mod Λ^{op} is the ordinary duality of Artin algebras, then D induces a duality D: gr $\Lambda \to$ gr Λ^{op} such that $FD = DF$. We remark that, consequently, $(\mathrm{gr}\, \Lambda)^{\mathrm{op}} \approx \mathrm{gr}\, \Lambda^{\mathrm{op}}$.

Dualizing Proposition 1.3, we get

PROPOSITION 1.4. *If Λ is a graded Artin algebra, then* gr Λ *has injective envelopes; and, if $Y \to^{\beta} E$ is an injective envelope in* gr Λ, *then $FY \to^{F(\beta)} FE$ is an injective envelope in* mod Λ.

Other results from [9], namely, Propositions 2.3 and 2.4 assert that if X is in gr Λ, then $\mathrm{End}_\Lambda X$ is a graded Artin algebra and the functor $\mathrm{Hom}_\Lambda(\ , FX)$: $(\mathrm{mod}\, \Lambda)^{\mathrm{op}} \to \mathrm{mod}(\mathrm{End}_\Lambda X)$ induces a functor $\mathrm{Hom}_\Lambda(\ , X)$: $(\mathrm{gr}\, \Lambda)^{\mathrm{op}} \to \mathrm{gr}(\mathrm{End}_\Lambda X)$ such that $F\, \mathrm{Hom}_\Lambda(\ , X) = \mathrm{Hom}_\Lambda(\ , FX)F$.

We apply this discussion to the definition of the transpose of an object X of gr Λ: let

$$P_1 \xrightarrow{\alpha} P_0 \to X \to 0$$

be a minimal projective presentation of X in gr Λ. Then, putting $\alpha^* = \mathrm{Hom}_\Lambda(\alpha, \Lambda)$, we have an exact sequence

$$\mathrm{Hom}_\Lambda(P_0, \Lambda) \xrightarrow{\alpha^*} \mathrm{Hom}_\Lambda(P_1, \Lambda) \to \mathrm{coker}\, \alpha^* \to 0$$

in gr Λ^{op} which, incidentally, can easily be seen to be a minimal projective presentation of coker α^*. We define the *transpose of* X to be coker α^*, and we write coker $\alpha^* = \mathrm{tr}\, X$. Of course, the isomorphism class of tr X depends only on X, and not on the particular minimal projective presentation of X chosen.

Now, from the exact sequence last displayed, we get the exact sequence

$$\mathrm{Hom}_\Lambda(FP_0, \Lambda) \xrightarrow{F(\alpha^*)} \mathrm{Hom}_\Lambda(FP_1, \Lambda) \to F\, \mathrm{tr}\, X \to 0$$

in $\operatorname{mod} \Lambda^{\mathrm{op}}$. But we know, by Proposition 1.3, that $FP_1 \to^{F(\alpha)} FP_0 \to FX \to 0$ is a minimal projective presentation of FX in $\operatorname{mod} \Lambda$. Thus, as $\operatorname{Hom}_\Lambda(F(\alpha), \Lambda) = F(\alpha^*)$, $\operatorname{tr} FX = \operatorname{coker} F(\alpha^*) = F \operatorname{tr} X$.

Since the duality and forgetful functors commute, we have proved

PROPOSITION 1.5. *If Λ is a graded Artin algebra and X is in* $\operatorname{gr} \Lambda$, *then* $F(D\operatorname{tr})(X) = (D\operatorname{tr})(FX)$. *Thus, the dual transpose of a finitely generated gradable Λ-module is gradable.*

Similarly, we have

PROPOSITION 1.6. *If Λ is a graded Artin algebra and X is in* $\operatorname{gr} \Lambda$, *then* $F(\operatorname{tr} D)(X) = (\operatorname{tr} D)(FX)$. *Thus, the transpose dual of a finitely generated gradable Λ-module is gradable.*

COROLLARY 1.7. *If $0 \to A \to B \to C \to 0$ is an almost split sequence over a graded Artin algebra, then A is gradable if and only if C is gradable.*

In Section 3 we will show that in the case when A or C is gradable, B is gradable too.

2. $\operatorname{Mod}_\infty(\Lambda)$

Several of our major results—see Section 4—concern the relationship between the categories $\operatorname{mod}_\infty(\Lambda)$ and $\operatorname{mod} \Lambda$. In this section, we give a preliminary analysis of certain aspects of the relationship.

