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The Cohen–Macaulay Type of Cohen–Macaulay Rings

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INTRODUCTION

Let R be an integrally closed complete local commutative noetherian Cohen–Macaulay domain with maximal ideal $\mathfrak{m}$ and $R/\mathfrak{m}=k$ an algebraically closed field. We abbreviate Cohen–Macaulay to CM and denote by CM(R) the category of finitely generated CM modules. In this paper we deal with the question of when R is of finite CM type, that is, has only a finite number of indecomposable CM modules. If $\dim R=2$, and R is a $\mathbb{C}$-algebra with $R/\mathfrak{m}=\mathbb{C}$, the complex numbers, then the R of finite CM type are exactly the fixed rings $\mathbb{C}[[X, Y]]^G$, where X, Y are indeterminates and $G\subset GL(2, \mathbb{C})$ is a finite group acting on $\mathbb{C}[[X, Y]]$ [17, 3, 15]. For hypersurfaces in characteristic zero the R of finite CM type are exactly the simple hypersurface singularities in the sense of Arnold [18, 13], so in this case there is a nice connection with algebraic geometry.

When $\dim R\geqslant 3$ and R is not a hypersurface, we know only two examples of finite CM type, and they both have dimension 3. These are $R_1=k[[X_0, X_1, X_2, Y_0, Y_1]]/(X_0X_2-X_1^2,\ X_0Y_1-X_1Y_0,\ X_1Y_1-X_2Y_0)$ and $R_2=k[[X, Y, Z]]^{Z_2}$, where the generator of Z_2 acts by sending each variable to its negative, and the characteristic of k is different from 2. Each of these rings belongs naturally to a larger class. R_1 is a scroll of type (2, 1). We define more generally a scroll of type $(m_1, ..., m_r)$ (see [14]) and show that when $\dim R \geqslant 3$, then all other scrolls (except type (1, 1), which is a hypersurface) are of infinite CM type. R_2 is among the rings $k[[X_1, ..., X_n]]^G$, where the order of $G\subset GL(n, k)$ is invertible in k and $n\geqslant 3$, and is the only one of finite CM type. These are the main results of this paper.

To prove that the scroll of type (2, 1) is of finite CM type we use the

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theory of almost split sequences, first developed for artin algebras [7, 8] and extended to this setting [2, 4]. We will assume that the reader is familiar with the basic theory of almost split sequences (see [19]). In particular, we generalize from artin algebras a criterion for having only a finite number of indecomposable modules, and this is a direct consequence of the theory of preprojective partitions from [12]. The same method can be used to show that our fixed ring is of finite CM type, but here we give in addition a different proof. Our study of the fixed rings is based on an investigation of the connection between the reflexive modules over R and the skew group ring $k[[X_1, ..., X_n]]\,G$.

We consider only faithful actions of G on $S = k[[X_1, ..., X_n]]$, and we also assume that the actions are linear.

The scrolls are investigated in Part I and the fixed rings in Part II. For the work on scrolls we are grateful to David Eisenbud and Finn Knudsen for several suggestions and helpful conversations.

Some of the results in this paper were anounced at Bielefeld (1984 and 1985), at the Seminaire Malliavin [20], and at the Durham Symposium on representations of algebras [5].

PART I: SCROLLS OF FINITE COHEN–MACAULAY TYPE

In this part we introduce scrolls (see [14]) and describe conditions under which they are of finite CM type.

1. A Criterion for Finite Representation Type

Let T be a complete regular local noetherian ring and let Λ be a T-order, that is, Λ is a finitely generated free T-module and $\operatorname{Hom}_{T_p}(\Lambda_p, T_p)$ is Λ_p^{op}-projective for all nonmaximal prime ideals p in T [2]. Denote by $\mathfrak{L}(\Lambda)$ the Λ-lattices, that is, the finitely generated Λ-modules M such that M is a free T-module and M_p is Λ_p-projective for all nonmaximal prime ideals p in T. The category $\mathfrak{L}(\Lambda)$ is known to have almost split sequences, that is, for each nonprojective indecomposable C in $\mathfrak{L}(\Lambda)$ (or each noninjective indecomposable A in $\mathfrak{L}(\Lambda)$), there is an almost split sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{L}(\Lambda)$. And if C is indecomposable projective, there is a minimal right almost split map $B \to C$, and for A indecomposable injective there is a minimal left almost split map $A \to B$ [2, 12, 14].

For indecomposable finite dimensional algebras we know that if $\mathscr{C}$ is a finite set of indecomposable modules closed under irreducible maps, then $\mathscr{C}$ consists of all indecomposables [2]. There is a similar criterion for classical orders [21], and here we give a related criterion for T-orders Λ. We say that a set of indecomposable lattices $\mathscr{C}$ is closed under almost split sequences if for each indecomposable nonprojective object C in $\mathscr{C}$ (and each indecomposable noninjective object A in $\mathscr{C}$) all indecomposable summands of the terms in the almost split sequence $0 \to A \to B \to C \to 0$ are in $\mathscr{C}$. We denote A by τC and C by $\tau^{-1}A$.

THEOREM 1.1. *Let Λ be T-order and $\mathscr{C}$ a finite set of indecomposable lattices containing all indecomposable projectives.*

(a) *If $\mathscr{C}$ is closed under irreducible maps, then $\mathscr{C}$ consists of all indecomposable Λ-lattices.*

(b) *Assume that there are no projective injective Λ-lattices. If $\mathscr{C}$ is closed under almost split sequences, then $\mathscr{C}$ consists of all indecomposable Λ-lattices.*

Proof. Let C be an indecomposable nonprojective preprojective lattice in the sense of Auslander and Smalø [12]. Then there is a chain of irreducible maps between indecomposables $P \to C_1 \to \cdots \to C = C_n$, where P is projective and the C_i are not projective. We want to show that C is in $\mathscr{C}$. This is clear for (a). In case (b) we know that P is not injective, so we have an almost split sequence $0 \to P \to C_1 \amalg X \to \tau^{-1}P \to 0$, which shows that C_1 is in $\mathscr{C}$. If C_1 is not injective, the same argument shows that C_2 is in $\mathscr{C}$. If C_1 is injective, it is not projective, so we have an almost split sequence $0 \to \tau C_1 \to \tau C_2 \amalg Y \to C_1 \to 0$. Since C_2 is not projective, τC_2 is in $\mathscr{C}$. It then follows that C_2 is also in $\mathscr{C}$. Continuing this way, we get $C \in \mathscr{C}$. This shows that all indecomposable preprojectives are in $\mathscr{C}$, and hence there is only a finite number of them. By [12] we then know that all Λ-lattices are preprojective, and this finishes the proof.

Theorem 1.1 can be applied to the case of the CM modules CM(R) over a complete local integrally closed noetherian CM domain R. In this case R is the only indecomposable projective module and the dualizing module ω is the only indecomposable injective. Part (b) applies if $R \not\simeq \omega$, that is, if R is not Gorenstein. If R is Gorenstein, we can apply part (a). It is then not enough to show that a given set $\mathscr{C}$ is closed under almost split sequences, but we also need to find the minimal right almost split map $B \to R$ (and the minimal left almost split map $R \to C$, which can be obtained by duality). In practice $B \to R$ can be constructed using a construction of CM modules of Buchweitz, and the application of his construction to this problem appeared through discussions with him. The pertinent result is the following, where $\Omega^d X$ denotes the dth syzygy module for X.

PROPOSITION 1.2. *Let R be a complete local noetherian Gorenstein integrally closed domain of dimension $d+1 \geqslant 2$ and with maximal ideal* $\mathfrak{m}$. *If*

$B \to R$ is minimal right almost split, then the nonprojective part of B is isomorphic to $\Omega^{-d}(\Omega^d \mathfrak{m})$.

Proof. Consider the minimal projective resolution $P_{d-1} \to \cdots \to P_1 \to P_0 \to \mathfrak{m} \to 0$. Since $\dim R = d+1$, $\Omega^d \mathfrak{m} = \Omega^{d+1}(R/\mathfrak{m})$ is CM. Write $X^* = \operatorname{Hom}_R(X, R)$ and let $Q_d \to \cdots \to Q_1 \to Q_0 \to (\Omega^d \mathfrak{m})^* \to 0$ be a minimal projective resolution. Since $(\Omega^d \mathfrak{m})^*$ is CM, $\operatorname{Ext}^i_R((\Omega^d \mathfrak{m})^*, R) = 0$ for $i > 0$, so that we have an exact sequence $0 \to \Omega^d \mathfrak{m} \to Q_0^* \to Q_1^* \to \cdots \to Q_{d-1}^* \to \Omega^{-d}(\Omega^d M) = E \to 0$. Since $\operatorname{Ext}^i(\mathfrak{m}, R) = 0$ for $1 \leqslant i < d$, we have the exact commutative diagram

$$\begin{array}{ccccccccc}
P_0^* & \longrightarrow & P_1^* & \longrightarrow \cdots \longrightarrow & P_{d-1}^* & \longrightarrow & (\Omega^d \mathfrak{m})^* & & \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
Q_{d-1} & \longrightarrow & Q_{d-2} & \longrightarrow \cdots \longrightarrow & Q_0 & \longrightarrow & (\Omega^d \mathfrak{m})^* & \longrightarrow & 0,
\end{array}$$

which gives rise to the commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^d \mathfrak{m} & \longrightarrow & P_{d-1} & \longrightarrow \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & \mathfrak{m} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^d \mathfrak{m} & \longrightarrow & Q_0^* & \longrightarrow \cdots \longrightarrow & Q_{d-2}^* & \longrightarrow & Q_{d-1}^* & \longrightarrow & E & \longrightarrow & 0.
\end{array}$$

Let $h\colon X \to \mathfrak{m}$, where X is CM. Since $\operatorname{Ext}^i(X, R) = 0$, for $1 \leqslant i < d$, we have natural isomorphisms modulo projectives $\underline{\operatorname{Hom}}(X, \mathfrak{m}) \simeq \underline{\operatorname{Hom}}(\Omega^d X, \Omega^d \mathfrak{m}) \simeq \underline{\operatorname{Hom}}(X, E)$ (see [6]). This shows that E is the nonprojective part of a minimal right almost split map to R.

