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Modified variational iteration method for Boussinesq equation

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Abstract

This paper applies the modified variational iteration method to solve a class of nonlinear partial differential equations. Boussinesq equation is used as a case-study to illustrate the simplicity and effectiveness of the method. Comparison between variational iteration method and Adomian decomposition method is made.

Keywords: Boussinesq equation; Variational iteration method (VIM); Nonlinear partial differential equations; Modified VIM (MVIM); Adomian method

1. Introduction

The variational iteration method (VIM) was proposed by Ji-Huan He in 1999 [1,2] and has been proved by many authors [3,4] to be a powerful mathematical tool for solving various types of nonlinear problems, which represent a plenty of modern science branches [1–7].

In some applications when series solution is searched for, variational iteration method has some drawbacks which reduce the efficiency of the method due to repeated calculation and calculation of unneeded terms. Such phenomena will not happen if an initial guess is chosen with some unknown parameters. To overcome the shortcoming arising in series solution, the present authors proposed a modified variational iteration method (MVIM) and used it to give approximate solutions for some well-known non-linear problems [8]. The authors also proposed treatments on MVIM results by using Padé approximants and Laplace transform [9,10]. The treatment improves convergence [7–10].

This paper is an extension of the work done in [8]. A new application of MVIM for a nonlinear problem is described and is used to give approximate solutions for well-known nonlinear problem that takes the form

$$\begin{aligned} &Lu(x,t) + Ru(x,t) + Nu(x,t) = 0,\\ &u(x,0) = f_0(x), \\ &u_t(x,0) = f_1(x), \end{aligned} \tag{1}$$

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where $L = \frac{\partial^2}{\partial t^2}$, R is a linear operator and $Nu(x,t)$ is a nonlinear term. $Ru(x,t)$ and $Nu(x,t)$ are free of partial derivative with respect to t.

A comparison was made among variational iteration method, modified variational iteration method and Adomian decomposition method.

2. Variational iteration method

Using variational iteration method [1,2] to solve the nonlinear partial differential equation (1), the following variational iteration formula can be obtained:

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda\{LU_n + \overset{\sim}{\overbrace{RU_n}} + \overset{\sim}{\overbrace{NU_n}}\}\mathrm{d}\tau, \tag{2}$$

where λ is called a general Lagrange multiplier [11] which can be identified optimally via variational theory, $\overset{\sim}{\overbrace{RU_n}}$ and $\overset{\sim}{\overbrace{NU_n}}$ are considered as restricted variations, i.e. $\delta\,\overset{\sim}{\overbrace{RU_n}} = 0$, $\delta\,\overset{\sim}{\overbrace{NU_n}} = 0$.

Calculating variation with respect to U_n,

$$\delta U_{n+1}(x,t) = \delta U_n(x,t) + \delta \int_0^t \lambda(\tau)\left\{\frac{\partial^2 U_n}{\partial \tau^2} + \overset{\sim}{\overbrace{RU_n}} + \overset{\sim}{\overbrace{NU_n}}\right\}\mathrm{d}\tau, \tag{3}$$

$$\delta U_{n+1}(x,t) = \delta U_n(x,t) + \delta \int_0^t \lambda(\tau)\frac{\partial^2 U_n(x,\tau)}{\partial \tau^2}\mathrm{d}\tau, \tag{4}$$

$$\delta U_{n+1}(x,t) = \delta U_n(x,\tau)(1-\lambda'(\tau))|_{\tau=t} + \delta\frac{\partial U_n(x,\tau)}{\partial \tau}(\lambda(\tau))\bigg|_{\tau=t} + \int_0^t \delta U_n(x,\tau)\frac{\partial^2\lambda(\tau)}{\partial\tau^2}\mathrm{d}\tau. \tag{5}$$

Consequently, the following stationary conditions are obtained:

$$\begin{aligned} &\delta u_n : \lambda''(\tau) = 0,\\ &\delta u_n' : \lambda(\tau)|_{\tau=t} = 0,\\ &\delta u_n : 1-\lambda'(\tau)|_{\tau=t} = 0. \end{aligned} \tag{6}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda(\tau) = \tau - t. \tag{7}$$

Substituting the identified multiplier into Eq. (2) results in the following iteration formula:

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t (\tau - t)\left\{\frac{\partial^2 U_n(x,\tau)}{\partial\tau^2} + R(U_n) + NU_n\right\}\mathrm{d}\tau. \tag{8}$$

The second term on the right is called the correction term. Eq. (8) can be solved iteratively using $U_0(x,t) = f_0(x) + f_1(x)t$ as an initial approximation.

