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Symmetries of the Fisher–Kolmogorov–Petrovskii–Piskunov equation with a nonlocal nonlinearity in a semiclassical approximation

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ABSTRACT

The classical group analysis approach used to study the symmetries of integro-differential equations in a semiclassical approximation is considered for a class of nearly linear integro-differential equations. In a semiclassical approximation, an original integro-differential equation leads to a finite consistent system of differential equations whose symmetries can be calculated by performing standard group analysis.

The approach is illustrated by the calculation of the Lie symmetries in explicit form for a special case of the one-dimensional nonlocal Fisher–Kolmogorov–Petrovskii–Piskunov population equation.

1. Introduction

The mathematical models used for studying nonlocal interactions in physical, chemical, and biological systems are based on nonlinear integro-differential equations (IDEs).

The important IDEs widely used in applications are kinetic equations. A detailed review of kinetic phenomena and their modeling in plasma physics, in rarefied gas dynamics, and the other physical systems can be found elsewhere [1,2].

The theory of Bose–Einstein condensates substantially employs the Gross–Pitaevskii equation (GPE) [3]. The nonlocal GPE describes the evolution of coherent quantum ensembles of dipolar quantum gases with long-range dipole–dipole interaction which gives rise to novel properties of quantum matter (see, e.g., [4], and references therein).

The Fokker–Planck equation with a nonlocal nonlinearity was used in a stochastic theory involving feedback and a nonlinear family of Markov diffusion processes [5].

The classical Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) population equation [6,7] has been used in mathematical biology to explain the space–time evolution of microbiological population densities (bacteria or cells) due to the diffusion mechanism. To take into consideration long-range interactions of individuals typical of the colonial organization of microbial populations [8], nonlocal generalizations of the FKPP equation are used [9]. Nonlocal models are aimed, in particular, at describing the pattern formation in bacterial colonies [9]. This contributes to the study of micro-morphogenesis, which is of particular interest in the fundamental problems of modern microbiology [8].

The symmetry groups of IDEs are calculated by direct or indirect methods [10]. Algorithms of indirect calculation are based on replacing the input nonlocal IDE with a system of partial differential equations (PDEs). The system is then analyzed by the standard methods of the classical Lie group analysis of PDEs [11–14]. Nonlocal equations can be reduced to PDEs by

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using the method of moments or a covering method, amongst others (see, e.g., [10]). The first method was used to calculate the Lie point symmetry group for the Vlasov–Maxwell equations in plasma theory [15] and for the Benney, Vlasov-type, and Boltzmann-type kinetic equations [16]. The covering method was developed [17] and applied to a coagulation kinetic equation.

Direct methods were developed and also applied to the Boltzmann equation, equations of motion of viscoelastic media, the Benney, and the Vlasov–Maxwell equations (see [1,10,18,19] and references therein).

Advances in group analysis were achieved in the theory of one-parameter and multi-parameter approximate transformation groups initiated by Baikov et al. [20]. Similar ideas were suggested by Fushchych et al. [21]. Approximate symmetries involve a small parameter and are calculated for PDEs with or without a small parameter. The approximate symmetries were found, for example, for the Boussinesq equation [22], for nonlinear wave equations [23], and for other types of equation [20].

The area of nonlocal methods has attracted research because of new prospects for the development of symmetry analysis. Back in the 1990s, Fushchych and Shtelen [24] considered nonlocal transformations generated by linear differential operators. Fushchych et al. [25] used nonlocal ansätze to reduce nonlinear PDEs to equations with a lower number of independent variables. The nonlocal ansätze were shown to relate to conditional (nonclassical) symmetries of PDEs [26–29]. Akhatov et al. studied contact and quasilocal symmetries for nonlinear diffusion equations [30].

These ideas have been developed by Bluman et al. [31], Bluman and Cheviakov [32], Popovych et al. [33], Kunzinger and Popovych [34], Boyko [35], Zhdanov [36], and others.

Note that, in all the above references, the approximate symmetries and solutions of the equations under consideration were regularly dependent on a small parameter.

However, the same operator is known to possess different properties in different classes of functions (e.g., a differentiation operator is bounded on $\mathcal{C}_2$ and unbounded on L_2). As a consequence, the symmetries themselves and the way of their calculation depend on the class of functions for which the equation operator is defined. This leads to the idea of defining the equation operator in a class of functions singularly depending on a small asymptotic parameter and of finding the relevant approximate solutions, symmetries, conservation laws, and symmetry operators.

Such classes of functions are used to construct solutions of PDEs in a semiclassical approximation in the context of the Maslov canonical operator method [37], of the complex germ method [38,39], or of the generalized adiabatic method [40].

The main idea of this paper is to consider a technique which admits classical group analysis methods to be applied to study the symmetries of IDEs with the use of a semiclassical approximation. We consider here a special class of equations: nearly linear IDEs. In a semiclassical approximation, an original IDE leads to a finite consistent system of differential equations whose symmetries can be investigated by standard group analysis [11–14].

In the next section, the construction of a consistent finite system is considered in the framework of a semiclassical approximation for a nearly linear one-dimensional IDE of general form.

The scheme for calculating the symmetries of the consistent system is given in Section 2, where the integral constraints that arise in this approach are also discussed.

In Sections 3 and 4, the general ideas are illustrated by a simple but nontrivial example of a one-dimensional nonlocal FKPP equation of particular type. The Lie symmetries and the corresponding similarity solution are found in explicit form.