2. Scrolls of Finite Cohen–Macaulay Type

Consider the matrix

$$\left(\begin{array}{c|c|c} Z_0^{(1)} \cdots Z_{m_1-1}^{(1)} & \multirow{2}{*}{\cdots} & Z_0^{(r)} \cdots Z_{m_r-1}^{(r)} \\ Z_1^{(1)} \cdots Z_{m_1}^{(1)} & & Z_1^{(r)} \cdots Z_{m_r}^{(r)} \end{array}\right),$$

where the $Z_i^{(j)}$ are indeterminates, and let k be an infinite field. We say that the ring $R = k[[Z_0^{(1)}, ..., Z_{m_1}^{(1)}, ..., Z_0^{(r)}, ..., Z_{m_r}^{(r)}]]/I$, where I is the ideal generated by the determinants of the 2×2 minors of the above matrix, is a scroll of type $(m_1, ..., m_r)$. For basic properties of such rings and their geometric interpretation we refer to [14] and the references given there. A scroll R is known to be an integrally closed CM noetherian domain which is an isolated singularity, and $\dim R = r+1$. It is not hard to see that if $r = 1$, then R is of the form $k[[X, Y]]^G$, and hence of finite CM type. The scroll of type (1, 1) is a hypersurface and is known to be of finite CM type by [18, 22, 23]. The aim of this section is to show the following.

THEOREM 2.1. *The ring* $R = k[[X_0, X_1, X_2, Y_0, Y_1]]/(X_0X_2 - X_1^2, X_0Y_1 - X_1Y_0, X_1Y_1 - X_2Y_0)$ *is of finite CM type.*

Proof. We denote the images of X_i and Y_i in R by x_i and y_i. It is easy to see that the ideals $A = (x_0, x_1)$, $B = (x_0, x_1, y_0)$, $C = (x_0, x_1, x_2) \simeq A^2$ are CM. For consider the exact sequence $0 \to A \to R \to R/A \to 0$. $R/A = k[[X_2, Y_0, Y_1]]/(X_2Y_0)$ is a complete intersection of dimension 2. Hence depth $R/A = 2$, so that depth $A = 3$ since depth $R = 3$. Similarly $R/B \simeq k[[X_2, Y_1]]$ gives depth $B = 3$ and $R/C \simeq k[[Y_0, Y_1]]$ gives depth $C = 3$. It is known that R, A, B, C are the only indecomposable CM R-modules of rank one, but we do not need to use this fact here. It is also known, or can be computed directly, that A is the dualizing module.

Denote by D the duality from $\mathrm{CM}(R)$ to $\mathrm{CM}(R)$ defined by $D(X) = \mathrm{Hom}_R(X, A)$. We recall that the transpose Tr_L is defined by $\mathrm{Tr}_L X = \Omega^1 X^*$ for X in $\mathrm{CM}(R)$, using that $\dim R = 3$ [2]. We know that D preserves indecomposable CM modules and Tr_L preserves indecomposable nonprojective CM modules, and we have $D^2(X) \simeq X$, and $\mathrm{Tr}_L^2 Y \simeq Y$ when Y is not projective. It is known, or can be computed directly, that $DB \simeq C$, and clearly $DR \simeq A$. We next compute Tr_L for our indecomposable CM modules of rank 1. This computation is needed in the construction of almost split sequences, since they are of the form $0 \to D\,\mathrm{Tr}_L X \to E \to X \to 0$ [2].

We have $B \simeq B' = (1, x_1/x_0, y_0/x_0)$, and $AB' = \mathfrak{m}$, so that $A^* \simeq B$. It follows from this, or can be seen directly, that we have an exact sequence $0 \to B \to R \amalg R \to A \to 0$. Then $\mathrm{Tr}_L B = \Omega^1 B^* = \Omega A = B$ and $\mathrm{Tr}_L A = \Omega^1 A^* = \Omega^1 B = K$, which must be an indecomposable CM module of rank 2. C^* is easily computed to be isomorphic to (x_0, y_0^2). Since it has a minimal set of two generators, $\mathrm{Tr}_L C = \Omega^1 C^* = C$.

We now want to construct candidates for almost split sequences. Considering the exact sequence $0 \to K \to R \amalg R \amalg R \to B \to 0$, we see that K is the submodule of R^3 generated by $(-y_0, 0, x_0)$, $(-y_1, 0, x_1)$, $(0, -y_0, x_1)$, $(0, -y_1, x_2)$, $(-x_1, x_0, 0)$, $(-x_2, x_1, 0)$. It is easy to see that we have maps f: $A \to K$ given by $f(x_0) = (-y_0, 0, x_0)$, $f(x_1) = (-y_1, 0, x_1)$, g: $A \to K$ given by $g(x_0) = (0, -y_0, x_1)$, $g(x_1) = (0, -y_1, x_2)$, and h: $B \to K$ given by $h(x_0) = (-x_1, x_0, 0)$, $h(x_1) = (-x_2, x_1, 0)$, $h(y_0) = (-y_1, y_0, 0)$. These maps induce an epimorphism $u = (f, g, h)$: $A \amalg A \amalg B \to K$. Considering the associated divisors in the class group of R, and using that $\mathrm{Ker}\, u$ has rank 1, we see that $\mathrm{Ker}\, u = R$. (This can also be computed directly.) Hence we have the exact sequence

$$0 \to R \to A \amalg A \amalg B \to K \to 0, \qquad (*)$$

and by dualizing, an exact sequence

$$0 \to DK \to R \amalg R \amalg C \to A \to 0. \qquad (**)$$

Since $D\operatorname{Tr}_L A = DK$, $(**)$ is a nonsplit exact sequence of CM modules with the correct end terms for being almost split, it follows as in [8] that it is enough to show $\underline{\operatorname{End}}(A) = k$ to show that $(**)$ is almost split. That $\underline{\operatorname{End}}(A) = k$ is a consequence of the following.

LEMMA 2.2. *Let R be a noetherian domain and X an ideal in R. Let Y be a fractional ideal such that $XY \subset R$. Then a map $f\colon X \to X$ which is multiplication by an element in XY factors through a projective module.*

Proof. Let m be an element in XY, that is, $m = a_1 b_1 + \cdots + a_r b_r$, where $a_1, ..., a_r$ is a set of generators for X, and b_i is in Y. Define $g\colon X \to R^r$ by $g(a) = (ab_1, ..., ab_r)$, and let $h\colon R^r \to X$ be the natural map sending the ith generator

$$\underset{}{(0, ..., \underset{i}{1}, ..., 0)}$$

in R^r to a_i. Then hg is multiplication by m, and we are done.

In our case $\operatorname{End} A \simeq R$ since R is integrally closed and A is a nonzero ideal. Since $AB' = \mathfrak{m}$, it follows from Lemma 2.2 that $\underline{\operatorname{End}}(A) = k$. Hence $(**)$ is almost split, and by duality $(*)$ is also almost split.

It is now easy to compute the almost split sequences with B and C on the right, by using basic properties of almost split sequences. We have $D\operatorname{Tr}_L C = B$, so the middle term of the almost split sequence with C on the right has rank 2. By $(*)$, K is a summand of the middle term, and by $(**)$, DK is. Hence $DK \simeq K$, and we have an almost split sequence

$$0 \to B \to K \to C \to 0.$$

Further we have $D\operatorname{Tr}_L B = C$, and by $(*)$ and $(**)$, A and R are both summands of the middle term for an almost split sequence with B on the right. A rank argument again gives an almost split sequence

$$0 \to C \to R \amalg A \to B \to 0.$$

We have now proved that the set $\{R, A, B, C, K\}$ of indecomposable CM modules contains R and is closed under almost split sequences. Since R is not injective, it follows from Theorem 1.1 that these are all indecomposable CM R-modules. This finishes the proof of Theorem 2.1.

3. Scrolls of Infinite Cohen–Macaulay Type

In this section we show that if R is a scroll with $\dim R \geqslant 3$ and not of type (1, 1) or (2, 1), then R is of infinite CM type. We do this by first studying the corresponding graded scrolls $R' = k[Z_0^{(1)}, ..., Z_{m_1}^{(1)}, ..., Z_0^{(r)}, ..., Z_{m_r}^{(r)}]/I$, where I is defined as before. We have chosen to deal with graded modules rather than with sheaves.

