On distribution of bivariate concomitants of records

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The distributions of two concomitants have been given when a random sample is available from a trivariate distribution. The illustration has been given by using a trivariate Pseudo-Exponential distribution.

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1. Introduction

Record values and record statistics have been widely used in the theory of statistical sciences. The record values are simply referred to as the smallest (largest) observation among all the previously recorded values. The concept of record values was first given by [1]. Since the development of the theory of records, lot of work have been done by using various probability distributions. The records have also been used to characterize many probability distribution. A large number of characterizations of Exponential distribution have been done by [2–4] by using the theory of records. The relationship among moments of exponential distribution have been studied by [5] on the basis of records. Characterizations of several other distributions by using the distribution of record values have been given by [6].

The probability distribution of kth upper record from a sample of size n is given by [7] as:

\[ f_{k:n}(x_k) = \frac{1}{I^*(k)} f(x_k) [R(x_k)]^{k-1}; \]

where \( R(x_k) = -\ln [1 - F(x)] \).

The joint distribution of kth and mth records is given by [7] as:

\[ f_{k,m:n}(x_k, x_m) = \frac{r(x_k)f(x_m)}{I^*(k) I^*(m-k)} [R(x_k)]^{k-1} [R(x_m) - R(x_k)]^{m-k-1}; \]

where \( r(x) = R'(x) \) and \( -\infty < x_m < x_k < \infty \).

When a sample is available from a bivariate population with density function \( f(x, y) \) than the sample can be arranged with respect to records of random variable \( X \). The random variable \( Y \) in this case is called the concomitants of record values. Specifically, the probability distribution of kth concomitant of record is given by [7] as:

\[ f(y_k) = \int_{-\infty}^{\infty} f(y_k|x_k)f_{k:n}(x_k) \, dx_k. \]

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The joint distribution of two concomitants of records is given as:
\[ f(y_k, y_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_k|x_k)f(y_m|x_m)f_{k,m:n}(x_k) \, dx_k \, dx_m. \]  
(1.4)

The distributions given in (1.3) and (1.4) can be found for any probability distribution.

Recently Filus and Filus [8–10] have introduced a new class of probability distributions known as Pseudo-distribution as a linear combination of several random variables having the same distribution. These distributions are effectively useful in a number of situations where the standard distributions are not applicable. In the following section we have first introduced the concept of bivariate concomitants if the random sample of size \( n \) is available from a trivariate distribution. The concept has been applied to trivariate Pseudo-Exponential distribution in Section 4.

2. Distribution of bivariate concomitants of records

Concomitants of record statistics have been widely studied by number of statisticians. It is often the case that we have a random sample from a trivariate distribution with density function \( f(x, y, z) \) and the sample is arranged with respect to records of random variable \( X \). In this case the random variables \( Y \) and \( Z \) are automatically arranged and are referred to as the bivariate concomitants of record statistics. The probability density function of \( k \)th pair of concomitants of record values is easily derived parallel to (1.3) and the joint distribution of \( k \)th and \( m \)th pair of concomitants of record values can be derived parallel to (1.4). Specifically, the joint distribution of \( k \)th pair of concomitants of record values is given as:
\[ f(y_k, z_k) = \int_{-\infty}^{\infty} f(y_k, z_k|x_k) f_{k:n}(x_k) \, dx_k; \]  
(2.1)

where \( f(y_k, z_k|x_k) \) is the conditional distribution of \( (y_k, z_k) \) given \( x_k \). The distribution of \( k \)th and \( m \)th pair of concomitants of record values is given as:
\[ f(w_k, w_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_k, z_k|x_k)f(y_m, z_m|x_m)f_{k,m:n}(x_k) \, dx_k \, dx_m; \]  
(2.2)

where \( w_k = [y_k \ z_k]^t \) and \( w_m = [y_m \ z_m]^t \). The distribution of bivariate concomitants for trivariate Pseudo-Exponential distribution have been obtained in Section 4.

3. The trivariate Pseudo-Exponential distribution

In this section we have introduced the trivariate Pseudo-Exponential distribution as a compound distribution of three random variables. The distribution is derived in the following:

Suppose that the random variable \( X \) has the exponential distribution with parameter \( \alpha \). The density function of \( X \) is:
\[ f(x; \alpha) = \alpha e^{-\alpha x}; \quad x > 0. \]  
(3.1)

Further, suppose that the random variable \( Y \) has the exponential distribution with parameter \( \phi_1(x) \), where \( \phi_1(x) \) is some function of random variable \( X \). The density function of \( Y \) is:
\[ f(y|x) = \phi_1(x) \exp \{-\phi_1(x)y\}; \quad y, \phi_1(x) > 0. \]  
(3.2)

Finally, suppose that the random variable \( Z \) also has the exponential distribution with parameter \( \phi_2(xy) \), where \( \phi_2(xy) \) is some function of random variables \( X \) and \( Y \). The density function of \( Z \) is therefore:
\[ f(z|x, y) = \phi_2(xy) \exp \{-\phi_2(xy)z\}; \quad z, \phi_2(xy) > 0. \]  
(3.3)

Now, the compound distribution of (3.1)–(3.3) is referred to as the trivariate Pseudo-Exponential distribution. The density function of the distribution is given as:
\[ f(x, y, z) = \alpha \phi_1(x) \phi_2(xy) \exp \{-\{\alpha x + \phi_1(x) y + \phi_2(xy) z\}\} \quad \phi_1(x) > 0; \quad \phi_2(xy) > 0; \quad x, y, z > 0. \]  
(3.4)

