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Classification of some countable descendant-homogeneous digraphs☆

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ABSTRACT

For finite q, we classify the countable, descendant-homogeneous digraphs in which the descendant set of any vertex is a q-valent tree. We also give conditions on a rooted digraph Γ which allows us to construct a countable descendant-homogeneous digraph in which the descendant set of any vertex is isomorphic to Γ.

1. Introduction

1.1. Background

A countable digraph is homogeneous if any isomorphism between finite (induced) subdigraphs extends to an automorphism. The digraphs with this property are classified by Cherlin in [7]. By analogy, the notion of descendant-homogeneity was introduced in [4]. A countable digraph is *descendant-homogeneous* if any isomorphism between finitely generated subdigraphs extends to an automorphism. Here, a subdigraph is finitely generated if its vertex set can be written as the descendant set of a finite set of vertices, that is, the set of vertices which are reachable by a directed path from the set.

Note that descendant-homogeneity can hold for trivial reasons: digraphs where the descendant set of any vertex is the whole digraph, or where no two vertices have isomorphic descendant sets are descendant-homogeneous. So it is reasonable to impose restrictions such as vertex transitivity and no directed cycles. We refer to [4] for further discussion.

In this paper we are particularly interested in vertex-transitive, descendant-homogeneous digraphs: so in this case, the descendant set of any vertex is isomorphic to some fixed digraph Γ. Examples of countable, vertex-transitive, descendant-homogeneous digraphs where Γ is a q-valent directed tree (for finite $q > 1$) were given in [9,4]. The main result of this paper is to show that the digraphs constructed in [9,4] constitute a complete list of all the countable descendant-homogeneous digraphs with descendant sets of this form (Theorem 2.2). In the final section of the paper, we give general conditions on Γ under which there is a countable, vertex-transitive, descendant-homogeneous digraph in which the descendant set of any vertex is isomorphic to Γ. In particular, these conditions are satisfied by certain 'tree-like' digraphs Γ studied in [1]. This gives new examples of descendant-homogeneous digraphs (and indeed, highly arc-transitive digraphs).

The first (non-trivial) examples of descendant-homogeneous digraphs known to the authors arose in the context of highly arc-transitive digraphs (those whose automorphism groups are transitive on the set of s-arcs for all s). In answer

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to a question of Cameron, Praeger, and Wormald in [6], the paper [9] gave a construction of a certain highly arc-transitive digraph D having an infinite binary tree as descendant set. The digraph was constructed as an example of a highly arc-transitive digraph not having the 'property Z', meaning that there is no homomorphism from D onto the natural digraph on $\mathbb{Z}$ (the doubly infinite path). However, we noted in [4] that it is also descendant-homogeneous. A more systematic analysis of this notion was carried out in [4], and further examples were given. The method of [9] immediately applies to q-valent trees for any finite $q > 1$ in place of binary trees, but in addition, it is shown in [4] that it is possible to omit certain configurations and still carry out a Fraïssé-type construction to give other examples of descendant-homogeneous digraphs whose descendant sets are q-valent trees. The classical Fraïssé theorem for relational structures provides a link between countable homogeneous structures (those in which any isomorphism between finite substructures extends to an automorphism) and amalgamation classes of finite structures. See [5,7,10] for instance. The analogue of Fraïssé's Theorem and the appropriate notion of amalgamation classes which applies to descendant-homogeneity is given in Section 2.1.

1.2. Notation and terminology

Let D a digraph with vertex and edge sets VD and ED, and let $u \in VD$. For ease, in this paper we require our digraphs to be asymmetric; that is, we never have a directed edge both from u to v, and from v to u, since this is true in the cases we consider anyway. For $s \geq 0$, an *s-arc* in D from u_0 to u_s is a sequence $u_0u_1 \ldots u_s$ of $s+1$ vertices such that $(u_i, u_{i+1}) \in ED$ for $0 \leq i < s$. We let

$$\mathrm{desc}^s(u) := \{v \in VD \mid \text{ there is an } s\text{-arc from } u \text{ to } v\},$$

and $\mathrm{desc}(u) = \bigcup_{s \geq 0} \mathrm{desc}^s(u)$, the *descendant set* of u (we also denote this by $\mathrm{desc}_D(u)$ if we need to emphasize that we are looking at descendants in D). If $X \subseteq VD$, we also let

$$\mathrm{desc}^s(X) := \bigcup_{x \in X} \mathrm{desc}^s(x),$$

and similarly $\mathrm{desc}(X) := \bigcup_{x \in X} \mathrm{desc}(x)$. The 'ball' of radius s at u is given by

$$B^s(u) := \bigcup_{0 \leq i \leq s} \mathrm{desc}^i(u).$$

For a digraph D we often write D in place of VD and use the same notation for a subset of the vertices and the full induced subdigraph. Henceforth, 'subdigraph' will mean 'full induced subdigraph' and an embedding of one digraph into another will always mean a full induced subdigraph.