Given a graded Artin algebra Λ, if the category $\operatorname{mod}_\infty(\Lambda)$ has finite representation type, then the ascending chain of categories

$$\operatorname{mod}_0(\Lambda) \subseteq \operatorname{mod}_1(\Lambda) \subseteq \cdots \subseteq \operatorname{mod}_d(\Lambda) \subseteq \cdots \tag{1}$$

must stabilize. Naturally, this is the same as saying that the chain eventually reaches $\operatorname{mod}_\infty(\Lambda)$. The reason for this phenomenon is that an object of $\operatorname{mod}_d(\Lambda)$ is a finite direct sum of indecomposable objects of $\operatorname{mod}_d(\Lambda)$, by [9, Theorem 3.3], and, conversely, any finite direct sum of objects of $\operatorname{mod}_d(\Lambda)$ is an object of $\operatorname{mod}_d(\Lambda)$. However, the stability of the chain (1) does not imply that $\operatorname{mod}_\infty(\Lambda)$ has finite representation type. Thus, for example, if Λ is the associated graded ring (see [9, Example 1.7]) of a radical squared zero Artin algebra of infinite representation type, then Λ has infinite representation type and $\operatorname{mod}_\infty(\Lambda) = \operatorname{mod}_1(\Lambda) = \operatorname{mod} \Lambda$.

To further analyze the stability of the chain of categories (1), we recall the notion of graded length set forth in [9, Sect. 4]. If X is an object of $\operatorname{gr} \Lambda$, then its graded length is defined by $\text{gr.l.}\, X = \bar{\delta} X - \underline{\delta} X + 1$. The graded length of a nonzero object A of $\operatorname{mod}_\infty(\Lambda)$ is defined to be the graded length of an indecomposable object X of $\operatorname{gr}\Lambda$ of largest graded length with the property that FX is a summand of A. The graded length of the zero module is defined to be 1. Now, by [9, Proposition 4.2], $\operatorname{mod}_d(\Lambda)$ is the full subcategory of $\operatorname{mod}_\infty(\Lambda)$ of objects of graded length at most $d+1$. In fact, if A is an object of $\operatorname{mod}_\infty(\Lambda)$, then gr.l. A is the least positive integer d such that A is an object of $\operatorname{mod}_{d-1}(\Lambda)$. In particular, we have shown that

PROPOSITION 2.1. *If Λ is a graded Artin algebra, then the ascending chain of categories*

$$\operatorname{mod}_0(\Lambda) \subseteq \operatorname{mod}_1(\Lambda) \subseteq \cdots \subseteq \operatorname{mod}_d(\Lambda) \subseteq \cdots$$

eventually reaches $\operatorname{mod}_\infty(\Lambda)$ *if and only if there is a bound on the graded lengths of indecomposable objects of* $\operatorname{mod}_\infty(\Lambda)$.

The stability of the chain does not ensure the existence of a bound on the lengths of indecomposable objects of $\operatorname{mod}_\infty(\Lambda)$, as evinced by the example presented above. However, insomuch as we will prove it next, the converse is valid.

When we write gr.l. Λ, we mean the graded length of Λ regarded as an object of $\operatorname{gr}\Lambda$. (This is generally larger than gr.l. $F\Lambda$.)

LEMMA 2.2. *If Λ is a graded Artin algebra and X is an indecomposable object of* $\operatorname{gr}\Lambda$ *of length at most K, then the graded length of X is at most* Kgr.l. Λ.

Proof. Let $L = \text{gr.l.}\Lambda$ and assume that $KL < \text{gr.l.}X$. Let $h_1, \dots, h_g$ be a minimal set of homogeneous generators of X. Then, since the choice of g implies that $g \leqslant l(X)$, it suffices to show that $K < g$. Setting $\deg h_i = d_i$, we may assume that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_g$. Plainly, $\underline{\delta}X \geqslant \underline{\delta}\Lambda + d_1$ and $\bar{\delta}X \leqslant \bar{\delta}\Lambda + d_g$; and we obtain the inequality

$$KL < d_g - d_1 + L.$$

If $g = 1$, this inequality implies that $K < g$.