To a monomial p in $k[Z_j^{(i)}]$ we define the number $d(p)$ to be the sum of the lower indices for the factors, and $t(p) = (c_1, ..., c_r)$, where c_i is the number of factors of the form $Z_j^{(i)}$ for a fixed i. From the relations for a graded scroll it follows directly that two monomials p and q are equal in R' if and only if $d(p) = d(q)$ and $t(p) = t(q)$. If for the monomials of degree i we choose one monomial for each given $d(p)$ and $t(p)$, we get a k-basis for $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, where $\mathfrak{m} = (Z_0^{(1)}, ..., Z_{m_1}^{(1)}, ..., Z_{m_r}^{(r)})$.

We shall prove the following.

Theorem 3.1. *Let R' be a graded scroll of type $(m_1 = n, m_2 = t, ..., m_r)$, $n \geqslant t \geqslant \cdots \geqslant m_r$. If $r \geqslant 2$ and R' is not of type (1, 1) or (2, 1), then R' has an infinite number of indecomposable graded CM modules, up to shifts.*

Proof. Write $m = m_1 + \cdots + m_r$, $A = (x_0, x_1)$, and $B = (b_1, ..., b_m)$, where $b_1, ..., b_m$ are the entries in the first row of the matrix. Then it is known that B is CM and A^i is CM for $0 \leqslant i \leqslant m-1$.

We shall first assume $m - n \geqslant 2$, and construct exact sequences in $\operatorname{Ext}^1(A^{n+i}, B)$ when $1 \leqslant i \leqslant m-n-1$. We have $A^{n+i} \simeq (x_0^i x_0 = a_1, x_0^i x_1 = a_2, ..., x_0^i x_n = a_{n+1}, ..., x_1^i x_n = a_{i+n+1})$. For $\lambda \in k$ we define $M_\lambda \subset R \amalg R$ to be generated by $u_j = (x_0^i b_j, 0)$, $1 \leqslant j \leqslant m$, $u_{m+j} = (0, a_j)$; $0 < j \leqslant i+n$, $u_{m+n+i+1} = (x_0^i x_n + \lambda x_0^i y_t, x_1^i x_n)$. Write $s = m+n+i+1$, and $B' = x_0^i B$.

We claim that the sequence $0 \to B' \to^\beta M_\lambda \to^\alpha A^{n+i} \to 0$ is exact, where β is the natural monomorphism and α is the natural epimorphism. Clearly $\alpha\beta$ is zero. Let $x = \sum_{j=m+1}^{s} r_j u_j$ be an element in M_λ such that $\alpha(x) = 0$. If $r_s = 0$, then clearly $x \in \operatorname{Im} \beta$, so we can assume $r_s \neq 0$. Since $\alpha(u_{m+j}) = a_j$, we have $\sum_{j=1}^{i+n+1} r_j a_j = 0$. Since the a_j are linearly independent in $\mathfrak{m}^{i+1}/\mathfrak{m}^{i+2}$, all r_j must have zero constant term. A monomial p in r_s cannot have only factors of the form $z_{m_k}^{(k)}$, since $d(pa_s)$ would be too large. Hence we must have $p = p' z_j^{(k)}$, $j < m_k$, and $z_j^{(k)}(x_0^i x_n + \lambda x_0^i y_t, 0) = z_{j+1}^{(k)}(x_0^i x_{n-1} + \lambda x_0^i y_{t-1}, 0) \in \beta(B')$. This shows that the sequence is exact, and consequently M_λ is CM.

We now assume that there is an isomorphism of degree zero f: $M_{\lambda'} \to M_\lambda$. A^{n+i} and B' both have all their generators, in a minimal set of generators, in degree $i+1$. A^{n+i} has $n+i+1$ generators, and B has $m \geqslant n+i+1$ generators. Since a nonzero map $B \to A^{n+i}$ must be a monomorphism, it follows that $\mathrm{Hom}_{\mathrm{gr}\,R}(B, A^{n+i})_0$, the graded maps of degree zero, is 0. This gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r] \arrow[d, "g"] & M_{\lambda'} \arrow[r] \arrow[d, "f"] & A^{n+i} \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B' \arrow[r] & M_\lambda \arrow[r] & A^{n+i} \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathrm{End}(B') = R'$ and $\mathrm{End}(A^{n+i}) = R'$, we see that $\mathrm{End}(B')_0 = k = \mathrm{End}(A^{n+1})_0$. Hence we can assume that g is the identity and h is the multiplication by some $c \neq 0$. Write $f(u_j) = a_{1j}u_1 + \cdots + a_{sj}u_s$ for $j < s$, $f(u_s') = a_{1s}u_1 + \cdots + a_{ss}u_s$, where $u_1, ..., u_{s-1}, u_s'$ are the generators for $M_{\lambda'}$ and $u_1, ..., u_s$ the generators for M_λ. Since $i \geqslant 1$, we have $x_0 u_{s-1} = x_n u_{s-n-1}$, and this shows $a_{n,s-1} = 0$. Consider the equality $x_0 u_s' = x_1 u_{s-1} + x_1 u_n + \lambda y_t u_1$. Applying f and comparing the coefficients of $x_1 u_n$ we get $a_{s,s} = a_{n,s-1} + 1 = 1$. Applying f and comparing the coefficients of $(x_0^{i+1} y_t, 0)$ gives $\lambda a_{ss} = \lambda'$, so that $\lambda = \lambda'$. This shows that different λ give nonisomorphic M_λ. Since k is an infinite field, we are done in this first case. (We point out that if we let $\lambda = (\lambda_1, ..., \lambda_r) \in \mathbf{P}^{r-1}(k)$, and in M_λ redefine u_s to be $(\sum_{j=1}^r \lambda_j z_{m_j}^{(j)} x_0^i, x_1^i x_n)$, we get similarly an infinite family of nonisomorphic M_λ indexed by $\mathbf{P}^{r-1}(k)$.)

Assume now that the graded scroll R' is of type $(n, 1)$, with $n \geqslant 3$. Let $B' = (x_0^2, x_0 x_1, ..., x_0 x_{n-1}, x_0 y_0)$ and $A^n = (x_0^2, x_0 x_1, ..., x_0 x_n)$. For $\lambda \in k$, define $M_\lambda \subset R \amalg R$ to be generated by $u_j = (x_0 x_{j-1}, 0)$, $1 \leqslant j \leqslant n$, $u_{n+1} = (x_0 y_0, 0)$, $u_j = (0, x_0 x_{j-n-2})$; $n+2 \leqslant j \leqslant 2n$, $u_{2n+1} = (x_0 y_1, x_0 x_{n-1})$, $u_{2n+2} = (x_1 y_1 + \lambda x_0 y_1, x_0 x_n)$.

We want to show that we have an exact sequence $0 \to B' \to^\beta M_\lambda \to^\alpha A^n \to 0$, where β is the natural inclusion and α the natural epimorphism. We clearly have $\alpha\beta$ is zero. Assume that $\alpha(x) = 0$, where $x = \sum_{j=1}^s r_j u_j$, $s = 2n+2$. As before, the r_j must have constant term zero, and if r_s and r_{s-1} are both zero, we are done. It is impossible that a monomial p in r_s has all factors of the form x_n and y_1. If p has one factor of the form x_{n-1} or y_0 and the rest of the form x_n and y_1, then a_{s-1} must have a monomial q with $x_0 q = x_1 p$. Then $q(x_0 y_1, 0) = p(x_1 y_1, 0)$ and $p(x_0 y_1, 0) = q(x_0 y_0, 0) \in \beta(B)$. If p does not satisfy any of the above conditions, the $p(x_1 y_1 + \lambda x_0 y_1, 0) \in \beta(B)$. If $r_s = 0$, we argue similarly for r_{s-1}, and get that the sequence is exact, and M_λ is CM.

Assume that f: $M_{\lambda'} \to M_\lambda$ is an isomorphism. We get as before a commutative diagram where the induced map g: $B' \to B'$ can be assumed to be the identity. We have $x_0 u_s = x_n(0, x_0^2) + y_1(x_0 x_1, 0) + \lambda' y_1(x_0^2, 0)$. Applying f and comparing the coefficients of $(x_1 y_1, 0)$ gives $a_{ss} = 1$. We note that we get no contribution from $x_n f(0, x_0^2)$ since $n > 1 + 1 = 2$. Comparing the coefficients of $(x_0 y_1, 0)$ gives $\lambda a_{ss} = \lambda'$, and hence $\lambda = \lambda'$. This finishes the proof.

We have the following main result of this part.