We say that $A \subseteq D$ is *descendant-closed* in D, written $A \leq D$ if $\mathrm{desc}_D(a) \subseteq A$ for all $a \in A$; and we say that an embedding $f : A \to B$ between digraphs is a *$\leq$-embedding* if $f(A) \leq B$. When A, B_1, B_2 are digraphs we say that $\leq$-embeddings $f_i : A \to B_i$ are *isomorphic* if there is an isomorphism $h : B_1 \to B_2$ with $f_2 = h \circ f_1$.

We say that $A \leq D$ is *finitely generated* if there is a finite subset $X \subseteq A$ with $A = \mathrm{desc}_D(X)$; in this case we refer to X as a generating set of A. If additionally no proper subset of X is a generating set, then X is called a minimal generating set. Clearly, in this case, no element in X is a descendant of any other element of X.

The digraph D is *descendant-homogeneous* if whenever $f : A_1 \to A_2$ is an isomorphism between finitely generated descendant-closed subdigraphs of D, there is an automorphism of D which extends f. The group of automorphisms of D is denoted by $\mathrm{Aut}(D)$.

We shall mainly be concerned with digraphs D where the descendant sets of single vertices are all isomorphic to a fixed digraph Γ: in this case we refer to Γ as 'the descendant set' of D. A subset of a digraph is *independent* if the descendant sets of any two of its members are disjoint. In any digraph in which the descendant sets are all isomorphic, for any two finite independent subsets X and Y, any bijection from X to Y extends to an isomorphism from $\mathrm{desc}(X)$ to $\mathrm{desc}(Y)$ since $\mathrm{desc}(X)$ and $\mathrm{desc}(Y)$ are both the disjoint union of $|X|$ descendant sets.

Throughout we fix an integer $q > 1$ and write $T = T_q$ for the q-valent rooted tree. So T has as its vertices the set of finite sequences from the set $\{0, \ldots, q-1\}$ and directed edges $(\bar{w}, \bar{w}i)$ (for $\bar{w}$ a finite sequence and $i \in \{0, \ldots, q-1\}$).

2. Amalgamation classes

2.1. The Fraïssé theorem

As in [4], the correct context for the study of descendant-homogeneous digraphs is a suitable adaptation of Fraïssé's notion of *amalgamation classes*. The reader who is familiar with this type of result (or with [4]) and who is mainly interested in the main classification result, Theorem 2.2, could reasonably skip to the next subsection. The extra generality which is given here is only needed in the final section of the paper.

Let $\mathcal{D}$ be a class of (isomorphism types of) digraphs. Then $\mathcal{D}$ has the *$\leq$-amalgamation property* if the following holds: if A, B_1 and B_2 lie in $\mathcal{D}$, and $\leq$-embeddings f_1 and f_2 of A into each of B_1 and B_2 are given, then there are a structure $C \in \mathcal{D}$ and $\leq$-embeddings g_1 and g_2 of B_1 and B_2 respectively into C such that $g_1 \circ f_1 = g_2 \circ f_2$. We say that g_1, g_2 solve the amalgamation problem given by f_1, f_2.