2. The consistent system and a semiclassical approximation

Consider an rth-order evolution IDE with a small asymptotic parameter ε as a factor of the partial derivatives, i.e.,

$$\hat{L}[u](t,x,\varepsilon) = 0, \tag{2.1}$$

$$\hat{L}[u](t,x,\varepsilon) = -\varepsilon u_t(t,x) + \hat{F}[u,I](t,x,\varepsilon), \tag{2.2}$$

$$\hat{F}[u,I](t,x,\varepsilon) = F(t,x,u(t,x),\varepsilon u_x(t,x),\ldots,\varepsilon^r u_{xx\cdots x}(t,x);\hat{I}[u](t,x),\varepsilon), \tag{2.3}$$

where the smooth real scalar function $u(t,x)$ depends on time t and belongs to the Schwarz space $\mathcal{S}$ in the space variable x. Here $\hat{I}[u](t,x) = (\hat{I}_1[u](t,x),\ldots,\hat{I}_l[u](t,x))$,

$$\hat{I}_k[u](t,x) = \int_{-\infty}^{\infty} b_k(t,x,y)u(t,y)dy, \quad k = \overline{1,l}; \tag{2.4}$$

$b_k(t,x,y)$ is a smooth function of t,x,y growing with x,y no faster than a polynomial; $u_x = \partial u(t,x)/\partial x$; and $u_{xx} = \partial^2 u(t,x)/\partial x^2, \ldots$

Denote by $t,x,u,u_1,u_2,\ldots,u_r; I_1,\ldots,I_l$ a set of real independent variables. A collection of variables $z_r = (x,u,u_1,u_2,\ldots,u_r)$ can be assigned to as a point of the jet space $J^{(r)}$ (see, e.g., [13]). The right-hand side of (2.3) is determined by a function

$$F(t,x,u,u_1,u_2,\ldots,u_r;I_1,\ldots,I_l) = F(t,z_r,I,\varepsilon), \tag{2.5}$$

smooth in its arguments, $I = (I_1, \dots, I_l)$. Function (2.5) is a symbol of the operator $\hat{F}$. Note that $u_1 = \hat{D}_x u, \dots, u_k = \hat{D}_x u_{k-1}$, where

$$\hat{D}_x = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \cdots \tag{2.6}$$

is the operator of total differentiation with respect to x acting on the functions depending on the jet space variables $x, u_1, u_2, \dots$

In the method of moments (e.g., [10]), a dynamic system of ODEs governing the evolution of the moments

$$\langle x^{(k)} \rangle (t) = \int_{-\infty}^{\infty} x^k u(t,x) dx, \quad k = \overline{0, \infty}, \tag{2.7}$$

is deduced from the input IDE and is examined by standard group analysis methods [11–14]. Practical use of the method of moments faces significant problems. The complete set of moments is infinite, so the existence of solutions, the integrability, and the calculation of symmetries involve fundamental problems. Another traditional problem with the method is how to recover a solution from its moments [41]. To get around these problems, we construct approximate solutions for Eqs. (2.1)–(2.3).

It can be thought that the small parameter ε implies small gradients in Eqs. (2.2) and (2.3). But this is true only if solutions of Eq. (2.1) are sought in a class of functions regular in the parameter ε. When solutions of Eq. (2.1) are constructed in a class of functions singular in ε, the partial derivatives are not supposed to be small. For a class of functions singular in ε, we take the class of trajectory concentrated functions

$$\mathcal{P}_t^{\varepsilon}(X(t,\varepsilon), S(t,\varepsilon)) = \left\{ u \,\middle|\, u(t,x,\varepsilon) = \varphi\left(\frac{\Delta x}{\sqrt{\varepsilon}}, t, \varepsilon\right) \exp\left[\frac{1}{\varepsilon} S(t,\varepsilon)\right] \right\}. \tag{2.8}$$

Here, the real function $\varphi(\eta, t, \varepsilon)$ belongs to the Schwartz space $\mathcal{S}$ in the variable $\eta \in \mathbb{R}$ and smoothly depends on $\sqrt{\varepsilon}$ as $\varepsilon \to 0$; $\Delta x = x - X(t,\varepsilon)$; and the real functions $S(t,\varepsilon)$ and $X(t,\varepsilon)$ characterizing the class $\mathcal{P}_t^{\varepsilon}(X(t,\varepsilon), S(t,\varepsilon))$ regularly depend on $\sqrt{\varepsilon}$ in a neighborhood of $\varepsilon = 0$, and are to be determined. Where this does not lead to confusion, we use the shorthand notation $\mathcal{P}_t^{\varepsilon}$ for (2.8), and write $X(t)$, $S(t)$ instead of $X(t,\varepsilon)$, $S(t,\varepsilon)$.

For the functions u of the class $\mathcal{P}_t^{\varepsilon}$, the following asymptotic estimates are valid [42]:

$$\frac{\|\hat{\Delta}_{k,l}(t,\varepsilon)u\|}{\|u\|} = O(\varepsilon^{(k+l)/2}), \qquad \frac{\|\widehat{T}_0 u\|}{\|u\|} = O(\varepsilon), \tag{2.9}$$

where

$$\widehat{T}_0 u = [\varepsilon \partial_t + \varepsilon \dot{X}(t) \partial_x - \dot{S}(t)]u, \tag{2.10}$$

$\|\cdots\|$ is the norm in L_2, and $\hat{\Delta}_{k,l}(t,\varepsilon)$ is a linear operator: $\Delta_{k,l}(\varepsilon\partial_x, t, x, \varepsilon) = (\varepsilon\partial_x)^k (x - X(t))^l$, $k, l \in \mathbb{N}$.