2.1. Remarks on variational iteration method results

Analysing the results obtained from iteration formula (8), it can be verified that U_n takes the following form:

$$\begin{aligned} &U_0 = B_0^0 + B_0^1 t,\\ &U_1 = B_1^0 + B_1^1 t + B_1^2 t^2 + B_2^3 t^2 + \cdots\\ &\vdots\\ &U_n = B_n^0 + B_n^1 t + B_n^2 t^2 + \cdots + B_n^{2n+1}t^{2n+1} + B_n^{2n+2}t^{2n+2} + B_n^{2n+3}t^{2n+3} + \cdots \end{aligned} \tag{9}$$

where B_n^m is the coefficient of t^m which is settled and takes the same value for each U_n as $(2n+1) \geq m$ and is not settled and doesn't take the same value for each U_n as $(2n+1) < m$.

2.1.1. Remark 1

Concerning our case of $L = \frac{\partial^2}{\partial t^2}$, there are repeated calculation in each step ($n > 0$). To cancel some of the repeated calculation, the iteration formula (8) can be handled as follows:

$$\begin{aligned} U_{n+1}(x,t) &= U_n(x,t) - \int_0^t t\,(U_n(x,\tau))_{\tau\tau}\,\mathrm{d}\tau + \int_0^t \tau\,(U_n(x,\tau))_{\tau\tau}\,\mathrm{d}\tau + \int_0^t (\tau - t)\,\{R(U_n) + NU_n\}\,\mathrm{d}\tau \\ &= U_n(x,t) - t\,(U_n(x,\tau))_\tau\,|_0^t + \tau t (U_n(x,\tau))_\tau\,|_0^t - U_n(x,\tau)\,|_0^t + \int_0^t (\tau - t)\,\{R(U_n) + NU_n\}\,\mathrm{d}\tau \\ &= U_n(x,t) - t\,(U_n(x,t))_t + t\,(U_n)_t\,(x,0) + t\,(U_n(x,t))_t - U_n(x,t) + U_n(x,0) \\ &\quad + \int_0^t (\tau - t)\,\{R(U_n) + NU_n\}\,\mathrm{d}\tau, \end{aligned}$$

but

$$U_n(x,0) + t\,(U_n)_t\,(x,0) = f_0(x) + f_1(x)t = U_0(x,t).$$

So

$$U_{n+1}(x,t) = U_0(x,t) + \int_0^t (\tau - t)\,\{R(U_n) + NU_n\}\,\mathrm{d}\tau. \tag{10}$$

Via the iteration formula (10) some repeated computations are cancelled.

2.1.2. Remark 2

If we rewrite Eq. (9) in the form

$$U_n(x,t) = U_n^{st}(x,t) + U_n^{ns}(x,t), \tag{11}$$

where U_n^{st} contains the settled terms in Eq. (9) and U_n^{ns} contains the non-settled terms in Eq. (9).

It is observed that the addition of the term $U_n^{ns}(x,t)$ deteriorates the convergence to the exact solution since the coefficients of t^sin $U_n^{ns}(x,t)$ are not the exact coefficients of t^s. So, $U_n^{ns}(x,t)$ doesn't lead to better convergence but deteriorates the convergence and should be cancelled.

To overcome this problem and eliminate $U_n^{ns}(x,t)$, the following modification on the recursive formula (10) is suggested:

$$U_{n+1}(x,t) = U_0(x,t) + \int_0^t (\tau - t)\,\{R(U_n) + G_n(x,\tau)\}\,\mathrm{d}\tau, \tag{12}$$

where $U_0 = f_0(x) + f_1(x)t$ and $G_n(x,t)$ is obtained from

$$NU_n(x,t) = G_n(x,t) + O(t^{2n+2}).$$

2.1.3. Remark 3

To cancel all repeated computation, let us rewrite Eq. (12) in the following iteration formula:

$$U_{n+1} = U_0 + \int_0^t \{R(U_{n-1}) + G_{n-1}\}\,\mathrm{d}\tau + \int_0^t \{R(U_n - U_{n-1}) + (G_n - G_{n-1})\}\,\mathrm{d}\tau. \tag{13}$$

But it is known from (12) that

$$U_n = U_0 + \int_0^t \{R(U_{n-1}) + G_{n-1}\}\,\mathrm{d}\tau. \tag{14}$$

Substituting by Eq. (14) in Eq. (13), we get:

$$U_{n+1} = U_n + \int_0^t \{R(U_n - U_{n-1}) + (G_n - G_{n-1})\}\,d\tau, \tag{15}$$

where $U_{-1} = 0$, $U_0 = f_0(x) + f_1(x)t$ and $G_n(x,t)$ is obtained from

$$NU_n(x,t) = G_n(x,t) + O(t^{2n+2}).$$

This final modified formula (15) cancels all the repeated calculations and the unsettled terms in VIM.

Accordingly, we are in the state to propose the modified variational iteration method. In the next section, the modified variational iteration method is summarized on the basis of our case.