THEOREM 3.2. *Let R be a scroll of type $(m_1, ..., m_r)$ over an infinite field k. Then R is of finite CM type if and only if R is of type (m), $(1, 1)$, or $(2, 1)$.*

Proof. We have seen that a scroll of type (m), $(1, 1)$, or $(2, 1)$ is of finite CM type. If R is of a different type, we consider $R' \to R$, where R' is the corresponding graded scroll. Let $M = M_0 \amalg M_1 \amalg \cdots$ be a graded R'-module, generated by M_0. R is the completion of R' with respect to the graded maximal ideal $\mathfrak{m}$. $M_i = \mathfrak{m}^{i+1} M_0$. The associated graded module of $R \otimes_{R'} M$ is then M. If $N = N_0 \amalg N_1 \amalg \cdots$ is another graded R'-module, generated by N_0, then an isomorphism $g\colon R \otimes_{R'} M \to R \otimes_{R'} N$ will induce an isomorphism of degree zero between M and N. Since by Theorem 3.1 we have an infinite number of nonisomorphic indecomposable CM modules $M = M_0 \amalg M_1 + \cdots$ generated by their degree zero component, R is of infinite CM type.

PART II: FIXED RINGS

Throughout this part let S be an integrally closed domain, and G a finite nontrivial group acting faithfully on S, such that $|G|$ is invertible in S. Denote by $R = S^G$ the fixed ring, and our main objective is to compare the modules over S and R. It will also be useful to study the modules over the skew group ring SG. When discussing the question of finite CM type, we shall in addition assume that S is complete local CM, with $S/\mathfrak{m} = k$ algebraically closed, where $\mathfrak{m}$ is the maximal ideal of S.

The main result of this part is that when S is the power series ring $k[[X_1, ..., X_n]]$ for $n \geqslant 3$, then $R = S^G$ is of finite Cohen–Macaulay type if and only if $n = 3$ and $G \simeq Z_2$, where the generator of Z_2 acts by sending each variable to its negative.

We start out in Section 1 by studying the relationship between the reflexive modules over SG and R. If the minimal primes in R are unramified in S, we show that the fixed point functor gives an equivalence of categories $\operatorname{Ref} SG \to \operatorname{Ref} R$ of reflexive modules. In the complete case, we use this to show that if R is of finite CM type, then S is. We also reduce the problem of showing that R is of infinite CM type to a similar problem for finite dimensional algebras, which we investigate in Section 2. As a consequence we get most of our results on $R = k[[X_1, ..., X_n]]^G$ $(n \geqslant 3)$ being of infinite CM type. The rest is given in Section 3 using different techniques.

Finally we show in Section 4 that $R=k[[X_1, X_2, X_3]]^{Z_2}$ is of finite CM type with the desired action. We give two different proofs, one in the spirit of the work in this part, and one using the same techniques as in Part 1.

In Section 5 we derive some interesting consequences of $R=S^G$ being of finite CM type and illustrate how our results may be used to show that a given $R=S^G$ is of infinite CM type.

1. Preliminary Results

Under our standard assumptions, the fixed ring $R=S^G$ is integrally closed. We want to study the relationship between Ref R and Ref SG. We shall need the following facts on reflexive modules.

Lemma 1.1. (a) *A module over R or S is reflexive if and only if it is a second syzygy module.*

(b) *S is a finitely generated reflexive R-module.*

(c) *If f: $B \to C$ is a map between reflexive modules in* mod R *or* mod S, *then* Ker f *is reflexive.*

(d) *An SG-module M is SG-reflexive if and only if it is R-reflexive if and only if it is S-reflexive.*

(e) *An SG-module M is reflexive if and only if it is a second syzygy module.*

We have the following direct consequence.

Lemma 1.2. *The fixed point functor F:* mod $SG \to$ mod R *takes reflexive modules to reflexive modules.*

Proof. If M is a reflexive SG-module, we have an exact sequence $0 \to M \to n(SG) \to m(SG)$, where $n(SG)$ denotes the direct sum of n copies of SG. The sequence $0 \to M^G \to n(SG)^G \to m(SG)^G$ is then an exact sequence of R-modules. There is an isomorphism of R-modules $S \to (SG)^G$ obtained by sending s to $\sum_{\sigma \in G} \sigma(s)\,\sigma$, and since S is a reflexive R-module, we conclude that M^G is a reflexive R-module.

We want to show that if height one primes in R are unramified in S, then F: Ref $SG \to$ Ref R in an equivalence. We recall that this assumption is equivalent to the natural ring map α: $SG \to \mathrm{End}_R(S) = \Gamma$ given by $\alpha(s\sigma)(x) = s\sigma(x)$ being an isomorphism [1]. Since R is integrally closed and S is a reflexive R-module, we know from [11, Chap. 1] that $\mathrm{Hom}_R(S^*, \ )$: mod $R \to$ mod Γ and $\mathrm{Hom}_\Gamma(S, \ )$: mod $\Gamma \to$ mod R induce inverse equivalences of categories between Ref R and Ref Γ, where $S^* = \mathrm{Hom}_R(S, R)$.

Hence we get the following consequence.

PROPOSITION 1.3. *Assume that height one primes in* $R = S^G$ *are unramified in* S. *Then the fixed point functor* F: mod $SG \to$ mod R *induces an equivalence of categories between* Ref SG *and* Ref R.

Proof. Because of the ring isomorphism $SG \to \mathrm{End}_R S = \Gamma$, the equivalence $\mathrm{Hom}_\Gamma(S, \ )$: Ref $\Gamma \to$ Ref R shows that we have an equivalence $\mathrm{Hom}_{SG}(S, \ )$: Ref $SG \to$ Ref R, and $\mathrm{Hom}_{SG}(S, \ )$ is the fixed point functor.

From the standard assumptions on S and $R = S^G$ it follows that R is an R-summand of S, by considering the maps $R \to^i S \to^t R$, where $t(s) = 1/|G| \sum_{\sigma \in G} \sigma(s)$. From this it follows that S is a projective Γ-module. We say that a Γ-module is CM if it is CM over the center of Γ. Since $\mathrm{Hom}_\Gamma(S, \ )$: mod $\Gamma \to$ mod R is an exact functor, it preserves CM modules. Hence we have the following relationship between the CM types.

PROPOSITION 1.4. *Assume that* S *is complete local and* $S/\mathfrak{m}$ *is algebraically closed.*

(a) *If* R *is of finite CM type, then* $\Gamma = \mathrm{End}_R(S)$ *is.*

(b) *If height one primes in* R *are unramified in* S, *and* R *is of finite CM type, then* S *is.*

Proof. Part (a) follows directly from the above comments. (b) Assume R is of finite CM type. Then by our assumption and (a), SG is of finite CM type. If M is an indecomposable CM S-module, then since $|G|$ is a unit in S, M is an S-summand of $SG \otimes_S M$. Since SG is right projective S-module, $SG \otimes_S M$ is a CM SG-module, and hence a CM S-module. This shows that S is of finite CM type.

We note that if we drop the assumption that height one primes in R are unramified in S, then R may be of finite CM type while S is of infinite CM type. For example, let $f(x, y, z)$ be such that the hypersurface $S = k[[x, y, z, u]]/(f + u^2)$ is of infinite CM type (see [18]) and let $G = Z_2$ act on S by sending u to $-u$ and leaving the other variables fixed. Then $R = S^G \simeq k[[x, y, z]]$, which is of finite CM type.

Even if some height one primes in R are ramified in S, there is some relationship between Ref SG and Ref R. We first formulate our result more generally for the situation of a ring map $\Lambda \to \Gamma$.

PROPOSITION 1.5. (a) *Let* $\Lambda \to \Gamma = \mathrm{End}_R(S)$ *be a ring map, such that* Γ *is a reflexive* Λ-*module,* S *is a projective* Λ-*module, and* $\mathrm{Hom}_\Lambda(S, \Gamma) = \mathrm{Hom}_\Gamma(S, \Gamma)$. *Then the restriction induces a fully faithful functor* Ref $\Gamma \to$ Ref Λ.

(b) *The natural ring map* $SG \to \operatorname{End}_R(S) = \Gamma$ *induces by restriction a fully faithful functor* $\operatorname{Ref} \Gamma \to \operatorname{Ref} SG$, *and hence we get by composition a fully faithful functor* $\operatorname{Ref} R \to \operatorname{Ref} SG$.

Proof. (a) Since S is projective both as Γ-module and Λ-module, we get, by using $\operatorname{Hom}_\Lambda(S, \Gamma) = \operatorname{Hom}_\Gamma(S, \Gamma)$, natural isomorphisms $\operatorname{Hom}_\Lambda(S, X) \to \operatorname{Hom}_\Gamma(S, X)$ for all X in $\operatorname{mod} \Gamma$. $\Gamma = \operatorname{End}_R(S)$ is clearly a finitely generated reflexive R-module and R is integrally closed. Then it is known that a Γ-module M is Γ-reflexive if and only if it is R-reflexive. If Y is in $\operatorname{Ref} \Gamma$, then $\operatorname{Hom}_\Lambda(\Gamma, Y)$ is also in $\operatorname{Ref} \Gamma$, since it is a reflexive R-module. We have $\operatorname{Hom}_\Gamma(S, \operatorname{Hom}_\Lambda(\Gamma, Y)) \simeq \operatorname{Hom}_\Lambda(S, Y) = \operatorname{Hom}_\Gamma(S, Y)$, and since $\operatorname{Hom}_\Gamma(S, \ )$: $\operatorname{Ref} \Gamma \to \operatorname{Ref} R$ is an equivalence, we have $Y \simeq \operatorname{Hom}_\Lambda(\Gamma, Y)$. We then have $\operatorname{Hom}_\Lambda(X, Y) \simeq \operatorname{Hom}_\Gamma(X, \operatorname{Hom}_\Lambda(\Gamma, Y)) \simeq \operatorname{Hom}_\Gamma(X, Y)$ for X in $\operatorname{mod} \Gamma$. Since if X is a reflexive Γ-module it is a reflexive Λ-module, a fully faithful functor $\operatorname{Ref} \Gamma \to \operatorname{Ref} \Lambda$ is induced.