With these estimates, Eq. (2.1) can be written as a formal power series,

$$\left(\hat{L}_0[u] + \sqrt{\varepsilon}\hat{L}_1[u] + \varepsilon \hat{L}_2[u] + \cdots\right)(t,x) = 0, \tag{2.11}$$

$$u(t,x) = u^{(0)}(t,x) + \sqrt{\varepsilon} u^{(1)}(t,x) + \varepsilon u^{(2)}(t,x) + \cdots, \tag{2.12}$$

where $u^{(k)}(t,x) \in \mathcal{P}_t^{\varepsilon}$ and

$$\frac{\|\hat{L}_k[u^{(k)}]\|}{\|u^{(k)}\|} = O(\varepsilon). \tag{2.13}$$

Note that the estimates (2.12) do not prevent the parameter ε from entering both in the operators $\hat{L}_k$ and in the functions $u^{(k)}(t,x)$. For the operator $\hat{L}$ of the form (2.3), the principal term $u^{(0)}(t,x)$ in the asymptotic expansion (2.12) is estimated as [43]

$$\hat{L} u^{(0)}(t,x) = O(\varepsilon^{3/2}). \tag{2.14}$$

Problem (2.11)–(2.13) is solved in the context of perturbation theory:

$$\hat{L}_0[u^{(0)}](t,x) = 0, \tag{2.15}$$

$$\hat{L}'_0[u^{(0)}](t,x) u^{(1)} + \hat{L}_1[u^{(0)}](t,x) = 0, \tag{2.16}$$

$$\dots,$$

where the linear operator $\hat{L}'[u]$ is the Frechet derivative of the nonlinear operator $\hat{L}$ calculated for $u \in \mathcal{P}_t^{\varepsilon}$.

It is seen from (2.15)–(2.16) that the principal term $u^{(0)}$ is found from the nonlinear problem (2.15) that should be solved exactly. The higher-order terms, $u^{(1)}, u^{(2)}, \ldots$, of the expansion (2.12) are found from the linear equations (2.16). Consequently, the nonlinear problem for the principal term $u^{(0)}$ is the key point in constructing an asymptotic solution, and below we focus on this problem. The solution of the linear problems is beyond of the scope of this work.

Let us obtain an analog to Eq. (2.15) in a semiclassical approximation. In constructing the semiclassical asymptotics (2.12), the moments $m_u(t)$, $x_u(t)$, and the centered moments $\alpha_u^{(k)}(t)$ are used instead of (2.7). These moments are defined as

$$m_u(t) = \int_{-\infty}^{\infty} u(t,x)dx, \tag{2.17}$$

$$x_u(t) = \frac{1}{m_u(t)} \int_{-\infty}^{\infty} xu(t,x)dx, \tag{2.18}$$

$$\alpha_u^{(k)}(t) = \frac{1}{m_u(t)} \langle \Delta x^k \rangle (t) = \frac{1}{m_u(t)} \int_{-\infty}^{\infty} (x - X(t,\varepsilon))^k u(t,x)dx, \quad k \geqslant 2. \tag{2.19}$$

For $u(t,x) \in \mathcal{P}_t^{\varepsilon}$, the integrals (2.17)–(2.19) do exist; then we can estimate $m_u(t)$, $x_u(t)$ for $\varepsilon \to 0$ as

$$m_u(t) = O(1), \qquad x_u(t) = O(1). \tag{2.20}$$

If $X(t,\varepsilon) = x_u(t)$, we have, from (2.9),

$$\alpha_u^{(k)}(t) = O(\varepsilon^{k/2}), \quad k \geqslant 2. \tag{2.21}$$

Expanding $b_k(t,x,y)$ in (2.4) into a formal power series in the variables $\Delta x = x - x_u(t)$ and $\Delta y = y - x_u(t)$,

$$I_k[u](t,x) = m_u(t) \sum_{l=j=0}^{\infty} \frac{1}{l!j!} \frac{\partial^{l+j} b_k(t,x,y)}{\partial x^l \partial y^j}\bigg|_{x=y=x_u(t)} \Delta x^l \alpha_u^{(j)}(t),$$

we rewrite the operator $\hat{F}$ (defined by (2.3)) in Eq. (2.1) as

$$\begin{aligned} \hat{F}[u,I](t,x,\varepsilon) &= \hat{\tilde{F}}[u,\Theta](t,x,\varepsilon) \\ &= \tilde{F}(t,x,u(t,x),\varepsilon u_x(t,x),\varepsilon^2 u_{xx}(t,x),\ldots,\varepsilon^r u_{xx\cdots x}(t,x);\Theta_u(t);\varepsilon), \end{aligned} \tag{2.22}$$

where

$$\Theta_u(t;\varepsilon) = \big(m_u(t;\varepsilon), x_u(t;\varepsilon), \alpha_u^{(2)}(t;\varepsilon), \ldots\big) \tag{2.23}$$

are the moments (2.17)–(2.19). In this notation, Eq. (2.1) reads

$$-\varepsilon u_t(t,x) + \hat{\tilde{F}}[u,\Theta](t,x,\varepsilon) = 0. \tag{2.24}$$