3. The proposed modified variational iteration method

Concerning the nonlinear partial differential equation (1) and following the same procedure as done in the variational iteration method to calculate the Lagrange multiplier λ and using the following modified variational iteration formula instead of the variational iteration formula (8):

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t (\tau - t)\{R(U_n - U_{n-1}) + (G_n - G_{n-1})\}\,d\tau, \tag{16}$$

where $U_{-1} = 0$, $U_0 = f_0(x) + f_1(x)t$ and $G_n(x,t)$ is obtained from

$$NU_n(x,t) = G_n(x,t) + O(t^{2n+2}). \tag{17}$$

The modified variational iteration formula (16) is used to obtain an approximate series solution for Eq. (1), which takes the

$$u(x,t) \simeq U_n(x,t),$$

where n is the final iteration step.

4. Case-study: "Good" Boussinesq equation

Consider the "good" Boussinesq equation [12]

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0, \quad x \in R, \tag{18}$$

with the constrains

$$\begin{aligned} u(x,0) &= \frac{-3c^2}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right], \\ u_t(x,0) &= \frac{3c^3\sqrt{1-c^2}}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right] \end{aligned} \tag{19}$$

where c is a constant.

It is solved using variational iteration method, modified variational iteration method and Adomian decomposition method.

Solving the "good" Boussinesq equation using VIM:

To solve Eq. (18) by means of the variational iteration method, we set

$$\begin{aligned} &RU_n = -(U_n)_{xx} + (U_n)_{xxxx}, \\ &\text{and} \quad NU_n = -(U_n^2)_{xx}. \end{aligned} \tag{20}$$

Substituting by (20) in (8), we get the following variational iteration formula:

$$U_{n+1} = U_n + \int_0^t (\tau - t)\{(U_n)_{\tau\tau} - (U_n)_{xx} + (U_n)_{xxxx} - (U_n^2)_{xx}\}d\tau. \tag{21}$$

Using (21), the approximate solutions $U_n(x,t)$ are obtained iteratively by substituting:

$$\begin{aligned} U_0(x,t) &= u(x,0) + u_t(x,0)t \\ &= \frac{-3c^2}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right] + \frac{3c^3\sqrt{1-c^2}}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right]t. \end{aligned} \tag{22}$$

Some approximate solutions are listed below:

$$\begin{aligned} U_1 &= U_0 + \frac{3}{8}c^4(-1+c^2)\,(-2+\mathrm{Cosh}\,[cx])\,\mathrm{Sech}^4\left[\frac{cx}{2}\right]t^2 \\ &\quad + \frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\mathrm{Sech}^5\left[\frac{cx}{2}\right]\left(-11\mathrm{Sinh}\left[\frac{cx}{2}\right] + \mathrm{Sinh}\left[\frac{3cx}{2}\right]\right)\right)t^3 \\ &\quad - \frac{3}{32}c^6(-1+c^2)^2\,(10 - 10\mathrm{Cosh}[cx] + \mathrm{Cosh}[2cx])\,\mathrm{Sech}^8\left[\frac{cx}{2}\right]t^4, \end{aligned} \tag{23}$$

$$\begin{aligned} U_2 &= U_0 + \frac{3c^4(-1+c^2)}{8}(-2+\mathrm{Cosh}[cx])\mathrm{Sech}^4\left[\frac{cx}{2}\right]t^2 + \frac{c^5(1-c^2)^{\frac{3}{2}}}{16}\mathrm{Sech}^5\left[\frac{cx}{2}\right] \\ &\quad \times\left(-11\mathrm{Sinh}\left[\frac{cx}{2}\right] + \mathrm{Sinh}\left[\frac{3cx}{2}\right]\right)t^3 - \frac{c^6(-1+c^2)^2}{128}(33 - 26\mathrm{Cosh}[cx] \\ &\quad + \mathrm{Cosh}[2cx])\mathrm{Sech}^6\left[\frac{cx}{2}\right]t^4 + \frac{c^7(1-c^2)^{\frac{5}{2}}}{1280}\mathrm{Sech}^7\left[\frac{cx}{2}\right]\left(302\mathrm{Sinh}\left[\frac{cx}{2}\right]\right. \\ &\quad \left. - 57\mathrm{Sinh}\left[\frac{3cx}{2}\right] + \mathrm{Sinh}\left[\frac{5cx}{2}\right]\right)t^5 + \frac{c^{10}(-1+c^2)}{5120}(1569 + 52\,971c^2 - 2(71 \\ &\quad + 33\,763c^2)\mathrm{Cosh}[cx] + (-1382 + 16\,718c^2)\mathrm{Cosh}[2cx] + 6(-53 + 227c^2) \\ &\quad \times\mathrm{Cosh}[3cx] + (-11 + 23c^2)\mathrm{Cosh}[4cx])\mathrm{Sech}^{12}\left[\frac{cx}{2}\right]t^6 + \frac{c^{11}(1-c^2)^{\frac{3}{2}}}{3584}(445 \\ &\quad - 7717c^2 + 2(62 + 3913c^2)\mathrm{Cosh}[cx] - 4(70 + 269c^2)\mathrm{Cosh}[2cx] + (40 + 14c^2) \\ &\quad \times\mathrm{Cosh}[3cx] + (-1 + c^2)\mathrm{Cosh}[4cx])\mathrm{Sinh}^{12}\left[\frac{cx}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right]t^7 + \cdots, \end{aligned} \tag{24}$$

$$\vdots$$

and so on.