(b) Γ is a reflexive R-module, and hence a reflexive SG-module, and S is a projective SG-module since it is a projective S-module. To finish the proof we need only show $\operatorname{Hom}_\Gamma(S, \Gamma) = \operatorname{Hom}_{SG}(S, \Gamma)$.

We have $\operatorname{Hom}_{SG}(S, \Gamma) = \Gamma^G$, where the action of G on Γ is given by $(gf)(s) = gf(s)$ for $f \in \operatorname{End}_R(S)$, $s \in S$. Then clearly $\Gamma^G = \operatorname{Hom}_R(S, S)^G = \operatorname{Hom}_R(S, R) \subset \Gamma$. Consider the map α: $\operatorname{End}_R(S) \to S$ given by $\alpha(f) = f(1)$. Then $\operatorname{Ker} \alpha = \{f\colon S \to S; f(1) = 0\} = \{f\colon S \to S; f(R) = 0\}$. $\operatorname{Hom}_\Gamma(S, \Gamma) = \{f\colon S \to S; (\operatorname{Ker} \alpha) f = 0\}$. If $\operatorname{Im} f \subset R$, then clearly $(\operatorname{Ker} \alpha) f = 0$. And if $\operatorname{Im} f \not\subset R$, choose $x \in R$, $x \notin \operatorname{Im} f$. Since R is an R-summand of S and S is R-reflexive, there is an R-map $h \in \operatorname{Ker} \alpha$ with $h(x) \neq 0$. Hence $\operatorname{Hom}_\Gamma(S, \Gamma) = \operatorname{Hom}_R(S, R) \subset \Gamma$, so that $\operatorname{Hom}_{SG}(S, \Gamma) = \operatorname{Hom}_\Gamma(S, \Gamma)$.

We mention without proof that also the following subcategory of $\operatorname{Ref} SG$ is equivalent to $\operatorname{Ref} R$ via the fixed point functor. The objects are the C such that there is an exact sequence of $S[G]$-modules $0 \to C \to nS \to mS$ such that the induced sequence by taking fixed points and dualizing $(mR)^* \to (nR)^* \to (C^G)^* \to 0$ is exact.

In the complete case, we want to establish a connection between $\mathrm{CM}(R)$ and modules over some finite dimensional algebra, which we shall apply to give necessary conditions for $\mathrm{CM}(R)$ to be of finite representation type. Here f.l. $S[G]$ denotes the SG-modules of finite length. Assume S is complete, $S/\mathfrak{m}$ is algebraically closed, and height one primes in R are unramified in S, and let $\dim S = d \geqslant 3$.

PROPOSITION 1.6. *Let* $\mathfrak{D}$ *be the subcategory of* f.l. SG *consisting of the* A *with* $A^G = 0$. *If* A *is in* $\mathfrak{D}$, *then* $\alpha(A) = (\Omega^2 A)^G$ *is CM.*

Further, A *is indecomposable if and only if* $\alpha(A)$ *is indecomposable, and if* A *and* B *are in* $\mathfrak{D}$, *then* $A \simeq B$ *if and only if* $\alpha(A) \simeq \alpha(B)$.

Proof. Let A be in $\mathfrak{D}$, and let $0 \to \Omega^2 A \to P_1 \to P_0 \to A \to 0$ be the beginning of a minimal projective resolution for A over SG, where $i \geqslant 1$. Then $0 \to (\Omega^2 A)^G \to P_1^G \to P_0^G \to 0$ is an exact sequence of R-modules since $|G|$ is invertible in S. The projective SG-modules P_j are CM since SG is CM. Hence P_j is a CM R-module, and so P_j^G is, since P_j^G is an R-summand of P_j, as is seen by considering $P_j^G \to^i P_j \to^t P_j^G$, where $t(p) = 1/|G| \sum_{\sigma \in G} \sigma(p)$. It follows that $(\Omega^2 A)^G$ is a CM R-module.

Since S is CM and A has finite length, $\mathrm{Ext}^i_S(A, S) = 0$ for $i = 0, 1$. Since SG is a free S-module, $\mathrm{Ext}^i_S(A, SG) = 0$, and hence $\mathrm{Ext}^i_S(A, SG)^G = 0$ for $i = 0, 1$. Since the fixed point functor is exact, $\mathrm{Ext}^i_{SG}(A, SG) \simeq \mathrm{Ext}^i_S(A, SG)^G$, so that $\mathrm{Ext}^i_{SG}(A, SG) = 0$ for $i = 0, 1$. We then get the exact sequence $0 \to A^* \to P_0^* \to P_1^* \to (\Omega^2 A)^* \to 0$, where $X^* = \mathrm{Hom}_{SG}(X, SG)$. Dualizing again gives $0 \to \Omega^2 A \to P_1 \to P_0 \to A \to 0$, so that $P_0^* \to P_1^* \to (\Omega^2 A)^* \to 0$ is a minimal projective presentation, and also $\mathrm{Ext}^1_{SG}((\Omega^2 A)^*, SG) = A$. If $\Omega^2 A$ is indecomposable, A must be, since a module of finite length cannot have projective dimension less then 3. Assume then that A is indecomposable and that $\Omega^2 A \simeq X \amalg Y$, where X and Y are not zero. Then we have $(\Omega^2 A)^* \simeq X^* \amalg Y^*$ with X^* and Y^* not zero. Considering a minimal projective presentation for X^* and Y^*, and dualizing, we get a contradiction to the fact that the presentation $P_1 \to P_0 \to A \to 0$ is minimal. By the above we have $\mathrm{Ext}^1_{SG}((\Omega^2 A)^*, SG) \simeq A$. The rest of our claim now follows by using that the fixed point functor gives an equivalence $\mathrm{Ref}\, SG \to \mathrm{Ref}\, R$.

Let $|G| = n$ and let e be the idempotent $1/n \sum_{\sigma \in G} \sigma$. If A is an SG-module, then $eA \subset A^G$, since $\sigma e = e$ for all $\sigma \in G$. And if $x \in A^G$, then $ex = 1/n \sum \sigma x = x$, so that $A^G \subset eA$. Hence the SG-modules A with $A^G = 0$ are the $SG/(e)$-modules, so that the above category $\mathfrak{D}$ is the category f.l. $SG/(e)$. We further have that $SG/\mathfrak{m}^2 SG = (S/\mathfrak{m}^2)G$, so that $(S/\mathfrak{m}^2)G$ is a factor ring of SG, and hence $(S/\mathfrak{m}^2)\, G/(e)$ is a factor ring of $SG/(e)$. Hence we have the following consequence of the above.

THEOREM 1.7. *Let S be a complete local ring with $S/\mathfrak{m}$ algebraically closed, and assume that height one primes in R are unramified in S. If R is of finite* CM *type, then the artin algebra $(S/\mathfrak{m}^2)\, G/(e)$ is of finite representation type.*

This result motivates investigating when, for an artin algebra Λ with $\mathfrak{r}^2 = 0$, $\Lambda/\mathfrak{r} = k$ algebraically closed, and G a finite group acting, $\Lambda G/(e)$ is of finite representation type. Here ΛG denotes the skew group ring and e the usual idempotent for G. We will deal with this question in the next section.

2. The Representation Type of $\Lambda G/(e)$

Let Λ be an artin algebra with radical $\mathfrak{r}$, such that $\Lambda/\mathfrak{r}=k$ is algebraically closed and $\mathfrak{r}^2=0$. Let G be a finite group acting on Λ such that $|G|$ is invertible in Λ. G induces an action on $\mathfrak{r}=V$, which is then a kG-module. We recall from [3, 10] that the Gabriel quiver for ΛG is the opposite of the McKay quiver for V, where the McKay quiver is defined as follows. The vertices are ine one–one correspondence with the simple kG-modules $V_1=k, ..., V_n$. There are n_{ij} arrows from V_j to V_i if n_{ij} is the multiplicity of V_j in a direct sum decomposition of $V\otimes V_i$. It is easy to see that leaving out the vertex of $V_1=k$ we get (the opposite of) the Gabriel quiver for $\Lambda G/(e)$. It is a well-known theorem of Gabriel that $\Lambda G/(e)$ is of finite (representation) type if and only if the underlying graph of the separated Gabriel quiver is a finite disjoint union of the Dynkin diagrams A_n, D_n, E_6, E_7, and E_8 [16]. The separated quiver is obtained by replacing each vertex v with two vertices v' and v'' and each arrow $v\to w$ with an arrow $v'\to w''$. We will use this to give a characterization of when $\Lambda G/(e)$ is of finite type.

The following lemma reduces our considerations to abelian groups.