To find the principal term $u^{(0)}$ of (2.12), we expand the coefficients of Eq. (2.24) in a power series of $\Delta x = x - x_u(t)$, truncating the series after the term of the order of $O(\varepsilon)$ according to estimates (2.9). In view of estimates (2.20), (2.21), we also keep only the moments $m_u(t)$, $x_u(t)$, $\alpha^{(2)}(t)$ in Eq. (2.24). As a result, the operator $\hat{\tilde{F}}$ is reduced to $\hat{\tilde{F}}_0$ quadratic in coordinates and derivatives. Then Eq. (2.24) will be written as

$$-u_t(t,x) + \hat{H}_0[u](t,x) = 0, \tag{2.25}$$

where $\hat{H}_0[u](t,x) = \varepsilon^{-1}\hat{\tilde{F}}_0[u,\Theta](t,x,\varepsilon)$. From Eq. (2.25), the dynamical system

$$-\dot{\Theta}_u(t;\varepsilon) + \Gamma(\Theta_u(t;\varepsilon),t;\varepsilon) = 0 \tag{2.26}$$

governing the evolution of the moments $\Theta_u(t) = (m_u(t), x_u(t), \alpha^{(2)}(t))$ is deduced. The right-hand side of (2.26),

$$\Gamma(\Theta_u(t;\varepsilon),t;\varepsilon) = (f(\Theta_u(t;\varepsilon),t;\varepsilon), g(\Theta_u(t;\varepsilon),t;\varepsilon), h(\Theta_u(t;\varepsilon),t;\varepsilon)), \tag{2.27}$$

consists of the functions f, g, and h, corresponding to the moments $m_u(t;\varepsilon)$, $x_u(t;\varepsilon)$, and $\alpha_u^{(2)}(t;\varepsilon)$, respectively. We call Eqs. (2.26) *the Einstein–Ehrenfest* (EE) dynamical system.

By introducing the aggregate variable

$$w(t,x) = (u(t,x), \Theta_u(t)), \tag{2.28}$$

we rewrite system (2.25) and (2.26) as a *consistent system* combining the partial differential equations (2.25) and the dynamical EE system (2.26):

$$\hat{\mathcal{L}}[w](t,x) = 0. \tag{2.29}$$

Here,

$$\hat{\mathcal{L}}[w] = (-u_t + \hat{H}^{(0)}[u], -\dot{\Theta}_u + \Gamma(\Theta_u, t)). \tag{2.30}$$

Therefore, in the semiclassical approximation, the original IDE (2.1) and (2.2) results in the consistent system (2.29) that determines the principal term in the semiclassical expansion (2.12).

Since the consistent system contains a finite number of equations, its symmetry can be investigated by a standard method of classical group analysis and can be found exactly [11–14].

Such schemes for constructing semiclassical asymptotic solutions have been developed for linear quantum mechanical equations (see, e.g., [44]). For the linear case $\hat{L}'[u] = \hat{L}$, the asymptotic solutions can be found explicitly if the unperturbed problem (2.15) can be solved exactly. Thus, to obtain an exact solution of the unperturbed problem (2.25) and (2.29) is the key point of the construction of an asymptotic solution to the input Eq. (2.1).

The nonlocal Gross–Pitaevskii equation (known as the Hartree-type equation in the mathematical literature) [42], the nonlocal Fokker–Planck equation [45], and the nonlocal FKPP equation [43] are examples of nonlinear equations for which semiclassical asymptotics have been constructed.

These equations belong to a special class of nearly linear IDEs. Eq. (2.1) is a nearly linear equation if the operator $\hat{H}$ (2.3) is linear in $u(t, x), u_x(t, x), \dots, u_{xx\cdots x}(t, x)$ with the coefficients depending on $t, x, I[u](t, x)$.

Note that by a nearly linear equation is meant a nonlinear equation admitting a class of solutions which tend to the solutions of the corresponding linear equation with $\varkappa \to 0$ [46].

Symmetry analysis [11–14] employed together with a semiclassical approximation formalism [38,39] provides an approach to find semiclassical asymptotics for a wide class of nonlinear IDEs. An investigation of this type was carried out for linear equations of nonrelativistic quantum mechanics [47].

The construction of semiclassical asymptotics [38,39] requires only a finite approximation of the consistent system (2.24) and (2.26).

Let us turn to calculating the symmetries of the consistent system.

3. Symmetries of the consistent system

The symmetry of Eq. (2.29) is a nonlinear operator $\hat{\chi}$ such that

$$\hat{\mathcal{L}}[w] = 0 \Rightarrow \hat{\mathcal{L}}[w + s\hat{\chi}[w]] = o(s). \tag{3.1}$$

Here, s is a real parameter of a local one-parametric Lie group of symmetry operators of Eq. (2.29) with the generator $\hat{\chi}$ acting on an arbitrary solution w from (2.28) of the consistent system (2.29), (2.30). The operator $\hat{\chi}$ is defined as $\hat{\chi}[w] = (\hat{\sigma}[u](t, x), \hat{\vartheta}[\Theta_u](t))$, where the operators $\hat{\sigma}$ and $\hat{\vartheta}$ act on $u(t, x)$ and $\Theta_u(t)$, respectively.

The determining equation for $\hat{\chi}$ is

$$\hat{\mathcal{L}}[w] = 0 \Rightarrow \hat{\mathcal{L}}'[w]\hat{\chi}[w] = 0, \tag{3.2}$$

where $\hat{\mathcal{L}}'$ is the Frechet derivative.

According to [11–14], the symbol χ of the operator $\hat{\chi}$ can be taken, without loss of generality for the q-order symmetry $\hat{\chi}$ of the consistent system (2.29), as

$$\chi(t, z_q, \Theta_u; \varepsilon) = \left(\sigma(t, z_q, \Theta_u; \varepsilon), \vartheta(t; \varepsilon)\right), \tag{3.3}$$

where

$$\sigma(t, z_q; \varepsilon) = \sigma(t, x, u, u_1, \dots, u_q, \Theta_u; \varepsilon), \tag{3.4}$$

$$\vartheta(\Theta_u, t; \varepsilon) = \left(\vartheta^{(m_u)}(\Theta_u, t; \varepsilon), \vartheta^{(x_u)}(\Theta_u, t; \varepsilon), \vartheta^{(\alpha_u^{(2)})}(\Theta_u, t; \varepsilon)\right) \tag{3.5}$$

are the symbols of the operators $\hat{\sigma}$ and $\hat{\vartheta}$, respectively. We denote the symmetry by the symbol $\chi = (\sigma, \vartheta)$.