Remark 1. Suppose A, B_1, and B_2 are digraphs and $\leq$-embeddings f_1 and f_2 of A into each of B_1 and B_2 are given. We can clearly find a solution $g_i : B_i \to C$ with the property that $C = g_1(B_1) \cup g_2(B_2)$, $g_1(B_1) \cap g_2(B_2) = g_1(f_1(A))$ and every directed edge is contained in $g_1(B_1)$ or $g_2(B_2)$. Moreover, this solution is uniquely determined up to isomorphism by the f_i. Informally, we can regard the f_i as inclusion maps and take C to be the disjoint union of B_1 and B_2 over A. We make this into a digraph by taking as edge set $EC = EB_1 \cup EB_2$. It is easy to see that $B_1, B_2 \leq C$ and the inclusion maps $g_i : B_i \to C$ satisfy $g_1 \circ f_1 = g_2 \circ f_2$. We say that the solution $g_i : B_i \to C$ to the problem $f_i : A \to B_i$ is the *free amalgam* of the f_i. When f_1, f_2 are inclusion maps (or are understood from the context) we shall abuse this terminology and say that C is the free amalgam of B_1 and B_2 over A.

Note that if $B_1, B_2 \leq C$ then $B_1 \cup B_2 \leq C$ and $B_1 \cup B_2$ is the free amalgam of B_1 and B_2 over $B_1 \cap B_2$: there can be no directed edges between elements of $B_1 \setminus B_2$ and $B_2 \setminus B_1$ as B_1, B_2 are descendant-closed.

When we come to count structures and embeddings up to isomorphism (as in Lemma 4.2), it will be useful to have a more precise notation for free amalgamation. Suppose in the above that f_1 is inclusion and f_2 is an arbitrary $\leq$-embedding $f_2 : A \to B_2$. The free amalgam $B_1 *_{f_2} B_2$ has as vertex set the disjoint union of $B_1 \setminus A$ and B_2 (and the 'obvious' directed edges). The embedding $g_2 : B_2 \to B_1 *_{f_2} B_2$ is inclusion and the embedding $g_1 : B_1 \to B_1 *_{f_2} B_2$ is given by $g_1(b) = b$ if $b \in B_1 \setminus A$ and $g_1(b) = f_2(b)$ if $b \in A$.

We remark that in general, if $A \leq B_1$ and $f_2, f_2' : A \to B_2$ are $\leq$-embeddings with the same image, then $B_1 *_{f_2} B_2$ and $B_1 *_{f_2'} B_2$ need not be isomorphic.

The analogue of Fraïssé's Theorem which we use is the following. Recall that the notion of isomorphism of two embeddings was defined in Section 1.2.

Theorem 2.1. *Suppose M is a countable descendant-homogeneous digraph. Let $\mathcal{C}$ be the class of digraphs which are isomorphic to finitely generated $\leq$-subdigraphs of M. Then*

(1) *$\mathcal{C}$ is a class of countable, finitely generated digraphs which is closed under isomorphism and has countably many isomorphism types;*
(2) *$\mathcal{C}$ is closed under taking finitely generated $\leq$-subdigraphs;*
(3) *$\mathcal{C}$ has the $\leq$-amalgamation property;*
(4) *for all $A, B \in \mathcal{C}$ there are only countably many isomorphism types of $\leq$-embeddings from A to B.*

Conversely, if $\mathcal{C}$ is a class of digraphs satisfying (1)–(4), then there is a countable descendant-homogeneous digraph M for which the class of digraphs isomorphic to finitely generated $\leq$-subdigraphs of M is equal to $\mathcal{C}$. Moreover, M is determined up to isomorphism by $\mathcal{C}$.

We refer to a class $\mathcal{C}$ of digraphs satisfying (1)–(4) as a $\leq$-amalgamation class. The digraph M determined by $\mathcal{C}$ as in the theorem is called the Fraïssé limit of $(\mathcal{C}, \leq)$.

Remark 2. It is easy to see that in place of (4) we can substitute the condition:

(4′) *if $A \leq B \in \mathcal{C}$ and A is finitely generated, then the subgroup of the automorphism group* Aut(A) *consisting of automorphisms which extend to automorphisms of B is of countable index in* Aut(A).

Indeed, we wish to consider the number of $\leq$-embeddings $f : A \to B$ up to isomorphism. As B is countable and A is finitely generated there are countably many possibilities for the image $f(A)$, so it will be enough to count isomorphism types of $\leq$-embeddings with fixed finitely generated image $Y \leq B$. Let H be the subgroup of Aut(Y) consisting of automorphisms which extend to automorphisms of B. It is straightforward to show that if $f, f' : A \to B$ have image Y, then f, f' are isomorphic if and only if the map $g \in \mathrm{Aut}(Y)$ given by $g(y) = f'(f^{-1}(y))$ is in H. Thus there is a bijection between the H-cosets in Aut(Y) and the isomorphism types.