Lemma 2.1. *Let Λ be as above. If $\Lambda G/(e)$ is of finite type, then $\Lambda H/(e)$ is of finite type for all subgroups H of G.*

Proof. Let M be an indecomposable ΛH-module with $M^H=0$. We have $\operatorname{Hom}_{\Lambda H}(\Lambda G, M)^G \simeq \operatorname{Hom}_{\Lambda G}(\Lambda, \operatorname{Hom}_{\Lambda H}(\Lambda G, M)) \simeq \operatorname{Hom}_{\Lambda H}(\Lambda, M) = M^H=0$. Since ΛH is a two-sided ΛH-summand of ΛG, M is a summand of the ΛH-module $\operatorname{Hom}_{\Lambda H}(\Lambda G, M)$. This shows that if $\Lambda G/(e)$ is of finite type, then $\Lambda H/(e)$ is.

We now state the main result in this section.

Theorem 2.2. *Let Λ be as before, and assume that* $\dim_k V\geqslant 3$. *Then $\Lambda G/(e)$ is of finite type if and only if $G\simeq Z_2$ and the multiplicity of the nontrivial simple kG-module k_- in V is at most one.*

Proof. Assume $\dim_k V\geqslant 3$, and assume that G has an abelian subgroup H of order at least 3. If some simple summand V_1 of V (as kH-module) occurs with multiplicity $\geqslant 2$, we have $V_1\cdot\rightrightarrows\cdot k$ in the McKay quiver. Let V_2 be a simple kH-module not isomorphic to k or V_1. Then $V_i\otimes_k V_1\not\simeq k$ for $i=1$ or for $i=2$, and we have $V_i\otimes_k V_1\cdot\rightrightarrows\cdot V_i$ in the MacKay quiver, after removing k. Hence we also have a double arrow in the separated McKay quiver, showing that $\Lambda G/(e)$ is of infinite type. If every summand of V occurs with multiplicity one, there are arrows from at least three vertices to k. Tensoring we see that for any vertex there are arrows from at least three vertices. Hence when we remove k there are arrows from at least two vertices to any given vertex. And similarly there are at least two arrows leaving a given vertex. Hence in the separated quiver of the McKay quiver minus k there are either two arrows leaving a given vertex, or two arrows entering. The underlying graph can then not be a Dynkin diagram, so that we have infinite type.

It follows that if $\Lambda G/(e)$ is of finite type, then every abelian subgroup of G must have order 2. This shows that $G \simeq Z_2$, which we now assume. Let m be the multiplicity of k in V. Then there are m arrows from k to k, hence also from k_- to k_-. The separated McKay quiver minus k is then

$$\bullet \quad \begin{array}{c}\longrightarrow\\ \vdots\\ \longrightarrow\end{array} \quad \bullet$$

(m arrows)

Hence $\Lambda Z_2/(e)$ is of finite type if and only if $m = 1$. This finishes the proof.

Combining with the results in Section 1 we get the following.

THEOREM 2.3. *Let S and G satisfy the satisfy the standard assumptions of the previous section, and assume also that S is complete local, $S/\mathfrak{m}$ is algebraically closed, height one primes in R are unramified in S, and* $\dim S \geqslant 3$.

If R is of finite CM type, then $G \simeq Z_2$, and the multiplicity of the trivial kG-module k in $\mathfrak{m}/\mathfrak{m}^2$ is at most one.

3. DIMENSION THREE

In this section we assume in addition that S is complete local, $S/\mathfrak{m}$ is algebraically closed, height one primes in R are unramified in S, and $\dim R \geqslant 3$. The aim is to improve Theorem 2.3 by proving the following.

THEOREM 3.1. *If R is of finite CM type, then* $\dim R = 3$.

Proof. Assume R is of finite CM type. By Theorem 2.3 we can assume that $G = Z_2$ and k occurs with multiplicity at most one in the kG-module $V = \mathfrak{m}/\mathfrak{m}^2$. We shall need the following.

LEMMA 3.2. *The trivial SG-module k occurs with multiplicity at least 2 in $L/\mathfrak{m}L$, where $L = \Omega^2_{SG} k$.*

Proof. The exact sequence $0 \to L \to S \otimes_k V \to \mathfrak{m} \to 0$ gives a projective cover both as SG-modules and S-modules. Tensoring with $\mathfrak{m}$ over S we

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get $\operatorname{Tor}_1^S(k, \mathfrak{m}) \simeq L/\mathfrak{m}L$. Hence $0 \to \mathfrak{m} \to S \to k \to 0$ gives rise to the exact sequence $0 \to L/\mathfrak{m}L \to \mathfrak{m} \otimes_S \mathfrak{m} \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$, which again gives $L/\mathfrak{m}L \to \mathfrak{m}/\mathfrak{m}^2 \otimes_k \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^2/\mathfrak{m}^3 \to 0$.

Considering the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & J \arrow[r] \arrow[d] & V \otimes_k V \arrow[r] \arrow[d] & S_2(V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \otimes_k \mathfrak{m}/\mathfrak{m}^2 \arrow[r] & \mathfrak{m}^2/\mathfrak{m}^3 \arrow[r] & 0,
\end{tikzcd}$$

where $S_2(V)$ is the second symmetric product, it is sufficient to show that k occurs in the kG-module J with multiplicity at least 2.

Assume first that $V = nk_-$, where $n \geqslant \dim R \geqslant 3$. Since $\dim_k V \otimes_k V = n^2$ and $\dim S_2(V) = 1 + \cdots + n = n(n+1)/2$, we have $\dim_k J = n(n-1)/2 \geqslant 3$. Hence k occurs with multiplicity at least 2 in J.

If $V = k \amalg (n-1)k_-$, the multiplicity of k in $V \otimes_k V$ is $1 + (n-1)^2$, and in $S_2(V)$ it is $(1 + \cdots + n-1) + 1 = n(n-1)/2 + 1$. Hence k occurs $(n^2 - 3n + 2)/2 \geqslant 2$ times in J. This finishes the proof of the lemma.

Denote by L_- the SG-module $L \otimes_k k_-$. We know by Proposition 1.6 that $L_- = \Omega^2 k_-$ is an indecomposable SG-module, and hence $\operatorname{End}_{SG}(L_-)$ is local. The SG-modules $\mathfrak{m}L_-$ and $\mathfrak{r}L_-$ are both $\operatorname{End}_{SG}(L_-)$-submodules of L_-, where $\mathfrak{r}$ denotes the radical of $\operatorname{End}_{SG}(L_-)$, and there is the following relationship between them.

LEMMA 3.3. *With the above notation,* $\mathfrak{r}L_- \subset \mathfrak{m}L_-$.

Proof. $\operatorname{Ext}^i_{SG}(k_-, SG) \simeq \operatorname{Ext}^i_S(k_-, SG)^G = 0$ for $i = 1, 2$, so that $\underline{\operatorname{End}}_{SG}(L_-) \simeq \underline{\operatorname{End}}_{SG}(k_-) = k$ [6], and hence $\mathfrak{r} = P(L_-, L_-)$, the SG-maps from L_- to L_-, which factor through a projective SG-module. Let $f \in P(L_-, L_-)$, and let $g\colon P \to L_-$ be a projective cover in mod SG. Choose $h\colon L_- \to P$ such that $gh = f$. We must have $\operatorname{Im} h \subset \mathfrak{m}P$, and hence $\operatorname{Im} f \subset \mathfrak{m}L_-$, and this shows $\mathfrak{r}L_- \subset \mathfrak{m}L_-$.

We want to consider modules M_i with $\mathfrak{m}L_- \subset M_i \subset L_-$ and $L_-/M_i \simeq k_-$. By Lemma 3.2 there is an infinite number of choices for M_i, and we show that if $\dim R \geqslant 4$ different M_i give rise to nonisomorphic CM modules.

LEMMA 3.4. *Let M_i be as above, and assume* $\dim R \geqslant 4$. *Then $\Omega(M_i)$ is an indecomposable reflexive SG-module, and hence $\Omega(M_i)^G$ is an indecomposable CM R-module.*

Proof. The exact sequence $0 \to M_i \to L_- \to k_- \to 0$ shows that M_i is not projective. Since $\mathrm{Ext}^i_{SG}(k_-, SG)=0$ for $i=0,1,2$, we have that $\mathrm{Ext}^i_{SG}(L_-, SG) \simeq \mathrm{Ext}^i_{SG}(M_i, SG)$ for $i=0,1$, $M_i^{**} \simeq L_-^{**} \simeq L_-$, so that M_i is indecomposable. Since $\dim R \geqslant 4$, we have $\mathrm{Ext}^1_{SG}(L_-, SG) \simeq \mathrm{Ext}^3_{SG}(k_-, SG)=0$, so that $\mathrm{Ext}^1_{SG}(M_i, SG)=0$. Since M_i is indecomposable nonprojective, ΩM_i is indecomposable since $\underline{\mathrm{End}}(M_i) \simeq \underline{\mathrm{End}}(\Omega M_i)$ [6]. Since $L_- = \Omega^2 k_-$, where $k_-^G = 0$, L_-^G is a Cohen-Macaulay R-module. The exact sequence $0 \to M_i \to L_- \to k_- \to 0$ shows that $M_i^G \simeq L_-^G$, so that M_i^G is CM, and hence clearly $(\Omega M_i)^G$ is CM.

