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Axial anomaly and the precise value of the $\pi^0 \to 2\gamma$ decay width

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Abstract

The anomaly in the vacuum expectation value of the product of axial and two vector currents (AVV) in QCD is investigated. The goal is to determine from its value the $\pi^0 \to 2\gamma$ decay width with high precision. The sum rule for AVV formfactor is studied. The difference $f_{\pi^0} - f_{\pi^+}$ caused by strong interaction is calculated and appears to be small. The π^0–η mixing is accounted. The $\pi^0 \to 2\gamma$ decay width determined theoretically from the axial anomaly is found to be $\Gamma(\pi^0 \to 2\gamma) = 7.93$ eV with an error $\sim 1.5\%$. The measurement of the $\pi^0 \to 2\gamma$ decay width at the same level of accuracy would allow one to achieve a high precision test of QCD.

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The statement that in massless QED the axial anomaly is contributed by massless pseudoscalar state had been first done by Dolgov and Zakharov in 1971 [1]. Later this problem was investigated in many papers. (See, e.g., the book [2] and the reviews [3,4]). Now it is known, that the anomaly for transition of axial isovector current into two massless vector currents, i.e., into two photons, is almost completely exhausted by the contribution of $\pi^0 \to 2\gamma$ decay. However, the accuracy of the calculation of the $\pi^0 \to 2\gamma$ decay contribution is about 5–7% [4] and of the same order is the experimental error in the determination of the $\pi^0 \to 2\gamma$ decay width $\Gamma(\pi^0 \to 2\gamma)$ [5]. The PriMex experiment at JLab [6], where it is planned to reduce the error in the experimental value of $\Gamma(\pi^0 \to 2\gamma)$ down to 1–2%, is under way. Also, it is desirable to have theoretical prediction at the same level of accuracy and therefore, to have a possibility of high precision test of the anomaly—a significant ingredient of the modern field theory.

The calculation of hadronic contribution to the AVV anomaly at the desired level of accuracy was done by Moussalam [7] and by Goity, Bernstein and Holstein [8] in the framework of the Chiral Effective Theory (CET) (in the second order) and of $1/N_c$ expansion. In these calculations the result is expressed through the parameters of the CET Lagrangian in the second order, which were determined from the set of the data. In the present paper we perform such calculation in QCD using the dispersion relation representation for the AVV formfactor, QCD sum rules to determine $f_{\pi^0} - f_{\pi^+}$ difference and accounting for $\pi^0 - \eta$ mixing. The only parameter, which enters the result, is the value $\Gamma(\eta \to 2\gamma)$.

The notation

$$T_{\mu\alpha\beta}(p, p') = \langle p, \varepsilon_\alpha; p', \varepsilon'_\beta | j^{(3)}_{\mu 5} | 0\rangle \tag{1}$$

is used for the matrix element of the transition of the isovector axial current

$$j^{(3)}_{\mu 5} = (\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d)/\sqrt{2} \tag{2}$$

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into two photons with momenta p, p' an polarizations ε_α, ε'_β. (Here u and d are the fields of u and d quarks.) The general form of $T_{\mu\alpha\beta}(p,p')$ is ($q = p + p'$) [4,9,10]:

$$T_{\mu\alpha\beta}(p,p') = F_1(q^2) q_\mu \varepsilon_{\alpha\beta\rho\sigma} p_\rho p'_\sigma + \frac{1}{2} F_2(q^2)(p_\alpha \varepsilon_{\mu\beta\rho\sigma} - p'_\beta \varepsilon_{\mu\alpha\rho\sigma}) p_\rho p'_\sigma. \tag{3}$$

The functions $F_1(q^2)$ and $F_2(q^2)$ can be represented by nonsubtracted dispersion relations and in QCD the anomaly condition can be written as the sum rule [4,9]:

$$\int\limits_{(m_u+m_d)^2}^{\infty} \mathrm{Im}\, F_1(q^2)\, dq^2 = \sqrt{2}\alpha(e_u^2 - e_d^2) N_c, \tag{4}$$

where e_u and $_d$ are u- and d-quarks electric charges, $e_u = 2/3$, $e_d = -1/3$, N_c is the number of colours, $N_c = 3$. Note, that at large q^2 the function $\mathrm{Im}\, F_1(q^2) \sim (1/q^4) \ln q^2$ and the integral in (4) is well converging. Emphasize, that in QCD there are no perturbative corrections to Eq. (4) [11] and it is expected, that nonperturbative corrections are absent also.