If $g > 1$, suppose that $d_i - d_{i-1} > L - 1$ for some i with $1 < i \leqslant g$. Then $\bar{\delta}(\sum_{j=1}^{i-1} \Lambda h_j) \leqslant \bar{\delta}\Lambda + d_{i-1}$ and $\underline{\delta}(\sum_{j=i}^{g} \Lambda h_j) \geqslant \underline{\delta}\Lambda + d_i$. But then, $(\underline{\delta}\Lambda + d_i) - (\bar{\delta}\Lambda - d_{i-1}) > 1 - L + L - 1 = 0$. It follows that $X = \sum_{j=1}^{i-1} \Lambda h_j \oplus \sum_{j=i}^{g} \Lambda h_j$, against the assumption that X is indecomposable. Thus, $d_i - d_{i-1} \leqslant L - 1$ for $1 < i \leqslant g$, and we get that $d_g - d_1 \leqslant (g-1)(L-1)$. Combining this with the displayed inequality above, we have $KL < (g-1)(L-1) + L \leqslant g(L-1) + g = gL$. So $K < g$. ∎

COROLLARY 2.3. *If a graded Artin algebra has gradable indecomposable modules of arbitrarily large graded length, then it has gradable indecomposable modules of arbitrarily large length.*

COROLLARY 2.4. *Let Λ be a graded Artin algebra. Then the category* $\mathrm{mod}_{\infty}(\Lambda)$ *has finite representation type if and only if there is a bound on the lengths of its indecomposable objects.*

Proof. If there is a bound on the lengths of indecomposable objects of $\mathrm{mod}_{\infty}(\Lambda)$, then, by the preceding result coupled with Proposition 2.1, $\mathrm{mod}_{\infty}(\Lambda) = \mathrm{mod}_d(\Lambda)$ for some finite d. But then, as we saw in Section 1, there is a bound on the lengths of indecomposable objects of $\mathrm{gr}^d(\Lambda)$; and, by [9, Theorem 6.6], $\mathrm{gr}^d(\Lambda)$ is equivalent to the category of modules of finite length over some Artin algebra. Thus, by Auslander's extension of Roiter's Theorem to Artin algebras [1], $\mathrm{gr}^d(\Lambda)$ has finite representation type. Therefore, $\mathrm{mod}_{\infty}(\Lambda)$ has finite representation type, by Proposition 1.1. ∎

Thus, it is natural to ask whether $\mathrm{mod}_{\infty}(\Lambda)$ having finite representation type implies that $\mathrm{mod}\, \Lambda$ has finite representation type; that is, whether the existence of a bound on the lengths of gradable indecomposable Λ-modules implies the existence of a bound on the lengths of all indecomposable Λ-modules. This is, in fact, true. However, the proof will not be completed until Section 4. In the next section we exploit our vehicle of proof, namely, the almost split sequences of M. Auslander and I. Reiten.

3. ALMOST SPLIT SEQUENCES

The main result of this section is that if either end term of an almost split sequence over a graded Artin algebra is gradable, then the middle term is gradable. Other consequences concerning almost split sequences are obtained from our proof of this result; for example, that the category of graded finitely generated modules over a graded Artin algebra has almost split sequences. We refer the reader unfamiliar with almost split sequences and irreducible morphisms to [4, 5, 6, 11] for background, including the definitions of these terms.

We start with some preliminary results.

LEMMA 3.1. *Suppose that Λ is a graded Artin algebra and that X and Y are objects of* $\mathrm{gr}\, \Lambda$ *such that* $\mathrm{Hom}_{\Lambda}(X, Y)_i \neq 0$. *Then*

(i) $\underline{\delta}\sigma(i)X > \underline{\delta}Y - \mathrm{gr.l.}\, X$ *and* $\bar{\delta}\sigma(i)X < \bar{\delta}Y + \mathrm{gr.l.}\, X$;

(ii) $\underline{\delta}\sigma(-i)Y > \underline{\delta}X - \mathrm{gr.l.}\, Y$ *and* $\bar{\delta}\sigma(-i)Y < \bar{\delta}X + \mathrm{gr.l.}\, Y$.

Proof. (i) By assumption, there is a j with $\underline{\delta}X \leqslant j \leqslant \bar{\delta}X$ such that $\underline{\delta}Y \leqslant j+i \leqslant \bar{\delta}Y$. Thus, $\underline{\delta}\sigma(i)X = \underline{\delta}X + i \geqslant \underline{\delta}X + \underline{\delta}Y - j \geqslant \underline{\delta}X + \underline{\delta}Y - \bar{\delta}X > \underline{\delta}Y - \text{gr.l.}\, X$. Also, $\bar{\delta}\sigma(i)X = \bar{\delta}X + i \leqslant \bar{\delta}X + \bar{\delta}Y - j \leqslant \bar{\delta}X + \bar{\delta}Y - \underline{\delta}X < \bar{\delta}Y + \text{gr.l.}\, X$.