The results obtained in (24) can be written as

$$\begin{aligned} U_0 &= B_0^0 + B_0^1 t, \\ U_1 &= B_1^0 + B_1^1 t + B_1^2 t^2 + B_1^3 t^3 + B_1^4 t^4, \\ U_2 &= B_2^0 + B_2^1 t + B_2^2 t^2 + B_2^3 t^3 + B_2^4 t^4 + B_2^5 t^5 + B_2^6 t^6 + B_2^7 t^7 + \cdots, \\ &\vdots \end{aligned} \tag{25}$$

where

$$\begin{aligned} B_0^0 &= B_1^0 = B_2^0 = B_3^0 = \cdots = B_n^0 = \frac{-3c^2}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right], \\ B_0^1 &= B_1^1 = B_2^1 = B_3^1 = \cdots = B_n^1 = \frac{3c^3\sqrt{1-c^2}}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right], \\ B_1^2 &= B_2^2 = B_3^2 = \cdots = B_n^2 = \frac{3}{8}c^4(-1+c^2)\,(-2+\mathrm{Cosh}[cx])\,\mathrm{Sech}^4\left[\frac{cx}{2}\right], \\ B_1^3 &= B_2^3 = B_3^3 = \cdots = B_n^3 = \frac{c^5(1-c^2)^{\frac{3}{2}}}{16}\mathrm{Sech}^5\left[\frac{cx}{2}\right]\left(-11\mathrm{Sinh}\left[\frac{cx}{2}\right] + \mathrm{Sinh}\left[\frac{3cx}{2}\right]\right), \\ B_1^4 &= \frac{-3}{32}c^6(-1+c^2)^2\,(10 - 10\mathrm{Cosh}\,[cx] + \mathrm{Cosh}[2cx])\,\mathrm{Sech}^8\left[\frac{cx}{2}\right], \end{aligned}$$

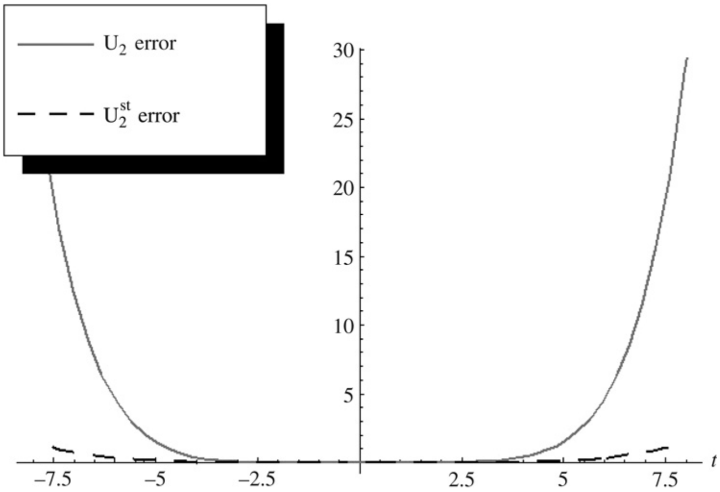

Fig. 1. The absolute error between the exact solution and U_2 and the absolute error between the exact solution and U_2^{st} at $x = 0$ and $c = 0.5$.

$$B_2^4 = B_3^4 = \cdots = B_n^4 = \frac{-c^6(-1+c^2)^2}{128}(33 - 26\text{Cosh}[cx] + \text{Cosh}[2cx])\text{Sech}^8\left[\frac{cx}{2}\right],$$
$$B_2^5 = B_3^5 = \cdots = B_n^5 = \frac{c^7(1-c^2)^{\frac{5}{2}}}{1280}\text{Sech}^7\left[\frac{cx}{2}\right]\left(302\text{Sinh}\left[\frac{cx}{2}\right] - 57\text{Sinh}\left[\frac{3cx}{2}\right] + \text{Sinh}\left[\frac{5cx}{2}\right]\right),$$
$$\begin{aligned} B_2^6 &= \frac{c^{10}(-1+c^2)}{5120}(1569 + 52\,971c^2 - 2(71 + 33\,763c^2)\text{Cosh}[cx] - (1382 - 16\,718c^2) \\ &\quad \times \text{Cosh}[2cx] - 6(53 - 227c^2)\text{Cosh}[3cx] - (11 - 23c^2)\text{Cosh}[4cx])\text{Sech}^{12}\left[\frac{cx}{2}\right], \end{aligned} \tag{26}$$
$$\vdots$$