Remark 3. The proof of Theorem 2.1 is reasonably standard, but we make some comments on the condition (4). First, suppose M and $\mathcal{C}$ are as in the statement. We show that (4′) in Remark 2 holds. So let $A \leq B \in \mathcal{C}$ and $H \leq \mathrm{Aut}(A)$ be the automorphisms of A which extend to automorphisms of B, as in (4′). We may assume $B \leq M$. Suppose $g_1, g_2 \in \mathrm{Aut}(A)$ lie in different H-cosets. As M is $\leq$-homogeneous we can extend g_i to $k_i \in \mathrm{Aut}(M)$. Then $k_1(B) \neq k_2(B)$. Otherwise $h = k_2^{-1}k_1$ stabilizes B and gives an automorphism of B which extends $h = g_2^{-1}g_1$; this implies $h \in H$ and $g_2H = g_1H$, which is a contradiction. As there are only countably many possibilities for the image of B under automorphisms of M, it follows that H is of countable index in Aut(A), as required.

The converse is a fairly standard construction, and can be read off from Theorem 2.18 of [11], which in turn is adapted from Theorem 1.1 of [8]. However, we give a few details of the proof. So suppose we have a class $\mathcal{C}$ of finitely generated digraphs satisfying (1)–(4). We construct a countable chain $C_1 \leq C_2 \leq C_3 \leq \cdots$ of digraphs in $\mathcal{C}$ with the property that if $A \leq C_i$ is finitely generated and $f : A \to B \in \mathcal{C}$ is a $\leq$-embedding, then there is $j \geq i$ and a $\leq$-embedding $g : B \to C_j$ with $g(f(a)) = a$ for all $a \in A$. The resulting digraph $\bigcup_i C_i$ will be descendant-homogeneous, by a back-and-forth argument. Note that by (4), we have only countably many f to consider (for any particular A). For if f, g are as above and $f' : A \to B$ is isomorphic to f with $f' = h \circ f$ for $h \in \mathrm{Aut}(B)$, then $g' = g \circ h^{-1} : B \to C_j$ satisfies $g'(f'(a)) = a$ for all $a \in A$.

2.2. The classification result

Recall that $q \geq 2$ is an integer and $T = T_q$ is the q-valent rooted tree. We shall classify countable, descendant-homogeneous digraphs M in which the descendant sets of vertices are isomorphic to T. Thus, by Theorem 2.1, we need to classify $\leq$-amalgamation classes of finitely generated digraphs with descendant sets isomorphic to T. In this case, we can replace condition (4) in Theorem 2.1 by the simpler condition:

$(4'')$ if $a_1, a_2 \in B \in \mathcal{C}$, *then* $\mathrm{desc}_B(a_1) \cap \mathrm{desc}_B(a_2)$ *is finitely generated*

as in Theorem 3.4 of [4]. Indeed, if $\mathcal{C}$ satisfies $(4'')$ then $(4')$ is a special case of Lemma 4.2 here. Conversely, if $(4')$ holds, then to see $(4'')$ let $B = \mathrm{desc}(a_1) \cap \mathrm{desc}(a_2)$ and $A = \mathrm{desc}(a_1)$. Let X be the minimal generating set for B, which just consists of its elements which are not (proper) descendants of any other members of B. Then X is independent and any automorphism of B which stabilizes A must fix X setwise. On the other hand, if Z is an infinite independent subset of A it is easy to see that the stabilizer of Z in $\mathrm{Aut}(A)$ is of index continuum (as there are continuum many translates of Z by automorphisms of A, since A is a regular rooted tree).

Thus we work with the class $\mathcal{C} = \mathcal{C}_\infty$ consisting of all digraphs A satisfying the following conditions:

- for all $a \in A$, $\mathrm{desc}(a)$ is isomorphic to T;
- A is finitely generated;
- for $a, b \in A$, the intersection $\mathrm{desc}(a) \cap \mathrm{desc}(b)$ is finitely generated.

Then $\mathcal{C}$ satisfies conditions (1), (2), (4) in Theorem 2.1 (cf. the above remarks and Lemma 4.2), and we are interested in the subclasses of $\mathcal{C}$ which satisfy (3). It is easy to see that $\mathcal{C}$ satisfies (3): in fact $\mathcal{C}$ is closed under free amalgamation. It follows that $(\mathcal{C}, \leq)$ is a $\leq$-amalgamation class. The Fraïssé limit D_∞ of $(\mathcal{C}, \leq)$ is the countable descendant-homogeneous digraph constructed in [9].