(ii) Similar to the proof of (i). ∎

COROLLARY 3.2. *If Λ is a graded Artin algebra, $0 \to X \to Y \to Z \to 0$ is an almost split sequence in $\mathrm{gr}_r^s(\Lambda)$, and Y' is an indecomposable summand of Y, then $\underline{\delta}Z > \underline{\delta}X - \text{gr.l.}\, Y' - \text{gr.l.}\, Z + 1$ and $\bar{\delta}Z < \bar{\delta}X + \text{gr.l.}\, Y' + \text{gr.l.}\, Z - 1$.*

Proof. By Theorems 2.4 and 2.14 of [5], there exist irreducible morphisms $X \to Y'$ and $Y' \to Z$ in $\mathrm{gr}_r^s(\Lambda)$. In particular, by the preceding result, $\underline{\delta}Z > \underline{\delta}Y' - \text{gr.l.}\, Z$ and $\underline{\delta}Y' \geqslant \underline{\delta}X - \text{gr.l.}\, Y' + 1$. Hence $\underline{\delta}Z > \underline{\delta}X - \text{gr.l.}\, Y' - \text{gr.l.}\, Z + 1$.

The assertion concerning the upper bound of Z is proved similarly. ∎

In [9] it was shown—see the proof of Theorem 6.6—that $\mathrm{gr}^d(\Lambda)$, for $d < \infty$, has a projective generator of the form $P_0 \oplus \cdots \oplus P_d$, where $P_i = \sigma(i)(\Lambda/J_i)$ and the J_i are certain specified homogeneous left ideals of Λ. Thus, apart from isomorphism, the only indecomposable projective objects of $\mathrm{gr}^d(\Lambda)$ are summands of the P_i. Therefore, using [9, Lemma 6.5], we obtain

LEMMA 3.3. *If I is an indecomposable object of $\mathrm{gr}_a^b(\Lambda)$ where a and b are finite, Λ is a graded Artin algebra, and I is either projective or injective, then $l(I) \leqslant l(\Lambda)$.*

We can now prove

THEOREM 3.4. *Let $0 \to A \to^g B \to^f C \to 0$ be an almost split sequence over a graded Artin algebra Λ. If either A or C is a gradable Λ-module, then so too is B.*

Proof. According to Corollary 1.7, A is gradable if and only if C is gradable. Thus, we may assume there are objects X and C' of $\mathrm{gr}\,\Lambda$ with $FX = A$ and $FC' = C$. Let

$$M = l(\Lambda)(l(\Lambda)^2 + 1)\,\text{gr.l.}\,\Lambda \qquad \text{and} \qquad \mathbb{N} = \mathrm{gr}_{\underline{\delta}X - 3M + 3}^{\bar{\delta}X + 3M - 3}(\Lambda).$$

We view $\mathbb{N}$, using [9, Theorem 6.6], as the category of modules of finite length over an Artin algebra; and we note that X is in $\mathbb{N}$.

We claim that X is not an injective object of $\mathbb{N}$. To see this, recall that by Proposition 1.4, there is an injective envelope $\mu: X \to E$ in $\mathrm{gr}\,\Lambda$. Choose indecomposable summands E' and E'' of E such that $\underline{\delta}E = \underline{\delta}E'$ and $\bar{\delta}E = \bar{\delta}E''$. Clearly, FE' and FE'' are indecomposable injective Λ-modules, so that E' and E'' both have length at most $l(\Lambda)$. In particular, each has graded length less than M, by Lemma 2.2. But, since $\operatorname{Hom}_{\operatorname{gr}\Lambda}(X, E') \neq 0$ and $\operatorname{Hom}_{\operatorname{gr}\Lambda}(X, E'') \neq 0$, $\underline{\delta}E > \underline{\delta}X - \text{gr.l.}\,E' \geqslant \underline{\delta}X - M + 1$ and $\bar{\delta}E < \bar{\delta}X + \text{gr.l.}\,E'' \leqslant \bar{\delta}X + M - 1$, by Lemma 3.1. But then, E is in $\mathbb{N}$. Now, $F(\mu)\colon A \to FE$ is an injective envelope in $\operatorname{mod}\Lambda$, by Proposition 1.4, and, by assumption, A is not an injective Λ-module. This substantiates our claim.