Fig. 1 shows the absolute error between the exact solution and U_2 and the absolute error between the exact solution and U_2^{st}, where the exact solution of Eq. (18) takes the form [12]

$$u(x,t) = \frac{-3c^2}{2}\text{Sech}^2\left[\frac{c}{2}(x + \sqrt{1-c^2}t)\right]. \tag{27}$$

Solving the "good" Boussinesq equation using MVIM:

Now, let us solve Eq. (18) using MVIM. By using the modified iteration formula (15), where

$$U_{-1} = 0,$$
$$U_0 = \frac{-3c^2}{2}\text{Sech}^2\left[\frac{cx}{2}\right] + \frac{3c^3\sqrt{1-c^2}}{2}\text{Sech}^2\left[\frac{cx}{2}\right]\text{Tanh}\left[\frac{cx}{2}\right]t,$$

and $G_n(x,t)$ is calculated from the relation

$$-\left((U_n(x,t))^2\right)_{xx} = G_n(x,t) + O(t^{2n+2}).$$

The following results are obtained:

$$\begin{aligned} U_1 &= U_0 + \frac{3}{8}c^4(-1+c^2)\,(-2 + \text{Cosh}\,[cx])\,\text{Sech}^4\left[\frac{cx}{2}\right]t^2 \\ &\quad + \frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\text{Sech}^5\left[\frac{cx}{2}\right]\left(-11\text{Sinh}\left[\frac{cx}{2}\right] + \text{Sinh}\left[\frac{3cx}{2}\right]\right)\right)t^3, \end{aligned}$$

$$U_2 = U_1 + \frac{c^6(1-c^2)^2}{128}(33 - 26\mathrm{Cosh}[cx] + \mathrm{Cosh}[2cx])\,\mathrm{Sech}^6\left[\frac{cx}{2}\right]t^4 + \frac{c^7(1-c^2)^{\frac{5}{2}}}{1280} \times \left(\mathrm{Sech}^7\left[\frac{cx}{2}\right]\left(302\mathrm{Sinh}\left[\frac{cx}{2}\right] - 57\mathrm{Sinh}\left[\frac{3cx}{2}\right] + \mathrm{Sinh}\left[\frac{5cx}{2}\right]\right)\right)t^5,$$

$$U_3 = U_2 + \frac{c^8(-1+c^2)^3}{15360}(-1208 + 1191\mathrm{Cosh}[cx] - 120\mathrm{Cosh}[2cx] + \mathrm{Cosh}[3cx])\mathrm{Sech}^8\left[\frac{cx}{2}\right]t^6 - \frac{c^9(1-c^2)^{\frac{7}{2}}}{215\,040}\mathrm{Sech}^9\left[\frac{cx}{2}\right]\left(-15\,619\mathrm{Sinh}\left[\frac{cx}{2}\right] + 4293\mathrm{Sinh}\left[\frac{3cx}{2}\right] - 247\mathrm{Sinh}\left[\frac{5cx}{2}\right] + \mathrm{Sinh}\left[\frac{7cx}{2}\right]\right)t^7,$$

$$U_4 = U_3 + \frac{c^{10}(-1+c^2)^4}{3440\,640}(-78\,095 - 88\,234\mathrm{Cosh}[cx] + 14\,608\mathrm{Cosh}[2cx] - 502\mathrm{Cosh}[3cx] + \mathrm{Cosh}[4cx])\mathrm{Sech}^{10}\left[\frac{cx}{2}\right]t^8 + \frac{c^{11}(1-c^2)^{\frac{9}{2}}}{61\,931\,520}\mathrm{Sech}^{11}\left[\frac{cx}{2}\right] \times \left(1310\,354\mathrm{Sinh}\left[\frac{cx}{2}\right] - 455\,192\mathrm{Sinh}\left[\frac{3cx}{2}\right] + 47\,840\mathrm{Sinh}\left[\frac{5cx}{2}\right] - 1013\mathrm{Sinh}\left[\frac{7cx}{2}\right] + \mathrm{Sinh}\left[\frac{9cx}{2}\right]\right)t^9, \quad (28)$$

$\vdots$

By observing the results and the calculation done, we found that the unneeded terms are omitted. This means that the error will coincide with the error curve of the settled terms in Fig. 1 (dashed line).

