Blow-up of solutions to semilinear parabolic equations on Riemannian manifolds with negative sectional curvature

Fabio Punzo
Dipartimento di Matematica “G. Castelnuovo”, Università di Roma “La Sapienza”, P.le A. Moro 5, I-00185 Roma, Italy

1. Introduction

We are concerned with finite time blow-up and global existence of solutions to semilinear parabolic Cauchy problems of the following type:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + h(t)|u|^{p-1}u \quad \text{in } M \times (0, T), \\
 u &= u_0 \quad \text{in } M \times \{0\},
\end{aligned}
\]

(1.1)

where \( M \) is a smooth \( N \)-dimensional complete noncompact Riemannian manifold with metric \( g \) and negative sectional curvature, \( \Delta \) is the Laplace–Beltrami operator on \( M \); \( h \) is a positive continuous function defined in \([0, \infty)\), the initial datum \( u_0 \) is continuous and bounded on \( M \), \( p > 1 \).

It is well known that the corresponding problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + |u|^{p-1}u \quad \text{in } \mathbb{R}^N \times (0, T), \\
 u &= u_0 \quad \text{in } \mathbb{R}^N \times \{0\}
\end{aligned}
\]

(1.2)

(\( u_0 \geq 0 \)) does not admit global solutions for \( 1 < p \leq 1 + \frac{2}{N} \) (see [8,14]). Instead, for \( p > 1 + \frac{2}{N} \) global solutions exist, provided that \( u_0 \) is sufficiently small. This dichotomy is usually said Fujita’s phenomenon.

Furthermore, it is proved in [16] that problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + h(t)|u|^{p-1}u \quad \text{in } \Omega \times (0, T), \\
 u &= 0 \quad \text{in } \partial \Omega \times (0, T), \\
 u &= u_0 \quad \text{in } \Omega \times \{0\},
\end{aligned}
\]

(1.3)

(\( u_0 \geq 0 \)) does not admit global solutions for \( 1 < p \leq 1 + \frac{\alpha}{N} \) (see [8,14]). Instead, for \( p > 1 + \frac{\alpha}{N} \) global solutions exist, provided that \( u_0 \) is sufficiently small. A key role will be played by the infimum of the \( L^2 \)-spectrum of the operator \( -\Delta \) on \( M \).
has a global solution, when $u_0$ is nonnegative and small enough, $\Omega$ is a bounded domain of $\mathbb{R}^N$; moreover, $h(t) \equiv 1$ ($t \geq 0$), or
\begin{equation}
\alpha_1 t^q \leq h(t) \leq \alpha_2 t^q \quad (t > t_0)
\end{equation}
for some $\alpha_1 > 0$, $\alpha_2 > 0$, $t_0 > 0$ and $q > -1$. On the contrary, if $h(t) = e^{\omega t}$ ($t \geq 0$), then a Fujita-type phenomenon holds for certain values of $\alpha$.

Results given in [8] are generalized to Riemannian manifolds $M$ in [19], provided there exist $C > 0$ and $\alpha > 2$ such that:

(a) $\mu(B(x, r)) \leq C r^d$, when $r$ is large and for all $x \in M$;
(b) $\frac{\alpha}{4} \log \frac{h}{r} \leq \frac{\sqrt{r}}{r}$, when $r = d(x_0, x)$, for some $x_0 \in M$, is smooth. Here $\mu$ is the Riemannian volume on $M$, $\sqrt{r}$ is the volume density of $M$, $B(x, r)$ is the geodesics ball with center $x \in M$ and radius $r > 0$.

Observe that if the Ricci curvature of $M$ is nonnegative, then (a)–(b) are satisfied. On the other hand (see [11, Section 10.1]), hypotheses (a)–(b) imply that $\lambda_1(M) = 0$, where $\lambda_1(M)$ is the infimum of the $L^2$-spectrum of the operator $-\Delta$ on $M$.

Moreover, results similar to those established for (1.3) are obtained in [1] for problem (1.1) on the hyperbolic space $\mathbb{H}^N$, that is
\begin{equation}
\begin{aligned}
\partial_t u &= \Delta u + h(t)|u|^{p-1}u \quad \text{in} \; \mathbb{H}^N \times (0, T), \\
u &= u_0 \quad \text{in} \; \mathbb{H}^N \times \{0\}
\end{aligned}
\end{equation}
($u_0 \geq 0$). To be specific, in [1] it is shown that if $h(t) \equiv 1$ ($t \geq 0$) or (1.4) is satisfied, then there exist global solutions for sufficiently small initial data $u_0$. Moreover, when $h(t) = e^{\omega t}$ ($t \geq 0$) for some $\alpha > 0$, we have the following results:

(i) if $1 < p < 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^N)}$, then every nontrivial solution of problem (1.5) blows up in finite time;
(ii) if $p > 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^N)}$, then problem (1.5) possesses global solutions for small initial data;
(iii) if $p = 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^N)}$ and $\alpha > \frac{4}{3}\lambda_1(\mathbb{H}^N)$, then there exist global solutions of problem (1.5) for small initial data.

Recall that
\[\lambda_1(\mathbb{H}^N) = \frac{(N - 1)^2}{4};\]

furthermore, note that $\mathbb{H}^N$ has constant sectional curvature $-1$.

The blow-up result in (i) is proved in [1] by means of the following estimate derived in [6]:
\[c_N^{-1} \kappa(d(x, y), t) \leq p(x, y, t) \leq c_N \kappa(d(x, y), t) \quad (x, y \in \mathbb{H}^N, \; t > 0),\]

where
\[\kappa(d, t) := (4\pi t)^{-N/2}(1 + d)(1 + d + t) \frac{N-3}{2} e^{-\lambda_1(\mathbb{H}^N) d - \frac{N-1}{2} d - \frac{d^2}{4}} \]

($d \geq 0$, $t > 0$, $c_N > 0$ and $p$ is the heat kernel on $\mathbb{H}^N$).

Moreover, in order to prove the global existence results in (ii)–(iii) a bounded supersolution to problem (1.1) is used. This supersolution is constructed by means of a bounded ground state on $\mathbb{H}^N$.