Since X is not injective in $\mathbb{N}$, there is an almost split sequence $0 \to X \to^{\beta} Y \to^{\alpha} Z \to 0$ in $\mathbb{N}$. Moreover, by Lemma 1.2, the exact sequence $0 \to FX \to^{F(\beta)} FY \to^{F(\alpha)} FZ \to 0$ is not split in $\operatorname{mod}\Lambda$; and so there is an exact commutative diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, equal] & B \arrow[r, "f"] \arrow[d, "k"] & FC' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & FX \arrow[r, "F(\beta)"'] & FY \arrow[r, "F(\alpha)"'] & FZ \arrow[r] & 0 \end{tikzcd}$$

in $\operatorname{mod}\Lambda$. We know that $h = \sum h_i$, where $h_i \in \operatorname{Hom}_\Lambda(C', Z)_i$. Also, by [9, Lemma 2.1], there are unique morphisms $h_i'\colon \sigma(i)C' \to Z$ in $\operatorname{gr}\Lambda$ with $F(h_i') = h_i$. We will show that each h_i' is a morphism in $\mathbb{N}$.

The fact that $Z \simeq \operatorname{tr} DX$ in $\mathbb{N}$ implies, by Lemma 3.3, that Z, and each indecomposable summand of Y, has length at most $l(A)(l(\Lambda)^2 + 1)$. Thus, by Lemmas 2.2 and 3.2, we have the inequalities $\underline{\delta}Z \geqslant \underline{\delta}X - 2M + 2$ and $\bar{\delta}Z \leqslant \bar{\delta}X + 2M - 2$. Similarly, since $FC' \simeq \operatorname{tr} DA$ in $\operatorname{mod}\Lambda$, we have inequalities $\underline{\delta}\sigma(i)C' \geqslant \underline{\delta}Z - M + 1$ and $\bar{\delta}\sigma(i)C' \leqslant \bar{\delta}Z + M - 1$, using Lemma 3.1. Thus, $\underline{\delta}\sigma(i)C' \geqslant \underline{\delta}X - 3M + 3$ and $\bar{\delta}\sigma(i)C' \leqslant \bar{\delta}X + 3M - 3$. Consequently, h_i' is in $\mathbb{N}$.

Suppose that no h_i' is a split epimorphism. Then for each i, there is a commutative diagram

$$\begin{tikzcd} & \sigma(i)C' \arrow[dl, "s_i"'] \arrow[d, "h_i'"] \\ Y \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

in $\mathbb{N}$; and hence a commutative diagram

$$\begin{tikzcd} & FC' \arrow[dl, "\sum F(s_i)"'] \arrow[d, "h"] \\ FY \arrow[r, "F(\alpha)"'] & FZ \end{tikzcd}$$

in $\operatorname{mod}\Lambda$. But then, although $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is assumed to be an almost split sequence, it follows that g is a split monomorphism.

It must be that, say, h_n', is a split epimorphism. Thus, h_n', and hence h_n, is an isomorphism. But $h_n^{-1}h = \sum_i h_n^{-1}h_i$, h_i has nontrivial kernel for $i \neq n$, and FC' is indecomposable. It follows that $h_n^{-1}h$ is an isomorphism. Thus, h is an isomorphism. But then, k is an isomorphism and B is gradable. ∎

THEOREM 3.5. *If Λ is a graded Artin algebra then the category* gr Λ *has almost split sequences. Moreover, if $0 \to X \to^{\beta} Y \to^{\alpha} Z \to 0$ is an almost split sequence in* gr Λ, *then $0 \to FX \to^{F(\beta)} FY \to^{F(\alpha)} FZ \to 0$ is an almost split sequence in* mod Λ.

Proof. Let X be a noninjective indecomposable object of gr Λ. By Lemma 1.2, FX is not an injective Λ-module. Thus, by [9, Theorem 3.2] and the proof of Theorem 3.4, there exists an exact sequence $0 \to X \to^{\beta'} Y' \to^{\alpha'} Z' \to 0$ in gr Λ such that, in mod Λ, the exact sequence $0 \to FX \to^{F(\beta')} FY' \to^{F(\alpha')} FZ' \to 0$ is almost split. By the essential uniqueness of almost split sequences, it clearly suffices to show that $0 \to X \to^{\beta'} Y' \to^{\alpha'} Z' \to 0$ is an almost split sequence in gr Λ. For this, let $v \in \mathrm{Hom}_{\mathrm{gr}\,\Lambda}(W, Z')$ such that v is not a split epimorphism. But then, it follows by Lemma 1.2 that $F(v) \in \mathrm{Hom}_\Lambda(FW, FZ')$ is not a split epimorphism either. Thus, $F(\alpha')g = F(v)$ for some $g \in \mathrm{Hom}_\Lambda(FW, FY')$; and then, as in the proof of Lemma 1.2, $\alpha' g' = v$ for some $g' \varepsilon \mathrm{Hom}_{\mathrm{gr}\,\Lambda}(W, Y')$. Therefore, since the exact sequence $0 \to X \to^{\beta'} Y' \to^{\alpha'} Z' \to 0$ is not split, it is almost split. ∎

