JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **158**, 448–455 (1991)

An Iterative Algorithm for Variational Inequalities

MUHAMMAD ASLAM NOOR

Mathematics Department, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

Submitted by E. Stanley Lee

Received July 25, 1989

In this paper, we introduce and study a new class of nonlinear variational inequalities. This new class enables us to apply variational techniques to the solution of differential equations of both odd and even orders. A projection method is used to suggest an iterative algorithm for finding the approximate solution of this class. We also discuss the convergence criteria of the proposed iterative algorithm. Several special cases are discussed, which can be obtained from the general result.
© 1991 Academic Press, Inc.

1. INTRODUCTION

Variational inequality theory has become an effective and powerful technique for studying a wide class of problems arising in various branches of mathematical and engineering sciences. The variety of problems to which variational inequality techniques may be applied is impressive and amply representative for the richness of the field. This theory has been developed in several directions. Some of these developments have made mutually enriching contacts with other areas of pure and applied sciences including elasticity, fluid dynamics, transportation and economics equilibrium, and operations research. In recent years, variational inequalities have been extended and generalized in various directions. It is worth mentioning that differential equations of odd order cannot be studied in the general framework of the variational inequalities.

Inspired and motivated by the recent research work going on in this field, we introduce and study a new class of variational inequalities. This class enables us to study differential equations of both odd and even order, which is the main motivation of this paper. A projection technique is used to suggest an iterative algorithm for finding the approximate solution. We also discuss the convergence criteria of the iterative algorithm.

In Section 2, after reviewing some basic notations and results, we introduce the nonlinear variational inequality problem. Algorithms and con-

0022-247X/91 $3.00

vergence results are considered and discussed in Section 3. In Section 4, we consider a simple example to illustrate the application of the results developed in Sections 2 and 3.

2. Formulation and Basic Facts

Let H be a real Hilbert space with its dual H', whose inner product and norm are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Let C be a closed convex set in H. We denote by $\langle \cdot, \cdot \rangle$ the pairing between H' and H.

Given continuous mappings $T, g\colon H \to H'$, we consider the problem of finding $u \in H$ such that $g(u) \in C$, and

$$\langle Tu, g(v) - g(u) \rangle \geqslant \langle A(u), g(v) - g(u) \rangle, \qquad \text{for all} \quad g(v) \in C, \tag{2.1}$$

where $A(u)$ is a nonlinear continuous mapping such that $A(u) \in H'$. The inequality (2.1) is known as the general mildly nonlinear variational inequality.

Special Cases

I. Note that, if $g(u) = u \in C$, then Problem (2.1) is equivalent to finding $u \in C$ such that

$$\langle Tu, v - u \rangle \geqslant \langle A(u), v - u \rangle, \qquad \text{for all} \quad v \in C. \tag{2.2}$$

Inequalities (2.2) are known as mildly (strongly) nonlinear variational inequalities, which were introduced and considered by Noor [1, 2] in the theory of constrained mildly (strongly) nonlinear partial differential equations. For the finite element error estimates of these variational inequalities, see Noor [3].

II. If the nonlinear transformation $A(u) \equiv 0$ (or $A(u)$ is independent of the solution u, that is $A(u) \equiv f$ (say)), then (2.1) is equivalent to finding $u \in C$ such that $g(u) \in C$ and

$$\langle Tu, g(v) - g(u) \rangle \geqslant 0, \qquad \text{for all} \quad g(v) \in C. \tag{2.3}$$

The variational inequalities of the type (2.3) were studied by Noor [4], whose solution can be obtained by an iterative method.

III. If $A(u) \equiv 0$ and $g = I$, the identity mapping, then problem (2.1) is equivalent to finding $u \in C$ such that

$$\langle Tu, v - u \rangle \geqslant 0, \qquad \text{for all} \quad v \in C. \tag{2.4}$$

The problem (2.4) is originally due to Stampacchia [5]; see also [4, 6].

IV. If $C^* = \{u \in H', \langle u, v \rangle \geqslant 0 \text{ for all } v \in C\}$ is a polar of the convex cone C in H, then Problem (2.1) is equivalent to finding $u \in H$ such that

$$g(u) \in C, \qquad (Tu - A(u)) \in C^*, \qquad \langle Tu - A(u), g(u) \rangle = 0, \tag{2.5}$$

which is known as the general mildly nonlinear complementarity problem. Problem (2.5) appears to be a new one.

It is clear that Problems (2.2)–(2.5) are special cases of Problem (2.1). In brief, Problem (2.1) is the most general and unifying one, which is one of the main motivations of this paper.