In this paper we shall extend results described in (i)–(ii) to Cartan–Hadamard Riemannian manifolds $M$ with sectional curvature bounded above by a negative constant; clearly $\lambda_1(\mathbb{H}^N)$ will be replaced by $\lambda_1(M)$. This class of Riemannian manifolds includes, in particular, $\mathbb{H}^N$.

For this type of Riemannian manifolds we have $\lambda_1(M) > 0$. Hence the hypotheses (a)–(b) cannot be satisfied.

The proof of finite time blow-up relies on (2.5)–(2.6) below. Furthermore, the global existence can be proved by means of the same arguments as in [1]. Let us underline that the inequality $\lambda_1(M) > 0$ will be crucial in the sequel, in order to prove both finite time blow-up and global existence.

Observe that for problem (1.1) we are not able to prove the counterpart of (iii); this is an open problem. We underline that the method used in [1] do not work in our general case. Indeed, in [1], the proof of the statement (ii) makes heavily use of the term $(1 + d + t) \frac{N-3}{2}$, which appears in the estimates from above in (1.6)–(1.7). Instead, for general $M$ there is not such a term in the estimate from above for the heat kernel (see (2.5)).

The paper is organized as follows. In Section 2 we recall preliminaries of heat semigroup and spectral analysis on $M$. In Section 3 we discuss some geometric conditions that ensure comparison principles on $M$. In Section 4 we state our results about finite time blow-up and global existence, that will be shown in Sections 5 and 6, respectively.
2. Mathematical background

2.1. Heat semigroup and spectral analysis on M

Let $\{e^{t\Delta}\}_{t \geq 0}$ be the analytical contraction semigroup generated by $-\Delta$ on $L^2(M)$ (see [11–13]). The semigroup $\{e^{t\Delta}\}_{t \geq 0}$ admits a heat kernel, more precisely there exists a function $p \in C^\infty(M \times M \times (0, \infty))$, $p > 0$ in $M \times M \times (0, \infty)$ such that

$$e^{t\Delta} f(x) = \int_M p(x, y, t) f(y) \, d\mu_y \quad (x \in M, \ t > 0) \quad (2.1)$$

for any $f \in L^2(M)$. Moreover, we have

$$p(x, y, t) = p(y, x, t) \quad \text{for all } x, y \in M, \ t > 0;$$

$$\int_M p(x, y, t) \, d\mu_y \leq 1 \quad \text{for all } x \in M, \ t > 0; \quad (2.2)$$

$$p(x, y, t + s) = \int_M p(x, z, t) p(z, y, s) \, d\mu_z \quad \text{for all } x, y \in M, \ t > 0. \quad (2.3)$$

Finally, for every $y \in M$ the function

$$u(x, t) := p(x, y, t) \quad (x \in M, \ t > 0)$$

is a classical solution to the heat equation

$$\partial_t u = \Delta u \quad \text{in } M \times (0, \infty);$$

furthermore, for any $f \in C_0^\infty(M)$

$$\int_M p(x, y, t) f(y) \, d\mu_y \rightarrow f(x) \quad \text{as } t \rightarrow 0 \text{ in } C_0^\infty(M).$$

Let $\text{spec}(-\Delta)$ be the spectrum in $L^2(M)$ of the operator $-\Delta$. Note that (see [13, Chapter 4])

$$\text{spec}(-\Delta) \subseteq [0, \infty).$$

Denote by $\lambda_1(M)$ the bottom of $\text{spec}(-\Delta)$, that is

$$\lambda_1(M) := \inf\text{spec}(-\Delta).$$

Let us recall (see [11]) next

**Definition 2.1.** A Cartan–Hadamard manifold is a geodesically complete, simply connected Riemannian manifold with non-positive sectional curvature.

For every $p \in M$ and for every plane $\pi \subseteq T_p M$ denote by $K_\pi(p)$ the sectional curvature of the plane $\pi$ (see [9]). Observe that when $M$ is a Cartan–Hadamard manifold and $K_\pi(p) \leq -k^2$ for some constant $k > 0$ and for any $p \in M$ and any plane $\pi \subseteq T_p M$, then (see [15]; see also [11])

$$\lambda_1(M) \geq \frac{(N - 1)^2}{4} - k^2. \quad (2.4)$$

Moreover, if $M$ is a Cartan–Hadamard manifold, then (see [13, Corollary 15.17 and Remark 14.6])

$$p(x, y, t) \leq \frac{C}{(\min(t, T))^N} \left(1 + \frac{d^2}{t}\right)^{N/2} \exp\left\{-\frac{d^2}{4t} - \frac{\lambda_1(M)(t - T)}{T}\right\} \quad (2.5)$$

for all $x, y \in M, \ t > 0, \ T > 0$ and for some positive constant $C$; here we have set $d \equiv \text{dist}(x, y)$.

Furthermore, let us recall that if $M$ is a noncompact Riemannian manifold, then (see [2, Corollary 1])

$$\lim_{t \rightarrow \infty} \frac{\log p(x, y, t)}{t} = -\lambda_1(M) \quad \text{locally uniformly in } M \times M. \quad (2.6)$$
2.2. Definition of solution

In what follows we always make the following assumption:

\begin{itemize}
  \item [(i)] \( h \in C([0, \infty)), h > 0 \) in \([0, \infty)\);
  \item [(ii)] \( u_0 \) is continuous and bounded in \( M \).
\end{itemize}

(A_0)

The identity (2.1) allows us to extend the definition of \( [e^{\Delta} t]_{t \geq 0} \) as follows (see [13, Chapter 7]):

\[ (e^\Delta f)(x) := \int_M p(x, y, t) f(y) \, d\mu_y \quad (x \in M, \ t > 0) \]  

(2.7)

for any function \( f \) such that the right-hand side in (2.7) makes sense. In particular, if \( f \in L^1_{loc}(M), f \geq 0 \) in \( M \), then the function \( (e^\Delta f)(x) \) is measurable in \( M \times (0, \infty) \). If, in addition, \( (e^\Delta f)(x) \in L^1_{loc}(M \times I) \), where \( I \) is an open interval in \( (0, \infty) \), then \( (e^\Delta f)(x) \) is a classical solution to the heat equation

\[ \partial_t u = \Delta u \quad \text{in } M \times I. \]

We give next definition.