Using [9, Lemma 6.5], an immediate consequence of the proof of Theorem 3.4 is that if X is an indecomposable noninjective object of gr Λ, then there are integers i and p, with $p \geqslant 0$, such that $0 \to FX \to^{F(\beta)} FY \to^{F(\alpha)} FZ \to 0$ is an almost split sequence in mod Λ whenever $0 \to \sigma(i)X \to^{\beta} Y \to^{\alpha} Z \to 0$ is an almost split sequence in $\mathrm{gr}^p(\Lambda)$. Naturally, there is a similar consequence involving indecomposable nonprojective objects Z of gr Λ. However, we should point out that p, obtained by the proof of Theorem 3.4, is not minimal. In particular, the proof of the theorem uses the bound obtained in Lema 2.2; and this bound is not sharp.

We wish to prove some better results. But first, we make the elementary observation that if X and Y are objects of gr Λ for which there is either a monomorphism $X \to Y$ or an epimorphism $Y \to X$, then $\underline{\delta} Y \leqslant \underline{\delta} X$ and $\bar{\delta} X \leqslant \bar{\delta} Y$.

LEMMA 3.6. *Let $0 \to U \to^{g} V \to^{f} W \to 0$ be an almost split sequence in* gr Λ *and let $d + 1 \geqslant \max\{\mathrm{gr.l.}\, FU, \mathrm{gr.l.}\, FV, \mathrm{gr.l.}\, FW\}$. Then, for some integer i, $\sigma(i)U$, $\sigma(i)V$, and $\sigma(i)W$ are in $\mathrm{gr}^d(\Lambda)$. In particular, $0 \to \sigma(i)U \to^{\sigma(i)g} \sigma(i)V \to^{\sigma(i)f} \sigma(i)W \to 0$ is an almost split sequence in $\mathrm{gr}^d(\Lambda)$.*

Proof. Write $V = V_1 \oplus \cdots \oplus V_n$, where the V_m are indecomposable objects of $\operatorname{gr} \Lambda$. Then

$$g = (g_1, \dots, g_n) \qquad \text{and} \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

where $g_m \colon U \to V_m$ and $f_m \colon V_m \to W$ are irreducible morphisms in $\operatorname{gr}(\Lambda)$. By [5, Proposition 2.6], each g_m and each f_m is either a monomorphism or an epimorphism. Since g is a monomorphism, we have $\underline{\delta} U \geqslant \underline{\delta} V$ and $\bar{\delta} U \leqslant \bar{\delta} V$. If $\underline{\delta} U > \underline{\delta} V$ and $\bar{\delta} U < \bar{\delta} V$, then it follows that $\underline{\delta} W = \underline{\delta} V$ and $\bar{\delta} W = \bar{\delta} V$. Choosing, in this case, $i = -\underline{\delta} V$ the result follows. If $\underline{\delta} U = \underline{\delta} V$ and $\bar{\delta} U = \bar{\delta} V$, then, choosing $i = -\underline{\delta} U$, the result again follows. Now suppose $\underline{\delta} U = \underline{\delta} V$ and $\bar{\delta} U < \bar{\delta} V$. Choose V_c such that $\bar{\delta} V_c = \bar{\delta} V$. Then, as noted above, g_c must be a monomorphism. Hence $\underline{\delta} U \geqslant \underline{\delta} V_c \geqslant \underline{\delta} V = \underline{\delta} U$. Thus $\underline{\delta} V_c = \underline{\delta} V$ and $\bar{\delta} V_c = \bar{\delta} V$. Choosing $i = -\underline{\delta} V$ the result follows. The final case, $\underline{\delta} U > \underline{\delta} V$ and $\bar{\delta} U = \bar{\delta} V$, is handled in a fashion similar to the case above. ∎

For the rest of the paper, we denote by G the supremum of the graded lengths of indecomposable gradable Λ-modules. We remark that

$$G = \inf_{0 \leqslant d \leqslant \infty} \{d \colon \operatorname{mod}_d(\Lambda) = \operatorname{mod}_\infty(\Lambda)\} + 1.$$

The next, and final, result of the section is now apparent.