The determining equation (3.2) in operator form reads

$$\frac{\partial\hat{\sigma}}{\partial t} = H'[u](t, x, \varepsilon)\hat{\sigma}, \qquad u_t(t, x) = \hat{H}[u](t, x, \varepsilon), \tag{3.6}$$

$$\dot{\vartheta} = \Gamma'(\Theta_u(t), t, \varepsilon)\vartheta, \qquad \dot{\Theta}_u = \Gamma(\Theta_u(t), t, \varepsilon). \tag{3.7}$$

Let the functions $H, \Gamma, \chi, \sigma, \vartheta$, and ζ depending on the real variables $t, z_q, \Theta_u; \varepsilon$ be the symbols of the corresponding operators, and let

$$\widehat{Y}_\zeta = \sum_{i=0} (\widehat{D}_x)^i(\zeta)\frac{\partial}{\partial u_i}, \tag{3.8}$$

where $\widehat{D}_x$ is the operator of total differentiation (2.6).

The determining equations (3.6) and (3.7) for the symmetry (σ, ϑ) can be written in terms of the operator symbols as [11–14]

$$\frac{\partial \sigma}{\partial t} + \widehat{Y}_H(\sigma) - \widehat{Y}_\sigma(H) + \frac{\partial \sigma}{\partial \Theta_u}\Gamma(\Theta_u(t), t) - \frac{\partial H}{\partial \Theta_u}\vartheta = 0, \tag{3.9}$$

$$\frac{\partial \vartheta}{\partial t} + \frac{\partial \vartheta}{\partial \Theta_u}\Gamma(\Theta_u(t), t) - \frac{\partial \Gamma}{\partial \Theta_u}\vartheta = 0. \tag{3.10}$$

By solving Eqs. (3.6) and (3.7) or (3.9) and (3.10), we obtain the intermediate symmetries [10] σ and ϑ. Similar to (2.17)–(2.19), we have additional integral constraints:

$$\begin{aligned}
\vartheta^{(m_u)}(t) &= \int_{-\infty}^{+\infty} \hat{\sigma}[u](t,x)dx,\\
\vartheta^{(x_u)}(t) &= \frac{\vartheta^{(m_u)}(t)}{m_u(t)}(x_\sigma(t) - x_u(t)),\\
\vartheta^{(\alpha^{(2)})}(t) &= -\frac{\vartheta^{(m_u)}(t)}{m_u(t)}\alpha_u^{(2)}(t) + \frac{1}{m_u(t)}\int_{-\infty}^{+\infty}(x - x_u(t))^2\hat{\sigma}[u](t,x)dx.
\end{aligned} \tag{3.11}$$

Here, $x_\sigma = \frac{1}{\vartheta^{(m_u)}(t)}\int_{-\infty}^{+\infty} x\hat{\sigma}[u](t,x)dx$.

Let the function σ be the symbol of the first-order operator; then, the relations [11–13]

$$\begin{aligned}
&\sigma = \eta^1 - \xi^1 u_t - \xi^2 u_x, \qquad \vartheta^{(m_u)} = \eta^2 - \xi^1 \dot{m}_u,\\
&\vartheta^{(x_u)} = \eta^3 - \xi^1 \dot{x}_u, \qquad \vartheta^{(\alpha_u^{(2)})} = \eta^4 - \xi^1 \dot{\alpha}_u^{(2)}
\end{aligned} \tag{3.12}$$

yield the generator

$$\widehat{X} = \xi^1\partial_t + \xi^2\partial_x + \eta^1\partial_u + \eta^2\partial_{m_u} + \eta^3\partial_{x_u} + \eta^4\partial_{\alpha_u^{(2)}} \tag{3.13}$$

of the point symmetry group for the consistent system (2.25) and (2.26) or (2.29).

4. The nonlocal FKPP equation with a quadratic operator

To illustrate the specificity of group-theoretical methods as applied to integro-differential equations by a simple example, we consider a one-dimensional FKPP equation with a nonlocal nonlinearity:

$$\left[-\partial_t + \varepsilon\partial_x^2 + a - \int_{-\infty}^{+\infty} b(t,x,y)u(t,y)dy\right]u(t,x) = 0. \tag{4.1}$$

Here, ε is the diffusion coefficient. In general, $b(t,x,y)$ is an influence function smooth in its arguments, and a is a positive constant. For simplicity, we consider the calculation of symmetries for the case $b(t,x,y) = b = \text{const}$.

For Eq. (4.1) written in the form (2.25), we have

$$-u_t(t,x) + \hat{H}[u](t,x) = 0, \tag{4.2}$$

$$\hat{H}[u](t,x) = \varepsilon u_{xx}(t,x) + au(t,x) - bm_u(t)u(t,x). \tag{4.3}$$

Differentiating Eq. (2.17) with respect to t and taking into account (4.2) and (4.3), we obtain

$$\dot{m}_u(t) = am_u(t) - bm_u(t)^2 = \varphi(m_u(t))m_u(t), \tag{4.4}$$

where

$$\varphi(m_u) = a - bm_u. \tag{4.5}$$

Similarly, from (2.18), (2.19), (4.2) and (4.3), we obtain for $x_u(t)$ and $\alpha_u^{(k)}(t)$

$$\dot{x}_u = 0, \tag{4.6}$$

$$\dot{\alpha}_u^{(2)} = 2\varepsilon m_u, \qquad \dot{\alpha}_u^{(3)} = 0, \qquad \dot{\alpha}_u^{(k)} = \varepsilon k(k-1)\alpha_u^{(k-2)}, \quad k \geqslant 4. \tag{4.7}$$

Consider Eqs. (4.4), (4.6) and (4.7) as the Einstein–Ehrenfest (EE) system describing the evolution of time-dependent variables $m_u(t), x_u(t)$ and $\alpha_u^{(k)}(t)$.