Definition 2.2. A mild solution to problem (1.1) is a function \( u \in C(M \times [0, \tau]) \cap L^\infty(M \times (0, \tau)) \) for any \( \tau \in [0, T) \) such that

\[ u(x, t) = e^{\Delta t} u_0(x) + \int_0^t (e^{(t-s)\Delta} h(s)(u|^{p-1} u)(x)) \, ds \]  

(2.8)

\((x, t) \in M \times [0, T)\).

Moreover, we shall deal with weak solutions to problem (1.1) meant in the following sense.

Definition 2.3. A weak solution to problem (1.1) is a function \( u \in C(M \times [0, \tau]) \cap L^\infty(M \times (0, \tau)) \) for any \( \tau \in [0, T) \) such that

\[ -\int_0^\tau \int_M u(x, t) \{ \Delta \psi(x, t) + \partial_t \psi(x, t) \} \, d\mu_x \, dt = \int_M u_0(x) \psi(x, 0) \, d\mu_x + \int_0^\tau \int_M h(t)(u(x))^{p-1} u(x)(x, t) \, d\mu_x \, dt \]

for any \( \tau \in [0, T) \), for any precompact set \( \Omega \subseteq M \) with smooth \( \partial \Omega \), and for any \( \psi \in C^{2,1}(M \times [0, \tau]) \) with \( \text{supp} \psi(\cdot, t) \subseteq M \) \((t \in [0, \tau])\) and \( \psi(\cdot, \tau) = 0 \).

Definition 2.4. A solution to problem (1.1) is called global, if it exists for any \( t > 0 \), that is if \( T = \infty \).

Instead, we say that a solution to problem (1.1) blows up in finite time, if

\[ \lim_{t \to T^-} \| u(\cdot, t) \|_{L^\infty(M)} = \infty, \]

for some \( T > 0 \).

3. Auxiliary results

The one point compactification of \( M \) is the topological space \( M \cup \{ \infty \} \), where \( \infty \) is the ideal infinity point (that does not belong to \( M \)) and the family of open sets in \( M \cup \{ \infty \} \) consists of open sets of the form \((M \setminus K) \cup \{ \infty \})\), where \( K \) is an arbitrary compact set of \( M \). This family of sets determines the Hausdorff topology in \( M \cup \{ \infty \} \) and the topological space \( M \cup \{ \infty \} \) is compact.

Let \( Z : M \to \mathbb{R} \) be a function. Since in our case \( M \) is noncompact, the definition of topology of \( M \cup \{ \infty \} \) suggests to write

\[ \lim_{x \to \infty} Z(x) = \infty, \]

(3.1)

if for any \( \alpha > 0 \) there exists a compact subset \( K_{\alpha} \subseteq M \) such that

\[ Z(x) > \alpha \quad \text{for any } x \in M \setminus K_{\alpha} \]

(see, e.g., Paragraph 5.4.3 in [13]).

We will show next comparison principle. A key role will be played by weak supersolutions \( Z \) to equation

\[ \Delta Z = \lambda Z \quad \text{in } M, \]

(3.2)

for some \( \lambda \in [0, \infty) \), such that (3.1) is satisfied.
Definition 3.1. Let $\lambda \in [0, \infty)$, $f \in C(\Omega)$. A weak supersolution to equation
\[
\Delta U - \lambda U = f \quad \text{in } M
\]
is a function $U \in C(M)$ such that
\[
\int_M U \Delta \psi \, d\mu \leq \int_M (\lambda U + f) \psi \, d\mu
\]
for any $\psi \in C^0_0(M)$, $\psi \geq 0$. Weak subsolutions and solutions of Eq. (3.3) are defined accordingly.

Proposition 3.2. Let assumption $(A_0)$ be satisfied. Moreover, suppose that
\[
\begin{cases}
\text{there exists a weak supersolution to Eq. (3.2) for some} \\
\lambda > 0, \text{ such that condition (3.1) is satisfied.}
\end{cases}
\]
Let $u$ be a weak subsolution and $v$ a weak supersolution to problem (1.1). Then
\[
u \leq u \quad \text{in } M \times (0, T).
\]

Proof. The conclusion follows by the same argument as in the proof of Theorem 2.6 in [17], since $u, v \in L^\infty(M \times (0, T))$ for any $t \in (0, T)$. \qed

Let us recall that assumption $(A_1)$ implies that $M$ is stochastically complete, i.e. (see [11])
\[
\int_M p(x, y, t) \, d\mu_y = 1 \quad \text{for any } x \in M, \ t > 0.
\]

In order to provide explicit conditions for the existence of such a supersolution $Z$, we need to introduce some preliminary material.

Take a point $o \in M$ and denote by $\text{Cut}(o)$ the cut locus of $o$. We can define the polar coordinates in $M \setminus \text{Cut}^*(o)$, where $\text{Cut}^*(o) := \text{Cut}(o) \cup \{o\}$. Indeed, to any point $x \in M \setminus \text{Cut}^*(o)$ we can associate the polar radius $\rho(x) := \text{dist}(x, o)$ and the polar angle $\theta \in S^{N-1}$, such that the minimal geodesics from $o$ to $x$ star at $o$ to the direction $\theta$.

The Riemannian metric $g$ in $M \setminus \text{Cut}^*(o)$ has, in the polar coordinates, the form:
\[
ds^2 = d\rho^2 + a_{ij}(\rho, \theta) \, d\theta^i \, d\theta^j,
\]
where $(\theta^1, \ldots, \theta^{N-1})$ are coordinates on $S^{N-1}$ and $(a_{ij}(\rho, \theta))_{i,j=1,\ldots,N-1}$ is a positive definite matrix.