For $n \geq 2$, we now define the amalgamation classes $\mathcal{C}_n \subseteq \mathcal{C}$ (from [4]). Let $\mathcal{T}_n$ be the element of $\mathcal{C}$ generated by n elements $x_1, \dots, x_n$, such that $\mathrm{desc}^1(x_i) = \mathrm{desc}^1(x_j)$ for all $i \neq j$. So $\mathcal{T}_n$ is like the tree T, except that there are n root vertices (all having the same out-vertices). Let $\mathcal{C}_n$ consist of the digraphs $A \in \mathcal{C}$ such that $\mathcal{T}_n$ does not embed in A (as a descendant-closed subdigraph). We remark that one can analogously define $\mathcal{T}_\infty$, but this does not lie in $\mathcal{C}$ since it is not finitely generated.

It is clear that $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ and $\mathcal{C}_n \subseteq \mathcal{C}$ for all n. In [4] it is shown that $(\mathcal{C}_n, \leq)$ is a $\leq$-amalgamation class, though it is clearly not a free amalgamation class. In particular, when we 'solve' an amalgamation problem $f_i : A \to B_i$ by maps $g_i : B_i \to C$, we may have $g_1(B_1) \cap g_2(B_2) \supset g_1(f_1(A))$. Informally, this means that points of B_1, B_2 outside A may need to become identified in the amalgam C.

For $n \geq 2$, let D_n be the Fraïssé limit of $(\mathcal{C}_n, \leq)$, as in Theorem 2.1. Then D_n is a countable descendant-homogeneous digraph. Our main result is:

Theorem 2.2. *Let D be a countable descendant-homogeneous digraph whose descendant sets are isomorphic to T. Then D is isomorphic to D_n for some $n \in \{2, \dots, \infty\}$.*

3. Proof of the main theorem

We know from [4] that each $\mathcal{C}_n \subseteq \mathcal{C}$ is a $\leq$-amalgamation class. From now on we shall consider an arbitrary subclass $\mathcal{D}$ of $\mathcal{C}$ which is itself a $\leq$-amalgamation class (that is, satisfies (1)–(4) of Theorem 2.1), with the goal of showing that $\mathcal{C}$ and $\mathcal{C}_n$ are the only possibilities for $\mathcal{D}$.

To understand the argument better, suppose that there is some integer $n \geq 2$ such that $\mathcal{T}_n \notin \mathcal{D}$. Choose n as small as possible: so in particular, $\mathcal{T}_{n-1} \in \mathcal{D}$ (where $\mathcal{T}_1 = T$) and $\mathcal{D} \subseteq \mathcal{C}_n$. To prove our main result it suffices to show that if $A \in \mathcal{C}_n$ then $A \in \mathcal{D}$, and this is done by induction on the number of generators of A. Let $\{a_1, \dots, a_k\}$ be the minimal generating set of A and let A_1 be the descendant-closed subdigraph of A with generating set $\{a_1, \dots, a_{k-1}\}$. Let $A_0 = A_1 \cap \mathrm{desc}(a_k)$. Then A is the free amalgam of A_1 and $\mathrm{desc}(a_k)$ over A_0. By the induction hypothesis, $A_1 \in \mathcal{D}$, and we know that $\mathrm{desc}(a_k) \cong T \in \mathcal{D}$. So there are $C \in \mathcal{D}$ and $\leq$-embeddings $f : A_1 \to C$ and $g : T \to C$ such that $f(a) = g(a)$ for all $a \in A_0$ (identifying $\mathrm{desc}(a_k)$ with T). However, *a priori* one cannot force C to be the free amalgam. So we replace A_1 by some $B \geq A_1$, T by $T' \geq T$ and A_0 by $A_0' \leq B$, T' in such a way that the amalgam in $\mathcal{D}$ of B and T' over A_0' is forced to be free. This is the point of Lemmas 3.1–3.3 (which do not need the extra assumption on $\mathcal{D}$).