Since height one primes in R are unramified in S, $\Omega M_i \simeq \Omega M_j$ if and only if $(\Omega M_i)^G \simeq (\Omega M_j)^G$. We have an epimorphism of SG-modules f: $L_- \to 2k_-$. For each one-dimensional subspace U_j of $2k_-$, let $M_j = f^{-1}(U_j)$. We then get a family of subspaces in one–one correspondence with $\mathbf{P}^1(k)$. To finish our proof, we want to show that different M_i give rise to nonisomorphic CM modules $(\Omega M_i)^G$.

LEMMA 3.5. *In the above notation, if $M_1 \neq M_2$, then $\Omega M_1 \not\simeq \Omega M_2$.*

Proof. Assume that $\Omega M_1 \simeq \Omega M_2$. Since $\mathrm{Ext}^1_{SG}(M_i, SG)=0$ for $i=1,2$, we know that we have some isomorphism f: $M_1 \to M_2$. This gives a commutative diagram

$$\begin{array}{ccc}
M_1 & \xrightarrow[\sim]{f} & M_2 \\
\cap & & \cap \\
M_1^{**} & \xrightarrow{f^{**}} & M_2^{**} \\
\| \wr & & \| \wr \\
L_- & \xrightarrow[\sim]{g} & L_-
\end{array}$$

Since $\mathrm{End}_{SG}(L_-)/\mathfrak{r} = k$, we have $g = z + h$ with $z \in k$ and $h \in \mathfrak{r}$. $M_2 = g(M_1) \subset zM_1 + h(M_1) \subset M_1 + \mathfrak{r}M_1 \subset M_1$, and similarly $M_1 \subset M_2$. This shows that $M_1 = M_2$, and the lemma is proved.

4. A THREE-DIMENSIONAL RING OF FINITE CM TYPE

Let the assumption on S and G be as in the previous section. In this section we give two different proofs of the following.

THEOREM 4.1. *Let $G = Z_2$ act on $S = k[[X_1, X_2, X_3]]$ such that the generator sends x_i to $-x_i$. Then $R = S^G$ has up to isomorphism only the three indecomposables R, S_-^G, $\Omega^2_{SG}(k_-)$, where $S_- = S \otimes_k k_-$.*

For the first proof the following lemma is useful, where $R=S^G$ is assumed to be as in the previous section.

LEMMA 4.2. *Assume that* $\dim R=3$ *and* R *is an isolated singularity, that is,* R_p *is regular local for all nonmaximal prime ideals* p *in* R. *If* A *is in* $\operatorname{Ref} SG$, *then* A^G *is a CM* R-*module if and only if* $\operatorname{Ext}^1_{SG}(A^*, SG)^G=0$.

Proof. Let A be in $\operatorname{Ref} SG$. Then the exact sequence $0\to \Omega A^*=K\to Q\to A^*\to 0$, where Q is a projective SG-module, gives rise to the exact sequence $0\to A\to Q^*\to K^*\to \operatorname{Ext}^1_{SG}(A^*, SG)\to 0$. Since R is an isolated singularity and height one primes in R are unramified in S, S is also an isolated singularity [1]. If p is a nonmaximal prime ideal in R, we then have $\operatorname{gl\,dim} S_pG=\operatorname{gl\,dim} S_p\leqslant 2$, so that A_p^* is a projective S_pG-module, since A^* is a reflexive SG-module. Hence $\operatorname{Ext}^1_{SG}(A^*, SG)$, and consequently $\operatorname{Ext}^1_{SG}(A^*, SG)^G$, has finite length. The exact sequence of R-modules $0\to A^G\to Q^{*G}\to K^{*G}\to \operatorname{Ext}^1_{SG}(A^*, SG)^G\to 0$ then shows that A^G is CM if and only if $\operatorname{Ext}^1_{SG}(A^*, SG)^G=0$, and we are done.

Proof of Theorem 4.1. We first note that $R=S^G$ is known to be an isolated singularity, and the height one primes in R are unramified in S since the action of Z_2 on $V=\mathfrak{m}/\mathfrak{m}^2$ is free. Now let A be an indecomposable nonprojective reflexive SG-module. Since $\operatorname{gl\,dim} SG=3$ and A^* is a reflexive SG-module, we have $\operatorname{pd}_{SG} A^*=1$. This shows that $\operatorname{Ext}^1_{SG}(A^*, SG)\neq 0$. From Section 2 we know that $SG/(e)/\mathfrak{m}^2SG/(e)=k$, so that $SG/(e)=k$. This means that k_- is the only indecomposable SG-module with $k_-^G=0$, so that $\operatorname{Ext}^1_{SG}(A^*, SG)=nk_-$ for some $n>0$. Since A is indecomposable and $\operatorname{pd}_{SG} k_-=3$, nk_- must be indecomposable. It follows that $A\simeq \Omega^2 k_-$, so that $A^G\simeq(\Omega^2 k_-)^G$. Since S and S_- are the only indecomposable projective SG-modules, we get our desired result from Lemma 4.2.

We now give a proof of Theorem 4.1 using the method discussed in Part I.

Let $V=\mathfrak{m}/\mathfrak{m}^2$, and consider the Koszul complex $0\to S\otimes_k \Lambda^3 V\to S\otimes_k \Lambda^2 V\to S\otimes_k V\to S\to k\to 0$. This is also an exact sequence of SG-modules, with the natural action of G on S and V. From this we get the exact sequence $0\to S\to S_-^3\to S^3\to S_-\to k_-\to 0$, and hence the exact sequence of R-modules $0\to R\to (S_-^G)^3\to (S^G)^3\to S_-^G\to 0$. Writing $S_-^G=\omega$, we get the two short exact sequences $0\to R\to \omega^3\to L^G\to 0$ and $0\to L^G\to R^3\to \omega\to 0$. Dualizing the first one we get an exact sequence $0\to DL^G\to R^3\to \omega\to 0$, which must be isomorphic to $0\to L^G\to R^3\to \omega$. Since the class group $C(R)$ is isomorphic to $G=Z_2$ (see [9]), $\omega^*\simeq\omega$. We then compute $\operatorname{Tr}_L \omega=\Omega\omega^*=\Omega\omega=L^G$. Hence we have $\operatorname{Tr}_L L^G=\omega$, so that

$D\operatorname{Tr}_L\omega = DL^G \simeq L^G$ and $D\operatorname{Tr}_L L^G \simeq R$. From the exact sequence $S^3 \to S_- \to k \to 0$ we see that any nonisomorphism $g\colon S_- \to S_-$ factors through $S^3 \to S_-$. Hence it follows from the equivalence $\operatorname{Ref} SG \to \operatorname{Ref} R$ that any nonisomorphism $h\colon \omega \to \omega$ factors through $R^3 \to \omega$. Since $D\operatorname{Tr}_L\omega \simeq L^G$, we get that $0 \to L^G \to R^3 \to \omega \to 0$ is almost split [8]. Then the dual sequence $0 \to L^G \to R^3 \to \omega \to 0$ is also almost split. It is now a direct consequence of Theorem 1.1 in Part I that R, ω, L^G are the only indecomposables in CM R.

5. A Consequence of Finite CM Type

Let the assumption on S and G be as in the previous section, and assume in addition that R is an isolated singularity and $\dim R = 3$. We shall derive a consequence of R being of finite CM type by considering the torsionless SG-modules. They have the following relationship with CM R-modules.

Lemma 5.1. *Let $\mathscr{T}$ be the category of torsionless SG-modules T such that* $\operatorname{Ext}^1_{SG}(T, SG) = 0$.

(a) *$\Omega\colon \underline{\mathscr{T}} \to \underline{\operatorname{Ref} SG}$ is an equivalence of categories.*

(b) *If T is in $\mathscr{T}$, then $\Omega(T)^G$ is CM if and only if $(T^{**}/T)^G = 0$.*

Proof. Part (a) follows from [6]. Since $\Omega(T)$ is reflexive, we know that $\Omega(T)^G$ is CM if and only if $\operatorname{Ext}^1_{SG}((\Omega T)^*, SG)^G = 0$. We know that $T^{**}/T \simeq \operatorname{Ext}^2_{SG}(\operatorname{Tr} T, SG)$ (see [6]). Since $\operatorname{Ext}^1_{SG}(T, SG) = 0$, the exact sequence $0 \to \Omega(T) \to P \to T \to 0$ with P projective gives an exact sequence $0 \to T^* \to P^* \to \Omega(T)^* \to 0$, so that $\Omega(T)^* = \Omega(\operatorname{Tr} T)$. Hence $T^{**}/T \simeq \operatorname{Ext}^1_{SG}((\Omega T)^*, SG)$, so that (b) follows.

Since the fixed point functor $\operatorname{Ref} SG \to \operatorname{Ref} R$ is an equivalence, it follows that there is an infinite number of indecomposable CM R-modules if there is an infinite number of indecomposable torsionless T such that $\operatorname{Ex}^1_{SG}(T, SG) = 0$ and $(T^{**}/T)^G = 0$. The following construction of new modules having this property, starting with a reflexive module, will be useful.

Lemma 5.2. *Let B be an indecomposable reflexive SG-module with* $\operatorname{Ext}^1_{SG}(B, SG) = 0$. *If T is a submodule of B with $(B/T)^G = 0$, then T is torsionless,* $\operatorname{Ext}^1_{SG}(T, SG) = 0$, *and* $(T^{**}/T)^G = 0$.