Let $a := \det(a_{ij})$, $B(\rho, r) := \{x = (\rho, \theta) \in M \mid \rho < r\}$. Then in $M \setminus \text{Cut}^*(o)$ we have
\[
\Delta = \frac{1}{\sqrt{\det a}} \frac{\partial}{\partial \rho} \left( \sqrt{\det a} \frac{\partial}{\partial \rho} \right) + \Delta_{\text{Bel}} = \frac{\partial^2}{\partial \rho^2} + m(\rho, \theta) \frac{\partial}{\partial \rho} + \Delta_{\text{Bel}}(\rho, \theta),
\]
where $\Delta_{\text{Bel}}(\rho, \theta)$ is the Laplace–Beltrami operator on the geodesics sphere $\partial B(\rho, \theta)$ and $m(\rho, \theta)$ is a smooth function on $(0, \infty) \times S^{N-1}$, which represents, from a geometrical viewpoint, the mean curvature of $\partial B(\rho, \theta)$ in the radial direction.

We say that $M$ is a manifold with a pole, if there exists a point $o \in M$ such that $\text{Cut}(o) = \emptyset$. Observe that any Cartan–Hadamard manifold is a manifold with a pole (see [10]).

Let $M$ be a manifold with a pole, $x \in M$. In the following we shall denote by $\omega$ the plane of $T_xM$ with basis $(\frac{\partial}{\partial \rho} , X)$, where $X$ is a unit vector orthogonal to $\frac{\partial}{\partial \rho}$. Furthermore, denote by $\text{Ric}_{(\rho)}(x)$ the Ricci curvature of $M$ at $x$ in the radial direction $\frac{\partial}{\partial \rho}$.

A manifold with a pole is called a spherically symmetric manifold or a model manifold if
\[
a_{ij}(\rho, \theta) \, d\theta^i \, d\theta^j = \sigma^2(\rho) \, d\rho^2,
\]
where $d\rho^2$ is the standard metric on $S^{N-1}$ and $\sigma$ is a function such that conditions
\[
\sigma \in C^\infty((0, R_0)) \quad \text{for some } R_0 \in (0, \infty], \sigma(0) = 0, \sigma'(0) = 1
\]
are satisfied. In this case we set $M \equiv M_{\sigma}$. As special cases, observe that if $\sigma(\rho) = \rho$ ($\rho \in [0, \infty)$), then $M = \mathbb{R}^N$; whereas, if $\sigma(\rho) = \sinh \rho$ ($\rho \in [0, \infty)$), then $M$ is the $N$-dimensional hyperbolic space $\mathbb{H}^N$.

Note that, by hypothesis (3.5), the metric
\[
ds^2 = d\rho^2 + \sigma^2(\rho) \, d\rho^2
\]
can be smoothly extended from $M \setminus \{o\}$ to the whole of $M$. 
Moreover, the area of the geodesic sphere \( \partial B(o, r) \) is

\[
S_{\sigma}(r) = \omega_N \sigma^{N-1}(r),
\]

where \( \omega_N \) is the area of the unit sphere of \( \mathbb{R}^N \), while the volume \( V(r) \) of the geodesic ball \( B(o, r) \) is

\[
V_{\sigma}(r) := \int_0^r S(\xi) \, d\xi = \omega_N \int_0^r \sigma^{N-1}(\xi) \, d\xi.
\]

From (3.6) it follows that the Laplace–Beltrami operator on \( M_{\sigma} \) can be written as

\[
\Delta = \frac{1}{\sigma} \frac{\partial^2}{\partial \rho^2} + (N - 1) \frac{\sigma'}{\sigma} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Delta_{\theta} = \frac{1}{\sigma^2} \frac{\partial^2}{\partial \rho^2} + \frac{S_{\sigma}'}{S_{\sigma}} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Delta_{\theta},
\]

(3.9)

where \( \Delta_{\theta} \) is the Laplace–Beltrami operator on \( S^{N-1} \). Furthermore, for every \( x \equiv (\rho, \theta) \in M_{\sigma} \) we have

\[
K_{\omega}(x) = -\sigma''(\rho) \frac{\sigma(\rho)}{\sigma(\rho)} \quad (x \equiv (\rho, \theta) \in M \setminus \text{Cut}^+(o)),
\]

(3.10)

\[
Ric_{\sigma}(x) = -(N - 1) \frac{\sigma''(\rho)}{\sigma(\rho)}.
\]

(3.11)

In the sequel we shall use the following known principle (see \cite{10,11}).

**Lemma 3.3.** Let \( M \) be a geodesically complete noncompact manifold. Suppose that

\[
Ric_{\sigma}(x) \geq -(N - 1) \frac{\sigma''(\rho)}{\sigma(\rho)} \quad (x \equiv (\rho, \theta) \in M \setminus \text{Cut}^+(o))
\]

(3.12)

for some function \( \sigma \) such that (3.5) with \( R_0 = \infty \) is satisfied. Then

\[
m(\rho, \theta) \leq (N - 1) \frac{\sigma'(\rho)}{\sigma(\rho)}
\]

(3.13)

for all \( (\rho, \theta) \) in the domain of the polar coordinates.

**Remark 3.4.** In connection with Lemma 3.3, observe that (see \cite{10,11}) if \( M \) is a manifold with a pole and

\[
K_{\omega}(x) \leq -\frac{\sigma''(\rho)}{\sigma(\rho)} \quad (x \equiv (\rho, \theta) \in M)
\]

(3.14)

for some function \( \sigma \) such that (3.5) with \( R_0 = \infty \) is satisfied, then

\[
m(\rho, \theta) \geq (N - 1) \frac{\sigma'(\rho)}{\sigma(\rho)} \quad (\rho > 0, \theta \in S^{N-1}).
\]

(3.15)

Observe that the function \( m(\rho, \theta) \) used in Lemma 3.3 and in Remark 3.4 is the same as in Eq. (3.4). Moreover, the right-hand sides of (3.12)–(3.15) have a geometrical meaning for model manifolds (see (3.9)–(3.11)).

We shall prove the following comparison principle.