Lemma 3.1. *Let $A \in \mathcal{C}$ and X be a finite independent subset of A. Then there is a finite independent subset Y of A containing X such that $A \setminus \mathrm{desc}(Y)$ is finite.*

Proof. Let $a, x \in VA$ and let S be the minimal generating set of $\mathrm{desc}(a) \cap \mathrm{desc}(x)$. Since S is finite, there is $n(a, x) \in \mathbb{N}$ such that $S \subseteq B^{n(a,x)}(a)$. Let $m \geq n(a, x)$, $y \in \mathrm{desc}^m(a)$ and $y \notin \mathrm{desc}(x)$. Then $\mathrm{desc}(y) \cap \mathrm{desc}(x) = \varnothing$: if not, let $u \in \mathrm{desc}(y) \cap \mathrm{desc}(x)$. As $y \in \mathrm{desc}(a)$, $u \in \mathrm{desc}(a) \cap \mathrm{desc}(x)$, so $u \in \mathrm{desc}(s)$ for some $s \in S$. As $\mathrm{desc}(a)$ is a tree, and by the choice of m, $y \in \mathrm{desc}(s) \subseteq \mathrm{desc}(x)$, which is a contradiction.

Let $a_1, \ldots, a_r$ be the minimal generating set for A. Now let $N \geq \max\{n(a_i, a_j), n(a_i, x) \mid i \neq j, x \in X\}$ and let $B = \bigcup_{i=1}^{r} B^N(a_i)$. So if $y \in A \setminus (B \cup \mathrm{desc}(X))$ then $\mathrm{desc}(y) \cap \mathrm{desc}(X) = \varnothing$ and if $y_1, y_2 \in A \setminus (B \cup \mathrm{desc}(X))$, and neither is a descendant of the other, then $\mathrm{desc}(y_1) \cap \mathrm{desc}(y_2) = \varnothing$.

To justify this last statement, note that $y_1 \in B^{n_1}(a_i), y_2 \in B^{n_2}(a_j)$ for some i, j, and $n_1, n_2 > N \geq n(a_i, a_j)$. If $i = j$ then $y_1, y_2 \in \mathrm{desc}(a_i)$ which is a tree, so one is a descendant of the other. Thus $i \neq j$ and we may assume (by the same argument) that $y_1 \notin \mathrm{desc}(a_j)$. By the first paragraph (with $a = a_i, x = a_j$ and $y = y_1$) we have $\mathrm{desc}(y_1) \cap \mathrm{desc}(a_j) = \emptyset$, so $\mathrm{desc}(y_1) \cap \mathrm{desc}(y_2) = \emptyset$.

Let $y_1, \ldots, y_t$ be the minimal generating set of $A \setminus (B \cup \mathrm{desc}(X))$. Then $Y := X \cup \{y_1, \ldots, y_t\}$ is independent and $A \setminus \mathrm{desc}(Y) \subseteq B$ is finite. $\square$

For a finite independent subset X of A, and Y given by the lemma, we say that $Y \setminus X$ is a *complement* of X in A.

For $X \subseteq D \in \mathcal{C}$, a *common predecessor* for X in D is a vertex $a \in D$ such that (a, x) is a directed edge for all $x \in X$. Let $A \in \mathcal{C}$ and let U, V be independent subsets of A and T respectively with $f : \mathrm{desc}(U) \to \mathrm{desc}(V)$ an isomorphism. Let Q be the set consisting of those q-element subsets p of U such that p has a common predecessor in A and $f(p)$ has a common predecessor in T. For $p \in Q$, let w_p and $w_{f(p)}$ be such common predecessors of p and $f(p)$ respectively. We note that as T is a tree, $w_{f(p)}$ is uniquely determined, but w_p may not be. Also, as T is a tree, any two members of Q are disjoint. Now let

$$U' := \left(U \setminus \bigcup Q\right) \cup \{w_p \mid p \in Q\} \quad \text{and} \quad V' := \left(V \setminus \bigcup f(Q)\right) \cup \{w_{f(p)} \mid p \in Q\}.$$

In words, U' is obtained from U by replacing the vertices in $p \subseteq U$ by their common predecessors w_p, for all $p \in Q$. Similarly V' is obtained from V. Clearly $|U'| = |V'|$, $\mathrm{desc}(U) \subseteq \mathrm{desc}(U')$ and $\mathrm{desc}(V) \subseteq \mathrm{desc}(V')$. Moreover,