Let us saturate the left-hand side (l.h.s.) of (4) by the π^0 contribution. Use the relation

$$\langle 0| j^{(3)}_{\mu 5} |\pi^0\rangle = i f_{\pi^0} q_\mu. \tag{5}$$

The general form of the π^0-contribution to $T_{\mu\alpha\beta}(p,p')$ is

$$T_{\mu\alpha\beta}(p,p') = -f_{\pi^0} \frac{1}{q^2 - m_\pi^2} A_\pi q_\mu \varepsilon_{\alpha\beta\lambda\sigma} p_\lambda p'_\sigma, \tag{6}$$

where A_π is a constant. In the approximation, when only pion contribution is accounted in the l.h.s. of (4) the constant A_π is found by substituting (6) into (4). The result is (numerical values of e_u, e_d, N_c were used):

$$A_\pi^{(1)} = \frac{\alpha}{\pi} \frac{1}{f_{\pi^0}}. \tag{7}$$

The $\pi^0 \to 2\gamma$ decay width is easily calculated from (6), (7),

$$\Gamma^{(0)}(\pi^0 \to 2\gamma) = \frac{\alpha^2}{32\pi^3} \frac{m^3_{\pi^0}}{f^2_{\pi^0}}. \tag{8}$$

Index (0) at Γ means that the saturation of the sum rule (4) by the π^0 state was exploited. At $f_{\pi^0} = f_{\pi^+} = 130.7 \pm 0.4$ MeV and $m_{\pi^0} = 135.0$ MeV [5] we get from (8):

$$\Gamma^{(0)}(\pi^0 \to 2\gamma) = 7.73 \text{ eV}. \tag{9}$$

Turn now to the calculation of correction to zero approximation. The first correction is due to the fact, that generally f_{π^0} is not equal to f_{π^+}. There are two sources of the $f_{\pi^0} - f_{\pi^+}$ difference: the electromagnetic interaction and violation of isospin in strong interaction. π^0 has no electromagnetic interaction and the electromagnetic interaction of π^+ was already accounted, when the value of f_{π^+} was found from the $\pi^+ \to \mu^+\nu$ decay data [5]. So, the electromagnetic interaction does not change the value of f_{π^0} in comparison with the presented above value of f_π^+. (The discussion of magnitude of $f_{\pi^0} - f_{\pi^+}$ difference, caused by electromagnetic interaction was done in Ref. [8].) In order to find $f_{\pi^0} - f_{\pi^+}$ caused by strong interaction consider the polarization operators $\Pi^{(3)}_{\mu\nu}$ and $\Pi^{(+)}_{\mu\nu}$ of axial currents $j^{(3)}_{\mu 5}$ (2) and $j^{(+)}_{\mu 5} = \bar{u}\gamma_\mu\gamma_5 d$. (The presented below method is exposed in [12,13] and reviewed in [14].) The general form of these polarization operators is

$$\Pi^{(i)}_{\mu\nu}(q) = \Pi^{(i)}_T(q^2)(q_\mu q_\nu - \delta_{\mu\nu} q^2) + q_\mu q_\nu \Pi^{(i)}_L(q^2), \quad i = 3, +. \tag{10}$$

The pion contribution to $\Pi_{\mu\nu}(q)$ is given by [12]:

$$\Pi_{\mu\nu}(q)_\pi = -\frac{f_\pi^2}{q^2}(q_\mu q_\nu - \delta_{\mu\nu} q^2) - \frac{m_\pi^2}{q^2} q_\mu q_\nu \frac{f_\pi^2}{q^2 - m_\pi^2}. \tag{11}$$

Consider the longitudinal polarization operator $\Pi^{(i)}_L(q^2)$ to which only pseudoscalar mesons are contributing. In order to separate the interesting for us second term in (11) let us multiply (11) by $q_\mu q_\nu / q^2$ and consider the difference

$$q^2(\Pi^{(3)}_L(q^2) - \Pi^{(+)}_L(q^2)) = -\frac{m^2_{\pi^0} f^2_{\pi^0}}{q^2 - m^2_{\pi^0}} + \frac{m^2_{\pi^+} f^2_{\pi^+}}{q^2 - m^2_{\pi^+}}. \tag{12}$$