Solving the "good" Boussinesq equation using ADM:

Following the analysis of Adomian decomposition method [13–15], Eq. (18) can be rewritten in an operator form as

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = 0, \quad (29)$$

where the differential operator $L = \frac{\partial^2}{\partial t^2}$, $R = -\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}$, and the nonlinear term $Nu = (u^2)_x$. It assumes that the unknown function $u(x,t)$ can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (30)$$

where the components $u_n(x,t)$ will be determined recurrently and the nonlinear term $Nu(x,t)$ can be decomposed by an infinite series of polynomials given by

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(x,t), \quad (31)$$

where A_n are the so-called Adomian polynomials of $u_0, u_1, \ldots, u_n$ defined by

$$A_n(x,t) = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i(x,t)\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3, \ldots. \quad (32)$$

The Adomian polynomials that represent the nonlinear term $(u^2)_x$ are given by

$$\begin{aligned}
A_0 &= 2(u_{0_x})^2 + 2u_0 u_{0_{xx}},\\
A_1 &= 4u_{0_x} u_{1_x} + 2u_1 u_{0_{xx}} + 2u_0 u_{1_{xx}},\\
A_2 &= 2(u_{1_x}^2 + 2u_{0_x} u_{2_x} + u_{0_{xx}} u_2 + u_1 u_{1_{xx}} + u_{2_{xx}} u_0),\\
A_3 &= 2(2u_{1_x} u_{2_x} + 2u_{0_x} u_{3_x} + u_{0_{xx}} u_3 + u_{1_{xx}} u_2 + u_{2_{xx}} u_1 + u_{3_{xx}} u_0),\\
&\vdots
\end{aligned} \tag{33}$$

Other polynomials can be generated in a like manner.
The inverse operator L^{-1} is an integral operator given by

$$L^{-1}(.) = \int_0^t \int_0^t (.)\mathrm{d}\tau \mathrm{d}\tau. \tag{34}$$

Applying L^{-1}on Eq. (29) and using the given constraints we find that

$$u(x,t) = f_0(x) + f_1(x)t - L^{-1}(Nu(x,\tau) + Ru(x,\tau)). \tag{35}$$

Substituting Eqs. (30) and (31) into Eq. (35) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f_0(x) + f_1(x)t - L^{-1}\left(\sum_{n=0}^{\infty} A_n(x,\tau) + R\left(\sum_{n=0}^{\infty} u_n(x,\tau)\right)\right). \tag{36}$$

The components $u_n(x,t)$follow immediately upon setting

$$\begin{aligned}
u_0(x,t) &= f_0(x) + f_1(x)t\\
&= \frac{-3c^2}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right] + \frac{3c^3\sqrt{1-c^2}}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right]t,
\end{aligned} \tag{37}$$

and substituting by (33) in the following iterative equation:

$$u_{n+1}(x,t) = -L^{-1}(A_n + Ru_n). \tag{38}$$

We can obtain the following components:

$$\begin{aligned}
u_1(x,t) &= \frac{3}{8}c^4(-1+c^2)(-2+\mathrm{Cosh}[cx])\,\mathrm{Sech}^4\left[\frac{cx}{2}\right]t^2\\
&\quad + \frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\mathrm{Sech}^5\left[\frac{cx}{2}\right]\left(-11\mathrm{Sinh}\left[\frac{cx}{2}\right] + \mathrm{Sinh}\left[\frac{3cx}{2}\right]\right)\right)t^3\\
&\quad - \frac{3}{32}c^8(-1+c^2)(10 - 10\mathrm{Cosh}[cx] + \mathrm{Cosh}[2cx])\,\mathrm{Sech}^6\left[\frac{cx}{2}\right]t^4,\\
u_2(x,t) &= -\frac{1}{512}c^6(-1+c^2)(-40 - 440c^2 + 15(-1+33c^2)\mathrm{Cosh}[cx] + 24(-1\\
&\quad + 3c^2)\mathrm{Cosh}[2cx] + (-1+c^2)\mathrm{Cosh}[3cx])\mathrm{Sech}^8\left[\frac{cx}{2}\right]t^4 + \frac{c^7(1-c^2)^{\frac{5}{2}}}{1280}\\
&\quad \times\left(\mathrm{Sech}^7\left[\frac{cx}{2}\right]\left(302\mathrm{Sinh}\left[\frac{cx}{2}\right] - 57\mathrm{Sinh}\left[\frac{3cx}{2}\right] + \mathrm{Sinh}\left[\frac{5cx}{2}\right]\right)\right)t^5\\
&\quad + \frac{1}{2560}c^{10}(-1+c^2)(600 + 26\,670c^2 - (59 + 33\,775c^2)\mathrm{Cosh}[cx]\\
&\quad + 4(-133 + 2050c^2)\mathrm{Cosh}[2cx] + (123 - 645c^2)\mathrm{Cosh}[3cx] - (4 - 10c^2)\\
&\quad \times\mathrm{Cosh}[4cx])\mathrm{Sech}^{12}\left[\frac{cx}{2}\right]t^6 + \frac{3c^{13}(1-c^2)^{\frac{3}{2}}}{3584}\left(-3749\mathrm{Sinh}\left[\frac{cx}{2}\right]\right.\\
&\quad \left. + 1551\mathrm{Sinh}\left[\frac{3cx}{2}\right] - 235\mathrm{Sinh}\left[\frac{5cx}{2}\right] + 9\mathrm{Sinh}\left[\frac{7cx}{2}\right]\right)\mathrm{Sech}^{13}\left[\frac{cx}{2}\right]t^7,
\end{aligned}$$