**Proposition 3.5.** Let assumption \( (A_0) \) be satisfied. Let \( M \) be a manifold with a pole. Suppose that condition (3.12) is satisfied; moreover, assume that

\[
\int_1^\infty \frac{V_{\sigma}(\rho)}{S_{\sigma}(\rho)} \, d\rho = \infty,
\]

(3.16)

where \( V_{\sigma} \) and \( S_{\sigma} \) are given by (3.8) and (3.7) with \( \sigma \) defined in (3.12).

Let \( u \) be a subsolution and \( v \) a supersolution to problem (1.1). Then \( u \leq v \) in \( M \times (0, T) \).

**Proof.** At first we construct a classical supersolution \( z = z(\rho(x)) \) to equation

\[
\Delta z = 1 \quad \text{in} \ M \setminus B(o, 1),
\]
such that
\[
\lim_{\rho \to \infty} z(\rho) = \infty.
\] (3.17)

To this aim, let us distinguish two cases.

(a) Assume that
\[
\int_1^\infty \frac{d\xi}{S_\sigma(\xi)} = \infty.
\] (3.18)

Define
\[
z(x) \equiv z(\rho(x)) := \int_1^{\rho(x)} \frac{d\xi}{S_\sigma(\xi)} \quad (x \in M \setminus B(o, 1)).
\]

Note that by (3.12) and Lemma 3.3, (3.13) holds. Since \(z_\rho \geq 0\), from (3.4) and (3.13) it follows
\[
\Delta z = z_{\rho\rho} + m(\rho, \theta)z_\rho \leq z_{\rho\rho} + (N - 1) \frac{\sigma'(\rho)}{\sigma(\rho)} z_\rho = 0 \quad \text{in } M \setminus \overline{B(o, 1)};
\]
moreover, (3.18) yields (3.17).

(b) Assume that
\[
\int_1^\infty \frac{d\xi}{S_\sigma(\xi)} < \infty.
\] (3.19)

Define
\[
z(x) \equiv z(\rho(x)) := \int_1^{\rho(x)} \left( \int_1^t \frac{1}{S_\sigma(t)} \int_1 S_\sigma(\xi) d\xi dt \right) \quad (x \in M \setminus B(o, 1)).
\]

Since \(z_\rho \geq 0\), by (3.4) and (3.13),
\[
\Delta z = z_{\rho\rho} + m(\rho, \theta)z_\rho \leq z_{\rho\rho} + (N - 1) \frac{\sigma'(\rho)}{\sigma(\rho)} z_\rho = 1 \quad \text{in } M \setminus \overline{B(o, 1)}.
\]
Furthermore, (3.16) and (3.8) imply (3.17).

Since \(z_\rho \geq 0\), we can construct in both cases (a) and (b) a solution to problem
\[
\begin{cases}
\Delta w = 0 & \text{in } M \setminus \overline{B(o, 1)}, \\
w = -z & \text{on } \partial B(o, 1)
\end{cases}
\] (3.20)
such that
\[
- \max_{\partial B(o, 1)} z \leq w \leq 0 \quad \text{in } M \setminus \overline{B(o, 1)}.
\] (3.21)

Then
\[
\tilde{Z} := z + w \quad \text{in } M \setminus \overline{B(o, 1)}
\]
is a solution to problem
\[
\begin{cases}
\Delta \tilde{Z} = 1 & \text{in } M \setminus \overline{B(o, 1)}, \\
\tilde{Z} = 0 & \text{on } \partial B(o, 1);
\end{cases}
\] (3.22)
moreover, from (3.17) it follows
\[
\lim_{x \to \infty} \tilde{Z}(x) = \infty.
\] (3.23)
Since $z_\rho \geq 0$, we have that
\[
\tilde{Z} = z + w \geq z(\rho(x)) - \max_{\partial B(o,1)} z \geq 0 \quad \text{in } M \setminus \overline{B(o,1)}.
\] (3.24)

Let $W \in C^2(B(o,1))$ be the solution to problem
\[
\begin{cases}
\Delta W = 1 & \text{in } B(o,1), \\
W = 0 & \text{on } \partial B(o,1).
\end{cases}
\] (3.25)

By the strong maximum principle,
\[
\frac{\partial W}{\partial \nu} \geq \alpha \quad \text{on } \partial B(o,1)
\] (3.26)
for some constant $\alpha > 0$; here $\nu$ is the outer normal to $\partial B(o,1)$.

Define
\[
Z := \begin{cases}
H \tilde{Z} + 1 + \max_{B(o,1)} |W| & \text{in } M \setminus B(o,1), \\
W + 1 + \max_{B(o,1)} |W| & \text{in } B(o,1).
\end{cases}
\] (3.27)

where $H > 0$ is a constant to be chosen.

Clearly, $Z \in C(M)$ and
\[
Z \geq 1 \quad \text{in } M
\] (3.28)
(see (3.24)); furthermore, (3.23) implies
\[
\lim_{x \to \infty} Z(x) = \infty.
\]

From (3.22), (3.25) and (3.26) it easily follows that there exists $H > 0$ such that
\[
\int_M Z \Delta \psi \, d\mu \leq \int_M \mu \psi \, d\mu
\]
for any $\psi \in C^2_0(M)$, $\psi \geq 0$; here $\mu := \max\{1, H\}$. Hence $Z$ is a weak supersolution to equation
\[
\Delta Z = \mu \quad \text{in } M
\]
(see Definition 3.1 with $\lambda = 0$ and $f \equiv \mu$).

By (3.27), $Z$ is also a supersolution to Eq. (3.2) with $\lambda = \mu$. Then the conclusion follows from Proposition 3.2. \qed

**Remark 3.6.** (i) If (3.18) holds, then hypothesis (3.16) of Proposition 3.5 is not used.

(ii) In the proof of Proposition 3.8, if (3.19) holds, we can also define
\[
Z(x) := \int_0^{\rho(x)} \frac{1}{S_\sigma(t)} \int_0^t S_\sigma(\xi) \, d\xi + 1 \, dt \quad (x \in M).
\]
It is direct to check that $z$ is a solution to equation
\[
\Delta z = 1 \quad \text{in } M,
\] and a supersolution to Eq. (3.2) with $\lambda = 1$, since $z \geq 1$; moreover, it satisfies (3.17). Thus the conclusion follows from Proposition 3.5.