As was demonstrated by Weinberg [15], in the first order in $m_u - m_d$ the π^+ and π^0 masses are equal and the experimentally observed π^+ and π^0 mass difference arises from electromagnetic interaction. In the second order in u- and d-quark masses $m^2_{\pi^0} - m^2_{\pi^+}$ is proportional to $(m_u - m_d)^2$ or $(m_u - m_d)[\langle 0|\bar{u}u|0\rangle - \langle 0|\bar{d}d|0\rangle]$ and, as can be shown, is very small. So, in (12) we can put $m^2_{\pi^+} = m^2_{\pi^0}$ and have

$$\Delta(q^2) \equiv q^2\left(\Pi_L^{(3)}(q^2) - \Pi_L^{(+)}(q^2)\right) = -\frac{m_\pi^2}{q^2 - m_\pi^2}\left(f^2_{\pi^0} - f^2_{\pi^+}\right). \tag{13}$$

Exploit the standard QCD sum rule technique and represent $\Delta(q^2)$ as an operator product expansion (OPE)

$$\Delta(q^2) = R_2(q^2) + R_4(q^2) + R_6(q^2), \tag{14}$$

where R_2 corresponds to the contribution of bare loop diagram, R_4—to the operator of dimension 4 and R_6—to the operator of dimension 6. Only the terms, proportional to $m_u - m_d$ and $\langle 0|\bar{u}u|0\rangle - \langle 0|\bar{d}d|0\rangle$ remain in (14). Note, that $\Delta(q^2)$ is even under interchange $u \leftrightarrow d$. For this reason no linear in $m_u - m_d$ or $\langle 0|\bar{u}u|0\rangle - \langle 0|\bar{d}d|0\rangle$ terms can survive in the right-hand side of (14). Calculating the terms of OPE we get after Borel transformation:

$$\frac{m_\pi^2}{M^2} e^{-m_\pi^2/M^2}\left(f^2_{\pi^0} - f^2_{\pi^+}\right) \approx \frac{m_\pi^2}{M^2}\left(f^2_{\pi^0} - f^2_{\pi^+}\right) = \frac{3}{8\pi^2}(m_u - m_d)^2 - \frac{m_d - m_u}{4\pi^2 M^2}(a_u - a_d), \tag{15}$$

where

$$a_q = -(2\pi)^2\langle 0|\bar{q}q|0\rangle, \quad q = u, d, \tag{16}$$

and M^2 is the Borel parameter. The contribution of the $d = 6$ operator vanishes. For the difference $f_{\pi^0} - f_{\pi^+}$ we have

$$\frac{\Delta f_\pi}{f_\pi} \equiv \frac{f_{\pi^0} - f_{\pi^+}}{f_\pi} = \frac{3}{16\pi^2}\frac{M^2}{m_\pi^2 f_\pi^2}\left[(m_u - m_d)^2 - \frac{2}{3}\frac{m_d - m_u}{M^2}\gamma a_q\right], \tag{17}$$

where

$$\gamma = \frac{a_u - a_d}{a_q}. \tag{18}$$

For numerical estimations we put $m_d - m_u = 3.5$ MeV $\pm 20\%$ [14], $\gamma = 6 \times 10^{-3}$ [16] or $\gamma = 3 \times 10^{-3}$ [17] and $M = 1$ GeV. The calculations give

$$\frac{\Delta f_\pi}{f_\pi} \simeq 2 \times 10^{-4} \text{ [16]}; \qquad \frac{\Delta f_\pi}{f_\pi} \simeq 4 \times 10^{-4} \text{ [17]}. \tag{19}$$

Therefore, the difference $f_{\pi^0} - f_{\pi^+}$ is negligible.

The additional source of the $f_{\pi^0} - f_{\pi^+}$ difference arises because there is the contribution of η-meson to the polarization operator $\Pi_L^{(3)}$, besides the π^0. This contribution appears from the η–π mixing and must be subtracted from the value of f_{π^0}. Omitting the details of simple calculations, we present the result

$$\left(\frac{\Delta f_\pi}{f_\pi}\right) = -\frac{1}{2}\frac{m_\eta^2}{m_\pi^2}\,\mathrm{Sin}^2\,\theta_{\eta\pi}\exp\left[-\left(m_\eta^2 - m_\pi^2\right)/M^2\right], \tag{20}$$

where $\theta\eta\pi$ is the $\eta - \pi$ mixing angle given below (Eq. (29)). Numerically

$$\left(\frac{\Delta f_\pi}{f_\pi}\right) = -1.3 \times 10^{-3} \tag{21}$$

and is also very small.