3. Iterative Algorithms

In recent years, various numerical methods have been developed and applied to find approximate solutions of variational inequalities, including the projection method. To suggest an iterative algorithm for solving (2.1), we need the following results.

LEMMA 3.1. *If C is a convex set in H, then $u \in H$ is the solution of Problem* (2.1) *if and only if u satisfies the relation*

$$g(u) = P_C[g(u) - \rho A(Tu - A(u)],\tag{3.1}$$

where $\rho > 0$ is a constant and P_C is the projection of H onto C. Here A is the canonical isomorphism from H' onto H such that for all $v \in H$ and $f \in H'$,

$$\langle f, v \rangle = (Af, v).\tag{3.2}$$

Proof. Its proof is similar to that of Lemma 3.1 in [7].

From Lemma 3.1, we conclude that Problem (2.1) can be transformed into the fixed point problem of solving

$$u = F(u),$$

where

$$F(u) = u - g(u) + P_C[g(u) - \rho A(Tu - A(u))].\tag{3.3}$$

This formulation is very useful in approximation and numerical analysis of variational inequalities. One of the consequence of this formulation is that we can obtain an approximate solution of (2.1) by an iterative algorithm.

ALGORITHM 3.1. Given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - g(u_n) + P_C[g(u_n) - \rho A(Tu_n - A(u_n)], \qquad n = 0, 1, 2, \tag{3.4}$$

where $\rho > 0$ is a constant.

Special Cases

(i) We note that if $g(u) = u \in C$, then Algorithm 3.1 reduces to

ALGORITHM 3.2 [8, 9]. Given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_C[u_n - \rho A(Tu_n - A(u_n)], \qquad n = 0, 1, 2, \ldots$$

(ii) If $A(u) \equiv 0$, then Algorithm 3.1 becomes

ALGORITHM 3.3 [4]. Given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - g(u_n) + P_C[g(u_n) - \rho A\ Tu_n], \qquad n = 0, 1, 2, \ldots$$

(iii) If $g(u) = u \in C$ and $A(u) \equiv 0$, then Algorithm 3.1 reduces to

ALGORITHM 3.4 [10, 11]. Given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_C[u_n - \rho A\ Tu_n], \qquad n = 0, 1, 2, \ldots$$

In brief, Algorithm 3.1 proposed here is more general and includes several previously known algorithms as special cases, which are mainly due to Glowinski, Lions, and Tremolieres [10] and Noor [9].

We also need the following concepts.

DEFINITION 3.1. A mapping $T: H \to H'$ is said to be

(a) *Strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geqslant \alpha \|u - v\|^2, \qquad \text{for all} \quad u, v \in H.$$

(b) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leqslant \beta \|u - v\|, \qquad \text{for all} \quad u, v \in H.$$

In particular, it follows that $\alpha \leqslant \beta$. We now study the conditions under which the approximate solution computed by Algorithm 3.1 converges to the exact solution of the variational inequality (2.1).

THEOREM 3.1. *Let the mappings $T, g: H \to H'$ be strongly monotone and Lipschitz continuous, respectively. If the mapping A is Lipschitz continuous, then*

$$u_{n+1} \to u \text{ strongly in } H,$$

for

$$\left|\rho - \frac{\alpha+\gamma(k-1)}{\beta^2-\gamma^2}\right| < \frac{\sqrt{(\alpha+\gamma(k-1)^2-(\beta^2-\gamma^2)\,k(2-k)}}{\beta^2-\gamma^2}, \qquad k<1,$$

$$\alpha > \gamma(1-k)+\sqrt{(\beta^2-\gamma^2)\,k(2-k)} \qquad \textit{and} \qquad \gamma(1-k)<\alpha,$$

where u_{n+1} *and* u *are solutions satisfying* (3.4) *and* (2.1), *respectively.*

Proof. From Lemma 3.1, we conclude that the solution u of (2.1) can be characterized by the relation (3.1). Hence from (3.1) and (3.3), we obtain

$$\begin{aligned}
\|u_{n+1}-u\| &= \|u_n-u-(g(u_n)-g(u))+P_C[g(u_n)-\rho A(Tu_n-A(u))] \\
&\quad - P_C[g(u)-\rho A(Tu-A(u))]\| \\
&\leqslant \|u_n-u-(g(u_n)-g(u)\| \\
&\quad + \|P_C[g(u_n)-\rho A(Tu_n-A(u_n))] \\
&\quad - P_C[g(u)-\rho A(Tu-A(u))] \\
&\leqslant 2\,\|u_n-u-(g(u_n)-g(u))\| \\
&\quad + \|u_n-u-\rho A(Tu_n-Tu)+\rho A(A(u_n)-A(u))\|, \qquad (3.5)
\end{aligned}$$

since P_C is a non-expansive mapping [10].