Proof. Since $(B/T)^G = 0$ and S is an isolated singularity, B/T has finite length. The exact sequence $0 \to T \to B \to B/T \to 0$ gives rise to the exact sequence $\operatorname{Ext}^1_{SG}(B, SG) \to \operatorname{Ext}^1_{SG}(T, SG) \to \operatorname{Ext}^2_{SG}(B/T, SG)$. We have that

$\operatorname{Ext}^2_{SG}(B/T, SG) = 0$ since B/T has finite length and $\dim S = 3$, and by assumption $\operatorname{Ext}^1_{SG}(B, SG) = 0$. Hence we get $\operatorname{Ext}^1_{SG}(T, SG) = 0$. That B/T has finite length also implies $T^{**} = B$, so that $(T^{**}/T)^G = 0$.

To find the reflexive modules B with the desired property we shall need the following.

LEMMA 5.3. *Let A be a reflexive SG-module. Then* $\operatorname{Ext}^1_{SG}(A, SG) = 0$ *if and only if A^* is a CM SG-module.*

Proof. Choose $x \neq 0$ in R. The exact sequence of SG-modules $0 \to SG \xrightarrow{\cdot x} SG \to (S/xS)\, G \to 0$ gives the exact sequence $0 \to \operatorname{Hom}_{SG}(A, SG) \xrightarrow{\cdot x} \operatorname{Hom}_{SG}(A, SG) \to \operatorname{Hom}_{SG}(A, (S/xS)\, G) \to \operatorname{Ext}^1_{SG}(A, SG) \to \cdots$. Since $(S/xS)G$ has an R-sequence of length 2, $\operatorname{Hom}_{SG}(A, (S/xS)\, G)$ also does. Hence $A^* = \operatorname{Hom}_{SG}(A, SG)$ is CM if $\operatorname{Ext}^1_{SG}(A, SG) = 0$. Since $\operatorname{Ext}^1_{SG}(A, SG)$ has finite length because S is an isolated singularity, A^* is not Cohen–Macaulay if $\operatorname{Ext}^1_{SG}(A, SG) \neq 0$.

We are now ready to prove the following consequence of finite CM type.

THEOREM 5.4. *In addition to the previous assumptions, assume that $G = Z_2$ and R is of finite CM type. Let A be an indecomposable reflexive SG-module where $G = Z_2$ such that A^* is Cohen–Macaulay. Then* $\dim_k A/(\mathfrak{r}A + \mathfrak{m}A) \leqslant 2$, *and if* $\dim_k A/(\mathfrak{r}A + \mathfrak{m}A) = 2$, *then* $A/(\mathfrak{r}A + \mathfrak{m}A) \simeq k \amalg k_-$. *Here* $\mathfrak{m} = \operatorname{rad} S$ and $\mathfrak{r} = \operatorname{rad} \operatorname{End}_{SG} A$.

Proof. Assume first that k_- occurs with multiplicity $m > 1$ in $A/(\mathfrak{r}A + \mathfrak{m}A)$. Then we have an infinite family $\{A_i\}$ of SG-submodules of A containing $\mathfrak{r}A + \mathfrak{m}A$. We want to show that if $A_i \neq A_j$, then $A_i \not\simeq A_j$. For assume $f\colon A_i \to A_j$ is an isomorphism. Consider as in Lemma 3.5 the diagram

$$\begin{array}{ccc} A_i & \xrightarrow[\sim]{f} & A_j \\ \cap & & \cap \\ A_i^{**} & \xrightarrow[\sim]{f^{**}} & A_j^{**} \\ \uparrow\wr & & \uparrow\wr \\ A & \xrightarrow[\sim]{g} & A. \end{array}$$

$\operatorname{End}_{SG} A$ is local and $\operatorname{End}_{SG}(A)/\mathfrak{r} = k$. We write $g = z + h$ with $z \in k$ and $h \in \mathfrak{r}$. Then we get $A_j = g(A_i) \subset zA_i + h(A_i) \subset A_i$, so that $A_i = A_j$. Hence an infinite number of different A_i gives rise to an infinite number of non-isomorphic reflexives ΩA_i, since $\operatorname{Ext}^1_{AG}(A_i, SG) = 0$. Since we know that the

$(\Omega A_i)^G$ are nonisomorphic indecomposable Cohen–Macaulay R-modules, this shows that $m \leqslant 1$.

By considering $A_- = A \otimes_k k_-$, we see that also k occurs with multiplicity at most one in $A/(mA + \mathfrak{r}A)$. This finishes the proof.

As an application of Theorem 5.4 we give the following.

PROPOSITION 5.5. *Let S be the scroll of type* (2, 1), *and let $G = Z_2$ act on $\mathfrak{m}/\mathfrak{m}^2$ by sending each element to its negative. Then $R = S^G$ is of infinite CM type.*

Proof. We recall that $S = k[[X_0, X_1, X_2, Y_0, Y_1]]/I$, where $I = (X_0 X_2 - X_1^2, X_0 Y_1 - X_1 Y_0, X_1 Y_1 - X_2 Y_0)$. Since the action on S is induced by a free action of G on $U = k[[X_0, X_1, X_2, Y_0, Y_1]]$, nonmaximal primes in U^G are unramified in U, and hence height one primes in $R = S^G$ are unramified in S. Consider the S-module $B = (x_0, x_1, y_0)$, which is also an SG-module, and it is reflexive. $\mathrm{Hom}_{SG}(B, SG) \simeq \mathrm{Hom}_S(B, S) = A = (x_0, x_1)$ is a CM SG-module. We also know that $\underline{\mathrm{End}}(B) = k$, so that $\mathfrak{r}B \subset \mathfrak{m}B$. Hence $\dim_k B/(\mathfrak{r}B + \mathfrak{m}B) = 3$, which shows that R is of infinite CM type.

Theorem 5.4 can also be used to show that $R = \mathbb{C}[[X, Y, Z, T]]/(X^{2m} + Y^2 + Z^2 + T^2)^G$ is of finite CM type, when the action of G is induced by sending each variable to its negative. If m is odd, we can choose the CM ideal $B = (X^m + iT,\ Y + iZ)$, which is then clearly an SG-module. If m is odd, one can use [18] to find a module B of rank 2 which can be used. We point out, however, that it is already known from [17] that R is of infinite CM type, since it is easy to see that R is a Gorenstein ring which is not an hypersurface.

We end this section with another sufficient condition for infinite CM type, based on using almost split sequences, similar to an argument of Bongartz for finite dimensional algebras.

PROPOSITION 5.6. *Let R be a complete integrally closed local CM domain, with $R/\mathrm{rad}\, R = k$ algebraically closed, which is an isolated singularity. Assume that there is some almost split sequence $0 \to A \to B \amalg B \amalg X \to C \to 0$, where B is indecomposable in CM R.*

(a) *If $rkB < rkC$ and B is not injective, then R is of infinite CM type.*

(b) *If $rkA > rkB$ and B is not projective, then R is of infinite CM type.*

Proof. (a) If B is not injective, we have an almost split sequence $0 \to B \to C \amalg C \amalg Y \to \tau^{-1}B \to 0$, and if $rkB < rkC$, then $rkC < rk(\tau^{-1}B)$. Since $rkC > rkB \geqslant 1$, C is not isomorphic to the dualizing module ω. Hence we have an almost split sequence $0 \to C \to \tau^{-1}B \amalg \tau^{-1}B \amalg Z \to \tau^{-1}C \to 0$. Continuing this way, we get indecomposable CM modules of arbitrarily large rank, and we are done. The proof of (b) is analogous.

We note that in both examples of rings R of finite CM type given in this paper, we have almost split sequences with a repeated term in the middle. For the scroll of type (2, 1) we have, in the notation of Part I, almost split sequences $0 \to R \to A \amalg A \amalg B \to K \to 0$ and $0 \to K \to R \amalg R \amalg C \to A \to 0$. But in the first case A is injective and $rkR = rkA$, and in the second case R is projective and $rkR = rkA$.

For the fixed ring in Section 4 we have the almost split sequences $0 \to L^G \to R^3 \to \omega \to 0$ and $0 \to R \to \omega^3 \to L^G \to 0$. In the first case R is projective, in the second case ω is injective, and $rkR = rk\omega$.

We illustrate Proposition 5.6 by sketching a different proof for the fact mentioned above that $R = \mathbb{C}[[X, Y, Z, T]]/(X^2 + Y^2 + Z^2 + T^2)^{Z_2}$ is of infinite CM type. R can be shown to be isomorphic to the subring $\mathbb{C}[[x^2s^2, xys^2, y^2s^2, x^2st, xyst, y^2st, x^2t^2, xyt^2, y^2t^2]]$ of $\mathbb{C}[[x, y, s, t]]$. Here there is an almost split sequence $0 \to E \to K \amalg K \amalg X \to \operatorname{Tr} E \to 0$, where $E = (xs, xt, ys, yt)$, $K = (xs^2, xst, xt^2, ys^2, yst, yt^2)$, and $rk \operatorname{Tr} E = 3$. Since K is not injective, Proposition 5.6 applies.

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