The solution of Eqs. (4.4), (4.6) and (4.7) has the form

$$m_u(t) = \frac{am_0e^{at}}{a+bm_0(e^{at}-1)}, \qquad x_u(t) = x_0,$$

$$\alpha_u^{(2)} = \frac{2\varepsilon}{b}\log\left(1+\frac{bm_0}{a}(e^{at}-1)\right)+\alpha_0^{(2)}, \qquad \alpha_u^{(3)} = \alpha_0^{(3)}, \tag{4.8}$$

$$\alpha_u^{(k)} = \varepsilon k(k-1)\int_0^t \alpha_u^{(k-2)}(t)dt + \alpha_0^{(k)}, \quad k \geqslant 4, \tag{4.9}$$

where

$$m_u(t)|_{t=0} = m_0, \qquad x_u(t)|_{t=0} = x_0, \qquad \alpha_u^{(k)}|_{t=0} = \alpha_0^{(k)}, \quad k \geqslant 2.$$

Note that Eqs. (4.2) and (4.3) depend only on the zero-order moment. The EE system (4.4), (4.6) and (4.7) is of recurrent type, i.e. any subsystem of the moment system, including the moments of the Mth order, does not include the moments of the higher order.

Therefore, we can take Eqs. (4.2)–(4.4) as the consistent system for calculating the symmetries.

5. Lie symmetries

The symmetries $\hat{\chi} = (\hat{\sigma}, \hat{\vartheta})$ of the consistent system (4.2)–(4.4) are given by the symbols (3.4) and (3.5),

$$\begin{aligned} &\sigma = \sigma(t, x, z_q, m_u), \\ &\vartheta = \vartheta(t, m_u). \end{aligned} \tag{5.1}$$

Here, $z_r = (x, u, u_1, u_2, \ldots, u_r)$; $t, x, u, u_1, u_2, \ldots, u_r$, and m_u are independent real variables.

The function $\vartheta(t, m_u(t))$, according to (3.11), is related to $\sigma(t, x, u(t, x), u_x(t, x), \ldots, u_{x\cdots x}(t, x), m_u(t))$ as

$$\vartheta(t) = \int_{-\infty}^{+\infty} \hat{\sigma}[u](t, x)dx. \tag{5.2}$$

The Lie symmetries of Eqs. (4.2) and (4.3) are related to σ of the second order [11–13]:

$$\sigma = \sigma(t, x, u, u_1, u_2, m_u), \qquad \vartheta = \vartheta(t, m_u). \tag{5.3}$$

The determining equations (3.9) and (3.10) for σ of the form (5.3) and for ϑ read

$$\begin{aligned} &\sigma_t + \varphi u\sigma_u + \varphi u_1\sigma_{u_1} + \varphi u_2\sigma_{u_2} - \varphi\sigma - \varepsilon(\sigma_{xx} + 2u_1\sigma_{xu} + 2u_2\sigma_{xu_1} + 2u_3\sigma_{xu_2} + 2u_1u_2\sigma_{uu_1} + 2u_1u_3\sigma_{uu_2} \\ &\quad + 2u_2u_3\sigma_{u_1u_2} + u_1^2\sigma_{uu} + u_2^2\sigma_{u_1u_1} + u_3^2\sigma_{u_2u_2}) + \varphi m_u\sigma_{m_u} + b\vartheta u = 0, \end{aligned} \tag{5.4}$$

$$\vartheta_t + \varphi m_u\vartheta_{m_u} = -b\vartheta m_u + \varphi\vartheta. \tag{5.5}$$

With the change of variables

$$\tau = t - \frac{1}{a}\log\frac{m_u}{a-bm_u}, \qquad z = \frac{1}{a}\log\frac{m_u}{a-bm_u}, \tag{5.6}$$

we have

$$\frac{\partial}{\partial t} + \varphi m_u\frac{\partial}{\partial m_u} = \frac{\partial}{\partial z}, \tag{5.7}$$

and Eqs. (5.4) and (5.5) take the form

$$\begin{aligned} &\sigma_z + \varphi u\sigma_u + \varphi u_1\sigma_{u_1} + \varphi u_2\sigma_{u_2} - \varphi\sigma - \varepsilon(\sigma_{xx} + 2u_1\sigma_{xu} + 2u_2\sigma_{xu_1} \\ &\quad + 2u_3\sigma_{xu_2} + 2u_1u_2\sigma_{uu_1} + 2u_1u_3\sigma_{uu_2} + 2u_2u_3\sigma_{u_1u_2} + u_1^2\sigma_{uu} + u_2^2\sigma_{u_1u_1} + u_3^2\sigma_{u_2u_2}) + b\vartheta u = 0, \end{aligned} \tag{5.8}$$

$$\vartheta_z = \vartheta\frac{a(1-be^{az})}{1+be^{az}}. \tag{5.9}$$

From (5.9), we have

$$\vartheta = S(\tau)\frac{e^{az}}{(1+be^{az})^2}, \tag{5.10}$$

where $S(\tau)$ is an arbitrary function of τ.