THEOREM 3.7. *Let Λ be a graded Artin algebra.*

(i) *If U is a noninjective indecomposable object of* $\operatorname{gr} \Lambda$, *then there is an integer i such that $\sigma(i)U$ is a noninjective object of* $\operatorname{gr}^{G-1}(\Lambda)$ *and* $0 \to FU \to^{F(g)} FV \to^{F(f)} FW \to 0$ *is an almost split sequence in* $\operatorname{mod} \Lambda$ *whenever* $0 \to \sigma(i)U \to^{g} V \to^{f} W \to 0$ *is an almost split sequence in* $\operatorname{gr}^{G-1}(\Lambda)$.

(ii) *If Z is a nonprojective indecomposable object of* $\operatorname{gr} \Lambda$, *then there is an integer j such that $\sigma(j)Z$ is a nonprojective object of* $\operatorname{gr}^{G-1}(\Lambda)$ *and* $0 \to FX \to^{F(\beta)} FY \to^{F(\alpha)} FZ \to 0$ *is an almost split sequence in* $\operatorname{mod} \Lambda$ *whenever* $0 \to X \to^{\beta} Y \to^{\alpha} \sigma(j)Z \to 0$ *is an almost split sequence in* $\operatorname{gr}^{G-1}(\Lambda)$.

We mention that there are graded Artin algebras Λ, such that G is finite, having almost split sequences $0 \to L \to M \to N \to 0$ in $\operatorname{gr}^{G-1}(\Lambda)$ for which the corresponding nonsplit exact sequences $0 \to FL \to FM \to FN \to 0$ in $\operatorname{mod} \Lambda$ fail to be almost split. Also, we remark that the part of [6, Proposition 3.4] dealing with almost split sequences is the instance of Theorem 3.7 obtained when $(\operatorname{rad} \Lambda)^2 = 0$ and Λ is the trivial extension $\Lambda/\operatorname{rad} \Lambda \ltimes \operatorname{rad} \Lambda$ of $\operatorname{rad} \Lambda$ by $\Lambda/\operatorname{rad} \Lambda$ (see [9, Example 1.8]).

4. Main Results

In this section, as promised at the end of Section 2, we affirm the result mentioned there using the theorems concerning almost split sequences of the last section. But actually, we have a more general result; and this is most easily stated in terms of Auslander–Reiten graphs. The reader will recall that the Auslander–Reiten graph of an Artin algebra Γ is the locally finite directed graph with vertices isomorphism classes $[X]$ of indecomposable Γ-modules X and edges between vertices $[X]$ and $[Y]$ corresponding to irreducible morphisms $X \to Y$ or $Y \to X$. Since, at the moment, we are not concerned with graph theory, we define a *component* of the Auslander–Reiten graph of Γ to consist of all indecomposable Γ-modules X with vertices $[X]$ lying in the same connected component of the graph.

For use later in the section, we state the following result concerning Auslander–Reiten graphs—see [3, 12].

Theorem 4.1 (M. Auslander). *If there is a bound on the lengths of modules belonging to a component of the Auslander–Reiten graph of an indecomposable Artin algebra, then every indecomposable module belongs to that component. Thus, if some component of the Auslander–Reiten graph of an indecomposable Artin algebra consists of modules of bounded length, then the algebra has finite representation type.*

Next, we expound the general result referred to at the start of the section.

Theorem 4.2. *If a component of the Auslander–Reiten graph of a graded Artin algebra contains a single gradable module, then every module belonging to that component is gradable. In particular, this is true of a component containing an indecomosable projective module, an indecomposable injective module, or a simple module.*

Proof. Given the first statement, the second is a consequence of Corollary 3.4 and Proposition 3.5 of [9]. Thus, it is enough to prove the first statement.

Let Λ be the graded Artin algebra. We must show that if A and B are indecomposable Λ-modules for which there exists an irreducible morphism $A \to B$, then A is gradable if and only if B is gradable. If A is not injective and B is not projective, this follows from Theorem 3.4, using [5, Propositions 3.2 and 3.3]. But, if A is injective, then B is isomorphic to a summand of $A/\operatorname{soc} A$; and, if B is projective, then A is isomorphic to a summand of $\operatorname{rad} B$, by the results of [5] just cited. In either case, since we know that injective modules and projective modules are gradable, A is gradable precisely when B is gradable, by [9, Proposition 3.5 and Theorem 3.3]. ∎

Since there are graded Artin algebras having finitely generated modules that are not gradable, there are, in general, components of the Auslander–Reiten graph that contain no gradable module. However, this can only be true of the Auslander–Reiten graph of a graded Artin algebra of infinite representation type. Indeed, as an immediate consequence of the foregoing results and [2, Theorem A], we have