Since T, g are both strongly monotone and Lipschitz continuous, by using the technique of Noor [7], we have

$$\|u_n-u-(g(u_n)-g(u))\|^2 \leqslant (1-2\delta+\sigma^2)\,\|u_n-u\|^2 \qquad (3.6)$$

and

$$\|u_n-u-\rho A(Tu_n-Tu)\|^2 \leqslant (1-2\rho\alpha+\rho^2\beta^2)\,\|u_n-u\|^2. \qquad (3.7)$$

From (3.5), (3.6), (3.7) and by using the Lipschitz continuity of A, we obtain

$$\begin{aligned}
\|u_{n+1}-u\| &\leqslant \{(2\sqrt{1-2\delta+\sigma^2})+\rho\gamma+\sqrt{1-2\alpha\rho+\rho^2\beta^2}\}\,\|u_n-u\| \\
&= \{k+\rho\gamma+t(\rho)\}\,\|u_n-u\| \\
&= \theta\,\|u_n-u\|,
\end{aligned}$$

where

$$k = 2\sqrt{1-2\delta+\sigma^2},$$

$$t(\rho) = \sqrt{1-2\alpha\rho+\rho^2\beta^2},$$

and

$$\theta = k + \rho\gamma + t(\rho).$$

Now $t(\rho)$ assumes its minimum value for $\bar{\rho} = \alpha/\beta^2$ with $t(\rho) = \sqrt{1 - \alpha^2/\beta^2}$. We have to show that $\theta < 1$. For $\rho = \bar{\rho}$, $k + \rho\gamma + t(\bar{\rho}) < 1$ implies that $k < 1$ and $\alpha > \gamma(1-k) + \sqrt{(\beta^2 - \gamma^2)\, k(2-k)}$. Thus it follows that $\theta = k + \rho\gamma + t(\rho) < 1$ for all ρ with

$$\left|\rho - \frac{\alpha + \gamma(k-1)}{\beta^2 - \gamma^2}\right| < \frac{\sqrt{(\alpha + \gamma(k-1))^2 - (\beta^2 - \gamma^2)\, k(2-k)}}{\beta^2 - \gamma^2}, \qquad k < 1,$$

$$\alpha > \gamma(1-k) + \sqrt{(\beta^2 - \gamma^2)\, k(2-k)}, \qquad \text{and} \qquad \gamma(1-k) < \alpha.$$

Since $\theta < 1$, so the fixed point problem (3.1) has a unique solution u and consequently, the iterative solution u_{n+1} obtained from (3.3) converges to u, the exact solution of the problem (2.1).

Remarks 3.1. 1. If $g = I$, the identity mapping, then we obtain a result of Noor [9]. In this case, $k = 0$ and (3.2) becomes

$$F(u) = P_C[u - \rho A(Tu - A(u))],$$

and $\theta = \rho\gamma + t(\rho) < 1$ for $0 < \rho < 2(\alpha - \gamma)/(\beta^2 - \gamma^2)$, $\rho\gamma < 1$, and $\gamma < \alpha$. Consequently, the mapping $F(u)$ has a fixed point, which is the solution of (2.2); see Noor [9].

2. If $g = I$, the identity mapping, and $A(u) \equiv 0$, then we obtain a classical result. In this case, $k = 0$, $\gamma = 0$, and (3.2) bcomes

$$F(u) = P_C[u - \rho A\; Tu]$$

and $\theta = t(\rho) < 1$ for $0 < \rho < 2\alpha/\beta^2$. Thus the mapping $F(u)$ has a fixed point, which is the solution of (2.4), see [10, 11].

3. If $A(u) \equiv 0$, then we obtain a result of Noor [4]. Then $\gamma = 0$ and (3.2) becomes

$$F(u) = u - g(u) + P_C[g(u) - \rho A\; Tu],$$

and

$$\theta = k + t(\rho) < 1 \qquad \text{for} \quad k < 1, \alpha > \beta\sqrt{k(k-2)},$$

$$\text{and} \left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \beta^2(2k - k^2)}}{\beta^2}.$$

Consequently the mapping $F(u)$ has a fixed point, which is the solution of the problem (2.3); see Noor [4].