(iii) Proposition 3.5 could also be shown for geodesically complete noncompact manifold. Since in this case $\text{Cut}(o) \neq \emptyset$, some difficulties arise. However they can be handled by the same method as in [3] (see also Theorem 15.1(i) in [11]). Moreover, in this case we must consider distributional solutions to Eq. (3.2) that are not necessarily continuous in $M$ (see Definition 3.1).

However, in Proposition 3.5 we have considered manifolds with a pole, since in the sequel we will apply it for Cartan–Hadamard manifolds, that are manifolds with a pole.

**Corollary 3.7.** Let assumptions $(A_0)–(A_1)$ be satisfied. Let $M$ be a manifold with a pole. Suppose that, for some $\beta > 0$,
\[
\text{Ric}_o(x) \geq -(N - 1)\beta^2 \quad \text{for any } x \in M.
\] (3.28)

Let $u$ be a subsolution and $v$ a supersolution to problem (1.1). Then $u \leq v$ in $M \times (0, T)$. 
Proof. Hypothesis (3.28) implies that (3.12) is satisfied with \( \sigma(\rho) = \frac{1}{\beta} \sinh(\beta \rho) \) for any \( \rho \geq 0 \). It is immediate to check that (3.16) is satisfied. Hence the conclusion follows from Proposition 3.5.

In the sequel, we shall use next result.

**Proposition 3.8.** Let assumptions \((A_0)\)–\((A_1)\) be satisfied. Let \( u \) be a weak solution to problem (1.1). Then \( u \) is also a mild solution to problem (1.1).

The proof of Proposition 3.8 makes use of comparison principle, which follows from assumption \((A_1)\) and Proposition 3.5. However, this proof is omitted, for it is similar to that of an analogous result given in [18] (see also [1]).

4. Main results

4.1. Local existence

We have next local existence result.

**Theorem 4.1.** Let assumption \((A_0)\) be satisfied; suppose that \( u_0 \geq 0 \) in \( M \). Then there exists a nonnegative weak solution to problem (1.1), for some \( T > 0 \). Either the solution is global, or it blows up in finite time. Furthermore, if assumption \((A_1)\) is also satisfied, then the weak solution is unique.

4.2. Finite time blow-up

In the sequel, we shall assume that

\[
\begin{align*}
(i) & \quad M \text{ is a Cartan–Hadamard manifold;} \\
(ii) & \quad \text{there exists } k > 0 \text{ such that for any } p \in T_pM \text{ and for any plane } \pi \subseteq T_pM \text{ there holds } K_\pi(p) \leq -k^2.
\end{align*}
\]

\((A_2)\)

Set

\[
H(t) := \int_0^t h(s) \, ds \quad \text{for any } t \geq 0.
\]

We shall prove the following finite time blow-up result.

**Theorem 4.2.** Let assumptions \((A_0)\)–\((A_2)\) be satisfied; suppose that \( u_0 \geq 0, u_0 \not\equiv 0 \). Moreover, assume that

\[
\lim_{t \to \infty} \frac{[H(t)]^{1/p}}{e^{\lambda_1(M) + \epsilon t}} = \infty
\]

(4.1)

for some \( \epsilon \in (0, \lambda_1(M)) \). Then the weak solution to problem blows up in finite time.

**Remark 4.3.** From the proof of Theorem 4.2 it follows that if \( u \) is a mild solution to problem (1.1), then \( u \) blows up in finite time. Furthermore, in this case, assumption \((A_1)\) can be removed.

**Remark 4.4.** (i) If for some \( \alpha_1 > 0, \alpha_2 > 0, t_0 > 0 \) and \( q > -1 \)

\[
\alpha_1 t^q \leq h(t) \leq \alpha_2 t^q \quad \text{for any } t > t_0,
\]

then assumption (4.1) is satisfied for \( \epsilon \in (0, \lambda_1(M)) \).

(ii) Let

\[
\frac{\alpha}{p - 1} > \lambda_1(M).
\]

If

\[
h(t) = e^{\alpha t} \quad \text{for any } t \geq 0,
\]

then (4.1) is satisfied for appropriate \( \epsilon \in (0, \lambda_1(M)) \).
4.3. Global existence

Let
\[ \tilde{h}(t) := h(t)e^{-(p-1)\lambda_1(M)t} \text{ for any } t \geq 0, \]
\[ \tilde{H}(t) := \int_0^t \tilde{h}(s) \, ds \text{ for any } t \geq 0, \]
\[ \tilde{H}_\infty := \lim_{t \to \infty} \tilde{H}(t). \]

Consider the elliptic equation
\[ \Delta \varphi + \lambda \varphi = 0 \text{ in } M. \tag{4.2} \]
It is well known that for any \( \lambda \leq \lambda_1(M) \) there exists a classical positive solution \( \varphi_\lambda \) to Eq. (4.2) (see [4,13]). When \( \lambda = \lambda_1(M) \), then \( \varphi_\lambda \) is called a ground state on \( M \).

Suppose that \( \tilde{H}_\infty < \infty; \) \tag{4.3}
furthermore, suppose that such a positive solution \( \varphi_{\lambda_1(M)} \equiv \varphi_1 \) is bounded in \( M \). Then choose \( C > 0 \) such that
\[ \| \varphi_1 \|_\infty < \frac{1}{C} \left[ \frac{1}{(p-1)\tilde{H}_\infty} \right]^{\frac{1}{p-1}}. \tag{4.4} \]

Then define
\[ \tilde{\varphi}_1(x) := C \varphi_1(x) \quad (x \in M). \]

We will show the following global existence result.