Eq. (5.8) is solved in the standard way [11–13], and its general solution, in view of (5.10), is obtained as

$$\sigma = \left(\frac{1}{2}A_1(\tau)z^2 + A_2(\tau)z + A_3(\tau)\right)u_2 + \left(\frac{1}{2\varepsilon}A_1(\tau)zx + \frac{1}{2\varepsilon}A_2(\tau)x + A_4(\tau)z + A_5(\tau)\right)u_1$$
$$+ \left(\frac{1}{8\varepsilon^2}A_1(\tau)x^2 + \frac{1}{2\varepsilon}A_4(\tau)x + \frac{1}{4\varepsilon}A_1(\tau)z + S(\tau)\frac{1}{a(1+be^{az})} + A_6(\tau)\right)u + R(\tau, z, x). \quad (5.11)$$

Here, $A_i(\tau)$, $i = 1, \dots, 6$, are arbitrary functions of τ, and $R(\tau, z, x)$ is an arbitrary solution of the equation

$$R_z = \varepsilon R_{xx} + \varphi R. \quad (5.12)$$

Defining $m_R = \int_{-\infty}^{+\infty} R(\tau, z, x)dx$, in view of (5.12), we obtain an equation for the function $m_R(\tau, z)$:

$$(m_R)_z = \varphi m_R. \quad (5.13)$$

In view of the integral relation (5.2), (5.10) and (5.11) yield

$$A_1(\tau) = A_4(\tau) = 0, \qquad A_6(\tau) = \frac{1}{2\varepsilon}A_2(\tau) - \frac{m_R}{m_u}. \quad (5.14)$$

In terms of the variables (t, m_u) related to (τ, z) by (5.6), relations (5.10), (5.11), in view of (5.14), take the form

$$\vartheta = S(\tau)\frac{m_u(a - bm_u)}{a^2}, \quad (5.15)$$

$$\sigma = A_2(\tau)\sigma_1 + A_3(\tau)\sigma_2 + A_5(\tau)\sigma_3 + S(\tau)\sigma_4 + \sigma_5. \quad (5.16)$$

Here,

$$\sigma_1 = \frac{1}{a}\log\frac{m_u}{a - bm_u}u_2 + \frac{1}{2\varepsilon}xu_1 + \frac{1}{2\varepsilon}u, \quad (5.17)$$

$$\sigma_2 = u_2, \qquad \sigma_3 = u_1, \quad (5.18)$$

$$\sigma_4 = \frac{a - bm_u}{a^2}u, \quad (5.19)$$

$$\sigma_5 = R - \frac{m_R}{m_u}u. \quad (5.20)$$

Note that the equality $\tau(t, m_u(t)) = \text{const}$ holds for the solutions of Eq. (4.4). In view of (5.13), we obtain that $m_R(t, m_u(t))/m_u(t) = \text{const}$.

From (5.15) and (5.17) to (5.20) we have the following symmetries: $\chi_i = (\sigma_i, \vartheta_i)$, $i = 1, \dots, 5$, for the consistent system (4.2)–(4.4):

$$\chi_1 = (\sigma_1, 0) \quad (5.21)$$

$$\chi_2 = (\sigma_2, 0), \qquad \chi_3 = (\sigma_3, 0), \quad (5.22)$$

$$\chi_4 = \left(\sigma_4, \frac{a - bm_u}{a^2}m_u\right), \quad (5.23)$$

$$\chi_5 = (\sigma_5, 0). \quad (5.24)$$

According to (3.12) and (3.13), the functions σ and ϑ given by (5.17)–(5.24) give rise to the following generators of the symmetry Lie group of point transformations for Eqs. (4.2) and (4.3):

$$\widehat{X}_1 = \frac{1}{a}\log\frac{m_u}{\varphi}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - \left(\frac{1}{2} - \frac{\varphi}{a}\log\frac{m_u}{\varphi}\right)u\frac{\partial}{\partial u} + \frac{\varphi m_u}{a}\log\frac{m_u}{\varphi}\frac{\partial}{\partial m_u},$$

$$\widehat{X}_2 = \varphi u\frac{\partial}{\partial u} + \varphi m_u\frac{\partial}{\partial m_u},$$

$$\widehat{X}_3 = \frac{\partial}{\partial x}, \qquad \widehat{X}_4 = \frac{\partial}{\partial t}, \qquad \widehat{X}_5 = \left(R - \frac{m_R}{m_u}u\right)\frac{\partial}{\partial u}. \quad (5.25)$$

The point transformations of dependent and independent variables generated by $\widehat{X}_i$, $i = 1, \ldots, 5$, are

$$\widehat{X}_1 : x' = xe^{\frac{s}{2}}, \qquad t' = t + (e^s - 1)\frac{1}{a}\log\frac{m_u}{\varphi},$$

$$u' = u\frac{m'_u}{m_u}e^{-s/2}, \quad m'_u = \frac{a(m_u/\varphi)^{e^s}}{1 + b(m_u/\varphi)^{e^s}},$$

$$\widehat{X}_2 : x' = x, \qquad t' = t, \qquad u' = u\frac{m'_u}{m_u}, \quad m'_u = \frac{am_u e^{sa}}{a - bm_u + bm_u e^{sa}},$$

$$\widehat{X}_3 : x' = x + s, \qquad t' = t, \qquad u' = u, \qquad m'_u = m_u,$$

$$\widehat{X}_4 : x' = x, \qquad t' = t + s, \qquad u' = u, \qquad m'_u = m_u,$$

$$\widehat{X}_5 : x' = x, \qquad t' = t, \qquad u' = R\frac{m_u}{m_R}\left(1 - e^{-(m_R/m_u)s}\right) + ue^{-(m_R/m_u)s}, \qquad m'_u = m_u,$$

where s is a group parameter.

Note that

$$\frac{1}{a}\log\frac{m_u}{\varphi} = t + \text{const}$$

is valid for the solutions of Eq. (4.4).