Several distributions can be derived from (3.4) for various choices of \( \phi_1(x) \) and \( \phi_2(xy) \). Using \( \phi_1(x) = x \) and \( \phi_2(xy) = xy \) we have obtained following trivariate Pseudo-Exponential distribution:
\[ f(x, y, z) = \alpha x^2 y \exp \{-x \{\alpha + y + yz\}\}; \quad x, y, z > 0. \]  
(3.4)

The product moments for (3.4) are given as:
\[ \mu_{r,s,t} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} x^r y^s z^t f(x, y, z) \, dx \, dy \, dz \]
\[ = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} x^r y^s z^t \alpha x^2 y \exp \{-x \{\alpha + y + yz\}\} \, dx \, dy \, dz. \]
After some calculus, the product moments turned out to be:

\[ \mu_{k,s,t} = \alpha^k \Gamma (1 + r - s) \Gamma (1 + s - t) \Gamma (1 + t) . \]  

(3.5)

The product moments (3.5) exist if \( r > s - 1 \) and \( s > t - 1 \). The marginal moments can be easily obtained from (3.5). The conditional distribution of \( X \) given \( Y \) and \( Z \) is obtained as:

\[ f (x|y, z) = \frac{f (x, y, z)}{f (y, z)} . \]  

(3.6)

Now

\[ f (y, z) = \int_0^\infty f (x, y, z) \, dx = \int_0^\infty \alpha x^2 y \exp \left\{ -x \{ \alpha + y + yz \} \right\} \, dx \]

\[ = \frac{2\alpha y}{(\alpha + y + yz)^2} . \]  

(3.7)

Using (3.4) and (3.7) in (3.6), the conditional distribution of \( X \) given \( Y \) and \( Z \) is given as:

\[ f (x|y, z) = \frac{1}{2} (\alpha + y + yz)^3 x^2 \exp \left\{ -x \{ \alpha + y + yz \} \right\} . \]

The \( h \)th conditional moment of \( X \) is:

\[ E (X^h|y, z) = \int_0^\infty x^h f (x|y, z) \, dx \]

\[ = \int_0^\infty x^h \frac{1}{2} (\alpha + y + yz)^3 x^2 \exp \left\{ -x \{ \alpha + y + yz \} \right\} \, dx \]

\[ = \frac{\Gamma (h + 3)}{2 (\alpha + y + yz)^h} . \]  

(3.8)

The conditional mean and variance can be readily obtained from (3.8). In the following section we have obtained the distribution of the concomitants of record statistics for (3.4).

4. Bivariate concomitants of upper records

The trivariate Pseudo-Exponential distribution is given in (3.4). The distribution of bivariate concomitants can be obtained by using (2.1). To obtain the distribution (2.1) we first compute the conditional distribution of \( (Y, Z) \) given \( X \) as:

\[ f (y, z|x) = \frac{f (x, y, z)}{f (x)} = x^2 y \exp \left\{ -xy (1 + z) \right\} ; \quad y, z > 0. \]  

(4.1)

Also for the random variable \( X \), the distribution of \( k \)th upper record is obtained by using (1.1). The distribution of \( k \)th upper record; \( X_k = X \); is:

\[ f_{x,x} (x_k) = \frac{\alpha^k}{\Gamma (k)} x^{k-1} e^{-\alpha x} ; \quad \alpha, x > 0. \]  

(4.2)

Using (4.1) and (4.2) in (2.1), the joint distribution of the pair of concomitants; \( Y_k = Y \) and \( Z_k = Z \); is obtained below:

\[ f (y_k, z_k) = \int_0^\infty x^2 y \exp \left\{ -xy (1 + z) \right\} \frac{\alpha^k}{\Gamma (k)} x^{k-1} e^{-\alpha x} \, dx \]

\[ = y \frac{\alpha^k}{\Gamma (k)} \int_0^\infty x^{k+1} \exp \left\{ -x (\alpha + y + yz) \right\} \, dx. \]

Making the transformation \( x (\alpha + y + yz) = w \) and simplifying, the joint distribution of two concomitants; \( Y_k = Y \) and \( Z_k = Z \); is obtained as:

\[ f (y_k, z_k) = \frac{k (k + 1) \alpha^k y}{(\alpha + y + yz)^{k+2}} ; \quad y, z > 0. \]  

(4.3)

The product moments of (4.3) are given by:

\[ \mu_{s,t} = E (Y_k^s, Z_k^t) = \int_0^\infty \int_0^\infty y^s z^t \frac{k (k + 1) \alpha^k y}{(\alpha + y + yz)^{k+2}} \, dydz. \]
Using [11] and some simplification, the product moments of two concomitants are given as:

$$
\mu_{s,t} = \frac{\alpha^k}{\Gamma(k)} \Gamma(k - s) \Gamma(s + 2) \Gamma(k + t).
$$

(4.4)

The means, variances and covariance can be easily obtained by using (4.4). It can be readily seen from (4.4) that mean and variance for \( Z_k \) does not exist. Now using (4.3), the marginal distribution of \( Z_k \) is given as

$$
f(z_k) = (1 + z)^{-1}; \ z > 0.
$$

(4.5)

The conditional moments of \( Y_k \) given \( Z_k = z \) are given as:

$$
E(Y_k | Z_k = z) = \frac{\alpha^s \Gamma(k - s) \Gamma(s + 2) \Gamma(k + t)}{(1 + z)^2 \Gamma(k)}; \ k > s.
$$

(4.6)

Using \( s = 1 \) in (4.6), the conditional expectation of \( Y_k \) given \( z \) is

$$
E(Y_k | z) = 2\alpha (k - 1)^{-1} (1 + z)^{-1}.
$$

References