\textbf{Theorem 4.5.} Let assumptions \((A_0)\)–\((A_2)\) be satisfied. Suppose that \( \varphi_1 \in L^\infty(M) \) and that conditions \((4.3)\)–\((4.4)\) are satisfied. Moreover, assume that \( 0 \leq u_0 \leq \varphi_1 \) in \( M \). Then the weak solution to problem \((1.1)\) is global; in addition, there exists \( \bar{C} > 0 \) such that
\[ \| u(\cdot,t) \|_{L^\infty(M)} \leq \bar{C} \quad \text{for all } t > 0. \tag{4.5} \]

\textbf{Remark 4.6.} Observe that in Theorems 4.2 and 4.5, we can remove: (a) hypothesis \((A_2)(i)\), if we require that the estimate \((2.5)\) is satisfied; (b) hypothesis \((A_2)(ii)\), if we assume that \( \lambda_1(M) > 0. \)

\textbf{Remark 4.7.} (i) Let
\[ \frac{\alpha}{p-1} < \lambda_1(M). \]

If
\[ h(t) = e^{\alpha t} \quad \text{for any } t \geq 0, \]
then hypothesis \((4.3)\) is satisfied.

(ii) Theorem 4.5 remains true, if we suppose that \( \phi \) is a positive bounded supersolution to Eq. (4.2) with \( \lambda = \lambda_1(M) \). This easily follows from its proof.

Sufficient conditions for \( \varphi_1 \in L^\infty(M) \) can be found in \([7]\), where specific hypotheses on \( \text{spec}(-\Delta) \) and \( \varphi_1 \in L^2(M) \) are made, and in \([5]\), where it is assumed that \( \varphi_1 \in L^2(M) \) and \( \mu(M) < \infty. \)

A case in which \( \varphi_1 \in L^\infty(M) \setminus L^2(M) \) will be addressed in Lemma 6.1 and Remark 6.2.

For general Riemannian manifolds it is an open problem to understand when \( \varphi_1 \in L^\infty(M) \), without requiring \( \varphi_1 \in L^2(M) \).
5. Local existence and finite time blow-up: proofs

Proof of Theorem 4.1. We can find a unique $T > 0$ such that

$$H(T) = \frac{1}{(p-1)\|u_0\|_{\infty}^{p-1}}.$$ 

Let $\{\Omega_n\}_{n \in \mathbb{N}}$ be a sequence of domains $\{\Omega_n\}_{n \in \mathbb{N}} \subseteq M$ such that $\bar{\Omega}_n \subseteq \Omega_{n+1}$ for every $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} \Omega_n = M$, $\partial \Omega_n$ is smooth for every $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$ let $u_n$ be the unique classical solution to problem

$$\begin{aligned}
\bar{\partial}_t u &= \Delta u + h(t)u^p \quad \text{in } \Omega_n \times (0, T), \\
u &= 0 \quad \text{in } \partial \Omega_n \times (0, T), \\
u &= u_0 \quad \text{in } \Omega_n \times \{0\}.
\end{aligned} \tag{5.1}$$

It is direct to show that

$$\bar{u}(t) := \|u_0\|_{\infty}[1 - (p - 1)\|u_0\|_{\infty}^{p-1}H(t)]^{-\frac{1}{p-1}} \quad (t \in [0, T))$$

is a classical supersolution to problem (5.1). On the other hand, $\bar{u} \equiv 0$ is a subsolution to the same problem. By comparison principle,

$$0 \leq u_n \leq \bar{u} \quad \text{in } M \times (0, T). \tag{5.2}$$

By standard compactness arguments, there exists a subsequence $\{u_{n_k}\} \subseteq \{u_n\}$, which converges locally uniformly in $M \times (0, T)$ to a weak solution $u$ of problem (1.1). Furthermore, by (5.2),

$$0 \leq u \leq \bar{u} \quad \text{in } M \times (0, T).$$

If, in addition, assumption $(A_2)$ is satisfied, then Proposition 3.2 implies that $u$ is unique. $\square$

In order to prove Theorem 4.2 we need two preliminary results.

Lemma 5.1. Let assumptions $(A_0)$ and $(A_2)$ be satisfied, $\varepsilon \in (0, \lambda_1(M))$; suppose $u_0 \neq 0$. Then there exist a precompact set $\Omega \subseteq M$, $t_0 > 0$ and $C_1 > 0$ such that

$$\left( e^{t \Delta} u_0 \right)(x) \geq \frac{C_1}{e^{(\lambda_1(M)+\varepsilon)t}} \quad \text{for any } x \in \Omega, \ t > t_0. \tag{5.3}$$

Proof. Let $\Theta$ be a precompact subset of $M$ such that $\inf_{\Gamma} u_0 > 0, \mu(\Omega) < \infty$. Let $\varepsilon \in (0, \lambda_1(M))$. By (2.6) there exists $t_0 > 0$ such that for any $x, y \in \Omega$ and $t > t_0$ there holds

$$p(x, y, t) \geq \frac{1}{e^{(\lambda_1(M)+\varepsilon)t}};$$

hence

$$\left( e^{t \Delta} u_0 \right)(x) \geq \int_{\Omega} p(x, y, t)u_0(y) d\mu_y \geq \frac{\mu(\Omega) \inf_{\Gamma} u_0}{e^{(\lambda_1(M)+\varepsilon)t}}.$$ 

This completes the proof. $\square$

Lemma 5.2. Let assumptions $(A_0)$, $(A_2)$ be satisfied. Let there exist a mild solution to problem (1.1). Then

$$\left( e^{\tau \Delta} u_0 \right)^{p-1} \leq \frac{1}{(p-1)H(\tau)} \quad \text{for any } x \in M, \ \tau \in (0, T). \tag{5.4}$$

Proof. Let $\tau \in (0, T)$. Let $u$ be a mild solution to problem (1.1). We multiply by $p(x, z, \tau - t)$ equality (2.8) with $x$ replaced by $z$, then integrate over $M$ and use (2.3). So, we get

$$\int_{M} p(x, z, \tau - t)u(z, t) d\mu_z = \int_{M} p(x, y, \tau)u_0(y) d\mu_y$$

$$+ \int_{0}^{\tau} \int_{M} p(x, y, \tau - s)h(s)u^p(y, s) d\mu_y ds \quad (t \in (0, \tau)).$$
that is

\[
\phi_t(x, t) = \phi_t(x, 0) + \int_0^t \int_M p(x, y, \tau - s) h(s) u^p(y, s) \, d\mu_y \, ds,
\]

where we have set

\[
\phi_t(x, t) := \int_M p(x, z, \tau - t) u(z, t) \, d\mu_z \quad (x \in M, \ t \in (0, \tau]).
\]