$$
\begin{aligned}
u_3(x,t) = {} & \frac{1}{245\,760}c^8(-1+c^2)(-2604-52392c^2-2562\,924c^4+6(-301+ \\
& \times 1546c^2+540\,099c^4)\mathrm{Cosh}[cx]+96(-17-498c^2+8183c^4) \\
& \times \mathrm{Cosh}[2cx]+(717-13\,242c^2+62\,637c^4)\mathrm{Cosh}\,[3cx]-(116-616c^2 \\
& +1076c^4)\mathrm{Cosh}\,[4cx]+(1-2c^2+c^4)\mathrm{Cosh}\,[5cx])\mathrm{Sech}^{12}\left[\frac{cx}{2}\right]t^6 \\
& +\frac{c^9(1-c^2)^{\frac{3}{2}}}{1\,720\,320}(-18\,774+37\,548c^2+3\,471\,786c^4-6(3157-6314c^2 \\
& +639\,157c^4)\mathrm{Cosh}[cx]+24(131-262c^2+27\,251c^4)\mathrm{Cosh}\,[2cx] \\
& +(3069-6138c^2+22\,851c^4)\mathrm{Cosh}[3cx]-(242-484c^2+2426c^4) \\
& \times \mathrm{Cosh}\,[4cx]+(1-2c^2+c^4)\mathrm{Cosh}\,[5cx])\mathrm{Sech}^{12}\left[\frac{cx}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right]t^7+\cdots,
\end{aligned}
\tag{39}
$$

and so on.

Considering these components, the solution can be approximated as

$$
u(x,t)\simeq \phi_n(x,t)=\sum_{m=0}^{n}u_m(x,t). \tag{40}
$$

$$
\begin{aligned}
\phi_1 = {} & \frac{-3c^2}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right]+\frac{3c^3\sqrt{1-c^2}}{2}\mathrm{Sech}^2\left[\frac{c\,x}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right]t \\
& +\frac{3}{8}c^4(-1+c^2)\,(-2+\mathrm{Cosh}[cx])\,\mathrm{Sech}^4\left[\frac{cx}{2}\right]t^2+\frac{1}{16}\Bigg(c^5(1 \\
& -c^2)^{\frac{3}{2}}\mathrm{Sech}^5\left[\frac{cx}{2}\right]\left(-11\mathrm{Sinh}\left[\frac{cx}{2}\right]+\mathrm{Sinh}\left[\frac{3cx}{2}\right]\right)\Bigg)t^3+O(t^4), \\
\phi_2 = {} & \frac{-3c^2}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right]+\frac{3c^3\sqrt{1-c^2}}{2}\mathrm{Sech}^2\left[\frac{c\,x}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right]t+\frac{3}{8}c^4(-1 \\
& +c^2)\,(-2+\mathrm{Cosh}[cx])\,\mathrm{Sech}^4\left[\frac{cx}{2}\right]t^2+\frac{1}{16}\Bigg(c^5(1-c^2)^{\frac{3}{2}}\mathrm{Sech}^5\left[\frac{cx}{2}\right] \\
& \times\left(-11\mathrm{Sinh}\left[\frac{cx}{2}\right]+\mathrm{Sinh}\left[\frac{3cx}{2}\right]\right)\Bigg)t^3-\frac{1}{128}c^6(-1+c^2)^2(33 \\
& -26\mathrm{Cosh}[cx]+\mathrm{Cosh}[2cx])\mathrm{Sech}^6\left[\frac{cx}{2}\right]t^4+\frac{c^7(1-c^2)^{\frac{5}{2}}}{1280}\Bigg(\mathrm{Sech}^7\left[\frac{cx}{2}\right] \\
& \times\left(302\mathrm{Sinh}\left[\frac{cx}{2}\right]-57\mathrm{Sinh}\left[\frac{3cx}{2}\right]+\mathrm{Sinh}\left[\frac{5cx}{2}\right]\right)\Bigg)t^5+O(t^6), \\
\phi_3 = {} & \frac{-3c^2}{2}\mathrm{Sech}^2\left[\frac{cx}{2}\right]+\frac{3c^3\sqrt{1-c^2}}{2}\mathrm{Sech}^2\left[\frac{c\,x}{2}\right]\mathrm{Tanh}\left[\frac{cx}{2}\right]t \\
& +\frac{3}{8}c^4(-1+c^2)\,(-2+\mathrm{Cosh}[cx])\,\mathrm{Sech}^4\left[\frac{cx}{2}\right]t^2+\frac{1}{16}\Bigg(c^5(1-c^2)^{\frac{3}{2}} \\
& \times\mathrm{Sech}^5\left[\frac{cx}{2}\right]\left(-11\mathrm{Sinh}\left[\frac{cx}{2}\right]+\mathrm{Sinh}\left[\frac{3cx}{2}\right]\right)\Bigg)t^3-\frac{1}{128}c^6(-1+c^2)^2 \\
& \times\,(33-26\mathrm{Cosh}[cx]+\mathrm{Cosh}[2cx])\,\mathrm{Sech}^6\left[\frac{cx}{2}\right]t^4+\frac{c^7(1-c^2)^{\frac{5}{2}}}{1280} \\
& \times\Bigg(\mathrm{Sech}^7\left[\frac{cx}{2}\right]\left(302\mathrm{Sinh}\left[\frac{cx}{2}\right]-57\mathrm{Sinh}\left[\frac{3cx}{2}\right]+\mathrm{Sinh}\left[\frac{5cx}{2}\right]\right)\Bigg)t^5 \\
& +\frac{c^8(-1+c^2)^3}{15\,360}(-1208+1191\mathrm{Cosh}[cx]-120\mathrm{Cosh}[2cx]
\end{aligned}
$$