4. Applications

A large number of differential equation problems of odd and even order can be characterized by a class of variational inequalities of the type (2.1). For simplicity, we consider the third order two-point boundary value problem

$$\left.\begin{array}{lr} Tu \geqslant f(x, u(x)) & \text{in } \ D \\ u(x) \geqslant \psi(x) & \text{in } \ D \\ [Au - f(x, u(x))][u(x) - \psi(x)] = 0 & \text{in } \ D \\ u = 0 \quad \text{and} \quad u' = 0 & \text{on } \ S \end{array}\right\}, \tag{4.1}$$

where D is a domain in R^2 with boundary $S = [0, 1]$, $T = -d^3/dx^3$ is the differential operator of third order, f is a given nonlinear function of x, and $\psi(x)$ is the given obstacle function. To study the problem (4.1) in the variational inequality framework, we define

$$C = \{u \in H_0^2(\Omega), u(x) \geqslant \psi(x) \text{ on } D\},$$

which is a closed convex set in $H_0^2(\Omega)$. Now using the technique of K-positive definite operators, as developed in [12], we can show that the problem (4.1) is equivalent to finding $u \in H_0^2(\Omega)$ such that $Ku \in C$ and

$$\langle Tu, Kv - Ku \rangle \geqslant \langle A(u), Kv - Ku \rangle, \qquad \text{for all} \quad Kv \in C, \tag{4.2}$$

where

$$\langle Tu, Kv \rangle = -\int_0^1 D^3u\, Dv\, dx = \int_0^1 D^2u\, D^2v\, dx \tag{4.3}$$

and

$$\langle A(u), Kv \rangle = \int_0^1 f(x, u(x))\, Dv\, dx \tag{4.4}$$

with $K = d/dx = D$.

It is clear that with $g = K$, we have the variational inequality problem (2.1). With proper choice of the mapping g and suitable assumptions on the operators T and A, we can verify all the hypothesis of Theorem 3.1.

5. Conclusion

In this paper, we have considered and studied a new class of variational inequalities, which includes the previously known ones as special cases. It is shown that the differential equations of odd order can be formulated in terms of this class. We have also suggested an iterative algorithm along with convergence criteria for finding the approximate solution of the variational inequalities. Development and improvement of an implementable algorithm for this class of variational inequalities deserve further research efforts. For related work, see also Noor [13, 14].

References

1. M. Aslam Noor, "On Variational Inequalities," Ph. D. thesis, Brunel University, 1975.
2. M. Aslam Noor, Mildly nonlinear variational inequalities, *Mathematica (Cluj)* **47** (1982), 99–110.
3. M. Aslam Noor, Error analysis of mildly nonlinear variational inequalities, *Caribbean J. Math.* **1** (1983), 49–62.
4. M. Aslam Noor, General variational inequalities, *Appl. Math. Lett.* **1** (1988), 119–122.
5. G. Stampacchia, Formes bilinéaires coercives sur les ensembles convexes, *C. R. Acad. Sci. Paris* (1964), 4413–4416.
6. J. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493–519.
7. M. Aslam Noor, An iterative scheme for a class of quasi variational inequalities, *J. Math. Anal. Appl.* **110** (1985), 463–468.
8. M. Aslam Noor and K. Inayat Noor, Iterative methods for variational inequalities and nonlinear programming, *Methods Oper. Res.* **31** (1979), 455–463.
9. M. Aslam Noor, Strongly nonlinear variational inequalities, *C. R. Math. Rep. Acad. Sci. Canada* **4** (1982), 213–218.
10. R. Glowinski, J. Lions, and R. Tremolieres, "Numerical Analysis of Variational Inequalities," North-Holland, Amsterdam, 1982.
11. K. Inayat Noor and M. Aslam Noor, Iterative methods for a class of variational inequalities, *in* "Numerical Analysis of Singular Perturbation Problems" (J. Miller and W. Hemker, Eds.), pp. 441–449, Academic Press, New York, 1979.
12. W. V. Petryshyn, Direct and iterative methods for the solution of linear operator equations in Hilbert Space, *Trans. Amer. Math. Soc.* **105** (1962), 136–175.
13. M. Aslam Noor, On nonlinear variational inequalities, *Int. J. Math. Math. Sci.* **14** (1991), to appear.
14. M. Aslam Noor, General algorithm and sensitivity analysis for variational inequalities, *J. Appl. Math. & Stoch. Anal.* **159** (1991), to appear.