By Jensen’s inequality,

\[
\left[ \phi_t(x, s) \right]^p \leq \int_M p(x, y, \tau - s) u^p(y, s) \, d\mu_y \quad (x \in M, \ s \in (0, \tau]).
\]

This combined with (5.5) yields

\[
\int_0^t h(s) \left[ \phi_t(x, s) \right]^p \, ds \leq \phi_t(x, t) - \phi_t(x, 0) \quad (x \in M, \ t \in (0, \tau]).
\]

Then by a Gronwall-type argument,

\[
(p - 1) H(t) \leq \frac{1}{\left[ \phi_t(x, 0) \right]^{p-1}} - \frac{1}{\left[ \phi_t(x, t) \right]^{p-1}} \quad (x \in M, \ t \in (0, \tau]),
\]

hence the conclusion immediately follows. □

Now we can show Theorem 4.2.

**Proof of Theorem 4.2.** By contradiction, suppose that the unique weak solution to problem (1.1) is a global solution.

Now, take \( \Omega \subseteq M \) and \( \varepsilon > 0 \) as in Lemma 5.1. Hence

\[
\phi_t(x, 0) \geq \frac{C_1}{e^{\lambda_1(M) + \varepsilon} t} \quad \text{for any } x \in \Omega, \ \tau > t_0
\]

for some \( C_1 > 0 \) and \( t_0 > 0 \).

Since \( u_0 \leq \tilde{\phi}_1 \) in \( M \), by Theorem 4.1 and Proposition 3.8, the unique weak solution to problem (1.1) is also a mild solution to the same problem. Hence by Lemma 5.2,

\[
\phi_t(x, 0) \leq \left( \frac{1}{p-1} \right)^{\frac{1}{p-1}} \left[ H(t) \right]^{-\frac{1}{p-1}} \quad \text{for any } x \in M, \ \tau > 0.
\]

From (5.6)–(5.7) it follows that for any \( \tau > t_0 \) we have

\[
\frac{[H(t)]^{\frac{1}{p-1}}}{e^{\lambda_1(M) + \varepsilon} t} \leq \frac{1}{C_1 \left( \frac{1}{p-1} \right)^{\frac{1}{p-1}}}. \]

If we send \( \tau \to \infty \) in the previous inequality, we get a contradiction with (4.1); hence the proof is complete. □

6. Global existence: proofs

**Proof of Theorem 4.5.** Let

\[
\xi(t) = \left[ 1 - (p - 1) \| \tilde{\phi}_1 \|_{\infty}^{p-1} H(t) \right]^{-\frac{1}{p-1}} \quad (t \in [0, \infty))
\]

(see (4.3)–(4.4)). It is easily seen that \( \xi \) solves problem

\[
\left\{ \begin{array}{ll}
\xi' = \| \tilde{\phi}_1 \|_{\infty}^{p-1} \tilde{h}(t) \xi^p, & t \in (0, \infty), \\
\xi(0) = 1.
\end{array} \right.
\]

Define

\[
\tilde{u}(t) := e^{-\lambda_1(M) t} \xi(t) \tilde{\phi}_1(x) \quad ((x, t) \in M \times [0, \infty))
\]
Since \( u_0 < \hat{\phi}_1 \) in \( M \), from (6.1), (4.2), we can infer that \( \bar{u} \) is a bounded classical supersolution to problem (1.1) with \( T = \infty \). From Proposition 3.2 the conclusion follows. \( \square \)

For every \( h > 0 \) define
\[
 f_h(\rho) := \frac{1}{h} \sinh(h \rho) \quad (\rho \geq 0).
\]

Then \( M_{f_h} \) is a model manifold with constant negative sectional curvature \( -h^2 \). In particular, for \( h = 1 \) we have \( M_{f_1} \equiv \mathbb{H}^N \).

We shall consider positive classical solutions to equation
\[
 \Delta \phi + \lambda \phi = 0 \quad \text{in} \ M_{f_h}.
\]

In particular, we shall only consider positive radial solutions \( \phi = \phi(\rho) \); then \( \phi \) solves
\[
\begin{align*}
 \phi'' + h(N - 1) \coth(h \rho) \phi' + \lambda \phi &= 0 \quad \text{in} \ (0, \infty), \\
 \phi'(0) &= 0, \\
 \phi &> 0 \quad \text{in} \ [0, \infty). 
\end{align*}
\]  

Set
\[
 \phi(\rho) = \sinh^{-\frac{N-1}{2}}(h \rho) u(h \rho) \quad (\rho \geq 0).
\]

If \( \phi \) is a solution to (6.3), then \( u \) satisfies
\[
\begin{align*}
 u''(h \rho) &= \frac{1}{h^2} \left[ \lambda_1(M_{f_h}) - \lambda + h^2 \frac{(N-2)^2 - 1}{4 \sinh^2(h \rho)} \right] u(h \rho) \quad \text{in} \ (0, \infty), \\
 u(0) &= 0, \\
 u &> 0 \quad \text{in} \ (0, \infty),
\end{align*}
\]  

since \( \lambda_1(M_{f_h}) = \frac{(N-1)^2}{4} h^2 \). Hence analogously to Lemma A.1 in [1] we have next

**Lemma 6.1.** For any \( \lambda \leq \lambda_1(M_{f_h}) \) and \( c > 0 \) there exists a unique positive classical radial solution to Eq. (6.2) such that \( \phi(0) = c \). Furthermore, if \( \lambda > 0 \), then
\[
 \lim_{\rho \to \infty} \phi(\rho) = 0.
\]

**Remark 6.2.** Note that, as observed in Remark A.1 in [1], in general \( \phi \notin L^2(M_{f_h}) \).

**References**