Let us dwell now on the main correction to the zero approximation result for $\Gamma(\pi^0 \to 2\gamma)$ arising from excited states contributions to the sum rule (4). The next after π^0 pseudoscalar meson is η. It can contribute to (4) because of isospin violation, caused by different masses of u- and d-quarks and resulting in π^0–η mixing. The problem of π^0–η mixing in QCD was considered in [18–20]. Following [18,20] introduce nonorthogonal states $|P_3\rangle$ and $|P_8\rangle$ and the corresponding fields φ_3, φ_8, coupled to $j^{(3)}_{\mu 5}$ and $j^{(8)}_{\mu 5}$:

$$\langle 0|j^{(k)}_{\mu 5}|P_l\rangle = i\delta_{kl} f_k q_\mu, \quad k = 3, 8. \tag{22}$$

Nonorthogonality of the fields φ_3, φ_8 corresponds to the non-diagonal term $\Delta H = m^2_{\eta\pi}\varphi_3\varphi_8$ in the effective interaction Hamiltonian. In the presence of such term the standard PCAC relations are modified in the following way [18]

$$\partial_\mu j^{(3)}_{\mu 5} = f_\pi\left(m_\pi^2\varphi_3 + m^2_{\eta\pi}\varphi_8\right),$$

$$\partial_\mu j_{\mu 5}^{(8)} = f_\eta \big(m_\eta^2 \varphi_8 + m_{\eta\pi}^2 \varphi_3\big). \tag{23}$$

The fields φ_3, φ_8 are expressed through the physical fields φ_π, φ_η as

$$\begin{aligned}\varphi_3 &= \mathrm{Cos}\,\theta_{\eta\pi}\varphi_\pi + \mathrm{Sin}\,\theta_{\eta\pi}\varphi_\eta,\\ \varphi_8 &= -\mathrm{Sin}\,\theta_{\eta\pi}\varphi_\pi + \mathrm{Cos}\,\theta_{\eta\pi}\varphi_\eta,\end{aligned} \tag{24}$$

where the mixing angle is given by

$$\theta_{\eta\pi} \approx \frac{m_{\eta\pi}^2}{m_\eta^2 - m_\pi^2} \approx \frac{m_{\eta\pi}^2}{m_\eta^2}. \tag{25}$$

In QCD the nondiagonal mass $m_{\eta\pi}^2$ is expressed through the difference of u- and d-quark masses [18,19]:

$$m_{\eta\pi}^2 = \frac{1}{\sqrt{3}} \frac{m_u - m_d}{m_u + m_d} m_\pi^2. \tag{26}$$

Now $\mathrm{Im}\, F_1(q^2)$ is given by the sum of contributions of π^0 and η mesons. In order to separate the formfactor $F_1(q^2)$ and to kill the contributions of axial mesons in the representation of $\mathrm{Im}\, T_{\mu\alpha\beta}(p, p')$ in terms of physical states, multiply Eq. (3) by q_μ/q^2. Using Eqs. (1), (23), (24) and taking the imaginary part, we get:

$$\begin{aligned}\mathrm{Im}\, q_\mu \frac{1}{q^2}\langle 2\gamma | j_{\mu 5}^{(3)}|0\rangle &= -\frac{1}{q^2}\,\mathrm{Im}\langle 2\gamma | m_\pi^2 (\mathrm{Cos}\,\theta_{\eta\pi}\varphi_\pi + \mathrm{Sin}\,\theta_{\eta\pi}\varphi_\eta) + m_{\eta\pi}^2(-\mathrm{Sin}\,\theta_{\eta\pi}\varphi_\pi + \mathrm{Cos}\,\theta_{\eta\pi}\varphi_\eta)|0\rangle\\ &\approx \pi\left[\delta\big(q^2 - m_\pi^2\big)A_\pi + \frac{m_\pi^2}{m_\eta^2}\frac{m_{\eta\pi}^2}{m_\eta^2}\delta\big(q^2 - m_\eta^2\big)A_\eta - \frac{(m_{\eta\pi}^2)^2}{m_\pi^2 m_\eta^2}\delta\big(q^2 - m_\pi^2\big)A_\pi + \frac{m_{\eta\pi}^2}{m_\eta^2}\delta\big(q^2 - m_\eta^2\big)A_\eta\right]\\ &\approx \pi\left\{\delta\big(q^2 - m_\pi^2\big)A_\pi\left[1 - \frac{m_\pi^2}{m_\eta^2}\frac{1}{3}\left(\frac{m_u - m_d}{m_u + m_d}\right)^2\right] + \frac{1}{\sqrt{3}}\frac{m_\pi^2}{m_\eta^2}\left(\frac{m_u - m_d}{m_u + m_d}\right)\delta\big(q^2 - m_\eta^2\big)A_\eta\right\},\end{aligned} \tag{27}$$