Let us find a group-invariant solution of Eqs. (4.2) and (4.3) using σ_1 of the form (5.17). To this end, we have to solve Eq. (4.2) together with

$$\hat{\sigma}_1[u](t, x) = 0. \tag{5.26}$$

Eq. (5.26), in view of (5.17), takes the form

$$\alpha_u(t)u_{xx}(t, x) + xu_x(t, x) + u(t, x) = 0, \tag{5.27}$$

where the coefficient

$$\alpha_u(t) = \frac{2\varepsilon}{a}\log\frac{m_u(t)}{a - bm_u(t)}, \tag{5.28}$$

depending on the moment $m_u(t)$, is a functional of $u(t, x)$.

To construct a solution of Eq. (5.27), we previously solve the linear equation

$$\alpha(t)u_{xx}(t, x) + xu_x(t, x) + u(t, x) = 0 \tag{5.29}$$

with a given arbitrary function $\alpha(t)$ having two independent solutions:

$$u_1(t, x, \alpha(t)) = v_1(t)\exp\left(-\frac{x^2}{2\alpha(t)}\right), \tag{5.30}$$

$$u_2(t, x, \alpha(t)) = v_2(t)\exp\left(-\frac{x^2}{2\alpha(t)}\right)\int_0^x \exp\left(\frac{\tilde{x}^2}{2\alpha(t)}\right)d\tilde{x}. \tag{5.31}$$

Here, $v_1(t)$ and $v_2(t)$ are arbitrary functions of t.

The function $\alpha_{u_1}(t)$ is related to the second moment $u_1(t, x)\alpha^{(2)}_{u_1}(t)$ as

$$\alpha_{u_1}(t) = \frac{\alpha^{(2)}_{u_1}(t)}{m_{u_1}(t)}.$$

For $\alpha(t) = \alpha_{u_1}(t)$ in (5.30), $u_1(t, x, \alpha_{u_1}(t))$ is a solution of (5.27).

From (2.17) and (5.30), we have $m_{u_1}(t) = v_1(t)\sqrt{2\pi\alpha_{u_1}(t)}$, or

$$v_1(t) = m_{u_1}(t)(2\pi\alpha_{u_1}(t))^{-\frac{1}{2}} = m_{u_1}(t)\left(\frac{4\pi\varepsilon}{a}\log\frac{m_{u_1}(t)}{a - bm_{u_1}(t)}\right)^{-\frac{1}{2}}.$$

It follows that

$$u_1(t, x, \alpha_{u_1}(t)) = m_{u_1}(t)\left(\frac{4\pi\varepsilon}{a}\log\frac{m_{u_1}(t)}{a - bm_{u_1}(t)}\right)^{-\frac{1}{2}}\exp\left(-\frac{x^2}{2\alpha_{u_1}(t)}\right), \tag{5.32}$$

where

$$\alpha_{u_1}(t) = \frac{2\varepsilon}{a}\log\frac{m_{u_1}(t)}{a - bm_{u_1}(t)}. \tag{5.33}$$

Direct calculation shows that (5.32) satisfies Eq. (4.2) in view of (4.4). Summarizing, we have that Eqs. (5.32) and (5.33) define a group-invariant solution of Eqs. (4.2) and (4.3).

For the solution $u_2(t, x, \alpha(t))$, (5.31), the Riemann integral determining the moment $m_{u_2}(t)$ by (2.17) does not exist. Consequently, $u_2(t, x, \alpha(t))$ is not a solution of Eq. (4.2).

6. Discussion

For nearly linear IDEs, the semiclassical approach allow us to bridge the original IDE and a finite consistent system of differential equations consisting of a PDE including nonlinear terms in the form of moments and a finite system of ODEs describing the evolution of the moments. Classical group analysis applied to the consistent system provides information on the original nonlinear IDE. Namely, the solution of the consistent system is a key point in constructing the principal term in semiclassical asymptotic expansions.

For the nonlocal FKPP equation, the consistent system is obtained in the class of trajectory concentrated functions $\mathcal{P}_t^{\varepsilon}$ [43], which ensures the existence of moments $m_u(t)$, $x_u(t)$, $\alpha_u^{(2)}(t)$ ((2.17)–(2.19)).

The specificity of the group analysis of a consistent system shows up, in particular, in the integral constraints (e.g., (5.2)) that reduce arbitrariness in constructing group-invariant solutions.

It should also be noted that the symmetry $\widehat{X}_5$ characteristic of nonlinear equations allows a linearization. An interesting example of this type of linearization is Burger's equation, whose solutions are reduced to positive solutions of the linear heat equation by using the Hopf–Cole transform.

Group analysis reveals the relationship between these equations and derives the transformation [13].

More complicated cases of linearization with contact transformations are considered elsewhere [48,49].

The methods of differential constraints, degenerate hodograph, and group analysis for constructing solutions of PDEs are considered comprehensively elsewhere [19].

The nonlocal FKPP equation (4.1) also admits a linearization. For the general case of an operator quadratic in the space variable and derivatives, the solution of the Cauchy problem has been constructed [43].

The approach considered, which is based on the reduction of IDEs to a finite consistent system followed by symmetry analysis, has a wide area of applications: symmetry analysis of the FKPP equation with coefficients of more general form, and other nonlinear equations, such as the Fokker–Planck equation and the Hartree-type equation. Calculations of high-order symmetries and conditional symmetries are also of interest.

Additional prospects for symmetry analysis applications arise when a semiclassical approximation is constructed in a class of functions other than $\mathcal{P}_t^{\varepsilon}$. For example, in a class of functions localized on curves or surfaces, the consistent system can consist of integro-differential equations, and this is a case absolutely other than that of $\mathcal{P}_t^{\varepsilon}$.

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