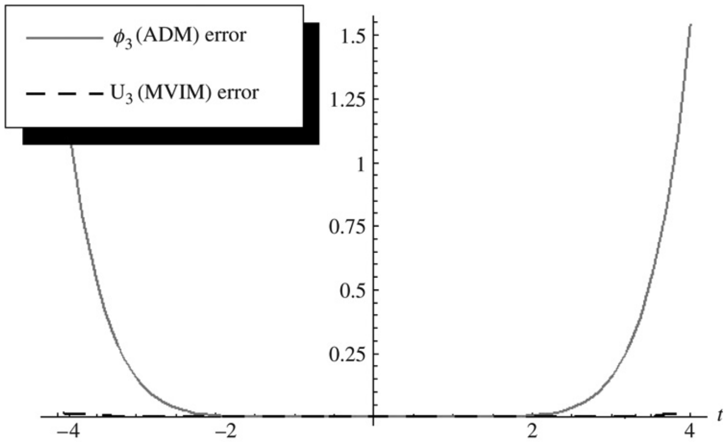

Fig. 2. The absolute error between the exact solution and ϕ_3 (ADM) and the absolute error between the exact solution and U_3 (MVIM) at $x = 0$ and $c = 0.5$.

Table 1
The time consumed in calculating $U_n(x, t)$ for Boussinesq equation using Mathematica -4 package

	$U_1(x,t)$	$U_2(x,t)$	$U_3(x,t)$	$U_4(x,t)$	$U_5(x,t)$
VIM	0.907	59.203	N/A	N/A	N/A
MVIM	0.672	1.719	4.562	8.281	34.11

$$\begin{aligned} &+\text{Cosh}[3cx])\text{Sech}^8\left[\frac{cx}{2}\right]t^6 - \frac{c^9(1-c^2)^{\frac{7}{2}}}{215\,040}\text{Sech}^9\left[\frac{cx}{2}\right] \\ &\times\left(-15\,619\text{Sinh}\left[\frac{cx}{2}\right] + 4293\text{Sinh}\left[\frac{3cx}{2}\right] - 247\text{Sinh}\left[\frac{5cx}{2}\right]\right. \\ &\left.+\text{Sinh}\left[\frac{7cx}{2}\right]\right)t^7 + \cdots, \end{aligned} \tag{41}$$

$\vdots$

By analysing the previous results, it can be noticed that ϕ_n composes of settled term and non-settled term, ϕ_n^{st} and ϕ_n^{ns} respectively, i.e.

$$\phi_n(x,t) = \phi_n^{st}(x,t) + \phi_n^{ns}(x,t). \tag{42}$$

Fig. 2 shows that the MVIM results are better than the ADM results in our case.

By observing the results obtained by MVIM, VIM and ADM in the case study, we found that the series solution obtained by MVIM converges faster than those by VIM and ADM, and MVIM eliminates all the non-settled terms in VIM and ADM.

5. Some remarks

By analysing the obtained results and procedures used in modified variational iteration method, variational iteration method and Adomian decomposition method we observe that:

1. Modified variational iteration method cancels all the unsettled terms in variational iteration method and also in Adomian decomposition method.
2. The series solution obtained by modified variational iteration method converges faster than that by variational iteration method. Table 1 shows the efficiency of the modified variational iteration method.

3. Modified variational iteration method eliminate the shortcoming in variational iteration method by cancellation of computation of the unsettled term, see Figs. 1 and 2.
4. Modified variational iteration method is easier than variational iteration method and Adomian decomposition method in calculation.

6. Conclusion

Modified variational iteration method (in the used cases) cancels the repeated calculation in the variational iteration method, also the computation of unnecessary terms in VIM and ADM. The obtained series solution converges faster than those obtained by VIM and ADM in the studied case.

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