where A_η is the amplitude $\eta \to 2\gamma$ decay. The ratio A_η/A_π is equal

$$\frac{A_\eta}{A_\pi} = \left[\frac{\Gamma(\eta \to 2\gamma)}{\Gamma(\pi^0 \to 2\gamma)}\frac{m_\pi^3}{m_\eta^3}\right]^{1/2}. \tag{28}$$

Numerically, at $\Gamma(\eta \to 2\gamma) = 510$ eV [5] and $\Gamma(\pi^0 \to 2\gamma) = 7.73$ eV, $A_\eta/A_\pi = 1.0$. According to (25), (26) the mixing angle $\theta_{\eta\pi}$ is given by (the numerical data were taken from Ref. [14]):

$$\theta_{\eta\pi} = \frac{1}{\sqrt{3}}\frac{m_u - m_d}{m_u + m_d}\frac{m_\pi^2}{m_\eta^2} = -0.0150 \pm 0.020 \tag{29}$$

and the total correction to the amplitude of $\pi^0 \to 2\gamma$ decay arising from η–π mixing is

$$\frac{(F_1)_\eta}{(F_1)_\pi} = -(1.2 \pm 0.25)\%. \tag{30}$$

Let us estimate the contributions to the sum rule (4) from higher pseudoscalar mesons. The η' meson contributes through $\eta\eta'$ mixing. Its contribution is suppressed by the ratio $(m_\eta/m_{\eta'})^2$ in comparison with η contribution and we have

$$\frac{(F_1)_{\eta'}}{(F_1)_\eta} \sim \sqrt{\frac{\Gamma(\eta' \to 2\gamma)}{\Gamma(\eta \to 2\gamma)}}\left(\frac{m_\eta}{m_{\eta'}}\right)^{3/2}\left(\frac{m_\eta}{m_{\eta'}}\right)^2 \mathrm{Sin}\,\vartheta_{\eta\eta'}, \tag{31}$$

where $\vartheta_{\eta\eta'}$ is the $\eta\eta'$ mixing angle. In accord with [21] we can put $\vartheta_{\eta\eta'} \approx -20^\circ$. Using $\Gamma(\eta' \to 2\gamma) = 4.29$ MeV [5] we have the estimation

$$(F_1)_{\eta'}/(F_1)_\eta \sim 0.15. \tag{32}$$

Finally, estimate the contribution of the resonance $\pi(1300)$. The contribution of this resonance to $T_{\mu\alpha\beta}(p, p')$ (1) (as well as any other excited states) should be proportional to $m_u + m_d$ or, what is equivalent, to m_π^2, otherwise the axial current would not conserve in the limit $m_u + m_d \to 0$. Therefore, for dimensional grounds,

$$\frac{(F_1)_{\pi(1300)}}{(F_1)_\pi} \sim \frac{m_\pi^2}{m_{\pi(1300)}^2} < 1\%. \tag{33}$$

In fact, probably, (33) overestimates the $\pi(1300)$ contribution, since, as was mentioned above, $F_1(q^2)$ decreases as $1/q^4$ at large q^2. So, in what follows we take $(F_1)_{\pi(1300)}/(F_1)_\pi \sim 0.5\%$. Summing all uncertainties in squares we have the total uncertainty in the amplitude about 0.7%. The final result for $\Gamma(\pi^0 \to 2\gamma)$ from the axial anomaly is (the correction, given by Eq. (21) is accounted):

$$\Gamma\left(\pi^0 \to 2\gamma\right) = 7.93\ \text{eV} \pm 1.5\%. \tag{34}$$

In the limit of errors this agrees with those found in [7] and [8]. The experimental test of this relation, which is planned by PriMex Collaboration would be very important: it would be a high precision test of QCD and even more general—the phenomenon of the anomaly. Not too many such high precision tests of QCD are known till now.

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