THEOREM 4.3. *Every module over a graded Artin algebra of finite representation type is gradable.*

Another immediate consequence of Theorems 4.1 and 4.2 is

THEOREM 4.4. *For a graded Artin algebra, the existence of a bound on the lengths of gradable indecomposable modules implies the existence of a bound on the lengths of all indecomposable modules.*

From this result, allied with the results of Section 2, we get

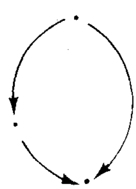
THEOREM 4.5. *The following are equivalent properties of a graded Artin algebra Λ.*

(i) *Λ has finite representation type.*

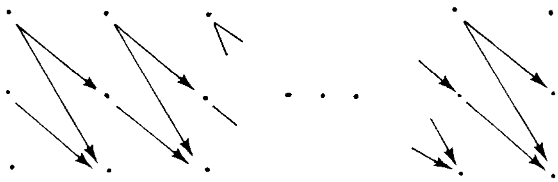
(ii) $\mathrm{mod}_{\infty}(\Lambda)$ *has finite representation type.*

(iii) $\mathrm{mod}_{d}(\Lambda)$ *has finite representation type for every nonnegative integer d, and G is finite.*

The assumption, in (iii), that G is finite is essential for the validity of the theorem. For, in the language of [8, p. 122], let Λ be the tensor k-algebra (k a field) of the quiver



Then—see [9, Example 1.9]—Λ is a positively graded Artin algebra of infinite representation type, by Gabriel's classification of quivers of finite type [7]. Also, by [10, Theorem, p. 151], for each nonnegative integer d, the tensor k-algebra, say, Ω^d of the quiver



(2)

$(3(d+1)$ points)

has the property that $\operatorname{mod} \Omega^d \approx \operatorname{gr}^d(\Lambda)$. By [7], each of the algebras Ω^d has finite representation type. Thus, by Proposition 1.1, the categories $\operatorname{mod}_d(\Lambda)$ each have finite representation type.

We remark, referring to the quiver (2), that if we put k at each nonisolated point but the two in the upper right-hand corner and put 0 at the rest of the points, then, assigning the identity map to each arrow (except for the one from 0 to k), we have, in effect, constructed an indecomposable gradable Λ-module of graded length $d+1$, by [9, Theorem 3.2]. It is of interest to note that, as pointed out to us by I. Reiten, the infinite strictly ascending chain of gradable indecomposable modules so obtained lies in a single component of the Auslander–Reiten graph of Λ. The existence of these modules is related to the fact that the indecomposable Λ-module gotten from the quiver of Λ by putting k at each point and taking arrows to be identity maps is not gradable.

In the vein of this example, we comment that when Λ is an arbitrary graded Artin algebra, we have described, in [9, Sect. 6], graded Artin algebras Ω^d with $\operatorname{gr}^d(\Lambda) \approx \operatorname{mod} \Omega^d$ canonically associated to Λ. The quivers of the Ω^d (in the sense of [8, p. 120]) depend, of course, on the given gradation of Λ, and yet can be shown to be related to the quiver of Λ in a manner not unlike the example. By [9, Corollary 6.4], the graded algebras Ω^d have, furthermore, the property that all their modules are gradable.

We infer that Theorem 4.5 admits the following ring theoretical amplification.

THEOREM 4.6. *If Λ is a graded Artin algebra then*

(i) *if $G = \infty$, then Λ has infinite representation type*;

(ii) *if $G < \infty$, then there is a morphism of graded rings $\Lambda \to \Omega$ for some graded Artin algebra Ω with the properties*:

(a) $\operatorname{mod}_{G-1}(\Omega) = \operatorname{mod} \Omega$,

(b) $\operatorname{mod} \Omega \approx \operatorname{gr}^{G-1}(\Lambda)$,

(c) *Ω has infinite representation type precisely when Λ has infinite representation type.*

To put Theorem 4.6 in perspective, consider the special case when Λ is the associated graded ring of an Artin algebra Γ of Loewey length 2. Then, plainly, $G=2$ and Theorem 4.6 reasserts the well-known result that $\Gamma/\mathrm{rad}\,\Gamma \ltimes \mathrm{rad}\,\Gamma$ has finite representation type if and only if $\begin{bmatrix} \Gamma/\mathrm{rad}\,\Gamma & 0 \\ \mathrm{rad}\,\Gamma & \Gamma/\mathrm{rad}\,\Gamma \end{bmatrix}$ has finite representation type. We are, incidentally, aware of no other previously known instance of Theorem 4.6.

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