Lemma 3.2. (a) *The sets U' and V' are independent subsets of A and T respectively, and the extension F of f which takes w_p to $w_{f(p)}$ for each $p \in Q$ is an isomorphism from* $\mathrm{desc}(U')$ *to* $\mathrm{desc}(V')$*;*
(b) *if $I \subseteq A$ is disjoint from U and $U \cup I$ is an independent subset of A, then $U' \cup I$ is also independent.*

Proof. (a) Let u_1 and u_2 be distinct members of U'. If neither lies in $\{w_p \mid p \in Q\}$, then they are in U, so $\mathrm{desc}(u_1) \cap \mathrm{desc}(u_2) = \varnothing$ is immediate. Next suppose that $u_1 = w_p$ for $p \in Q$ and $u_2 \notin \{w_p \mid p \in Q\}$. Then $\mathrm{desc}(u_1) = \{u_1\} \cup \mathrm{desc}(p)$, and as U is independent, $\mathrm{desc}(u) \cap \mathrm{desc}(u_2) = \varnothing$ for each $u \in p$, and also $u_1 \notin \mathrm{desc}(u_2)$, and it follows that $\mathrm{desc}(u_1) \cap \mathrm{desc}(u_2) = \varnothing$. Finally, if $u_1 = w_{p_1}$ and $u_2 = w_{p_2}$, then $\mathrm{desc}(u_1) = \{u_1\} \cup \mathrm{desc}(p_1)$ and $\mathrm{desc}(u_2) = \{u_2\} \cup \mathrm{desc}(p_2)$. Now for each $u \in p_1$ and $u' \in p_2$, $\mathrm{desc}(u) \cap \mathrm{desc}(u') = \varnothing$ by the independence of U, and $u_1 \notin \mathrm{desc}(p_2)$ and $u_2 \notin \mathrm{desc}(p_1)$ are clear, from which it follows that $\mathrm{desc}(u_1) \cap \mathrm{desc}(u_2) = \varnothing$. This shows that U' is independent, and the proof that V' is independent is similar.

To see that F is an isomorphism, note that the only new points in its domain are w_p, and F maps w_p to $w_{f(p)}$, and $f(\mathrm{desc}^1(w_p)) = \mathrm{desc}^1(w_{f(p)})$.

(b) Since $\mathrm{desc}(w_p) = \{w_p\} \cup \mathrm{desc}(p)$ for each $p \in Q$, and $p \subseteq U$ and $U \cup I$ is independent, it follows that $\mathrm{desc}(w_p) \cap \mathrm{desc}(x) = \varnothing$ for all $x \in I$, so $U' \cup I$ is also independent. $\square$

Lemma 3.3. *Let $A \in \mathcal{D}$ and let U be a finite independent subset of A. Let M be the maximal number of common predecessors in A of q-element subsets of U, and let $N \geq M$ be such that $\mathcal{T}_N \in \mathcal{D}$. Then there is $B \in \mathcal{D}$ with $A \leq B$ and such that every q-element subset of U has at least N common predecessors in B.*

Proof. Let $P = \{p_1, \ldots, p_t\}$ be the set of all q-element subsets of U. (Note that, unlike in the previous proof, the members of P need not be pairwise disjoint.) We construct a sequence $B_0 \leq B_1 \leq B_2 \leq \cdots \leq B_t$ in $\mathcal{D}$, such that p_i has at least N common predecessors in B_l for all $i \leq l$ and $l \leq t$. We start with $B_0 := A$ and assume inductively that we have constructed B_l, where $l < t$. Let $p_{l+1} = \{u_1, \ldots, u_q\}$ and consider a copy of $\mathcal{T}_N$ with generating set $G = \{g_1, \ldots, g_N\}$. Let $\mathrm{desc}^1(G) = \{h_1, \ldots, h_q\}$. Both sets $\bigcup_{j=1}^{q} \mathrm{desc}(u_j)$ and $\bigcup_{j=1}^{q} \mathrm{desc}(h_j)$ are the union of q disjoint copies of T, so there is an isomorphism taking the first to the second such that u_j is sent to h_j for each j. Let B_{l+1} be an amalgam in $\mathcal{D}$ of B_l and $\mathcal{T}_N$ with $\bigcup_{j=1}^{q} \mathrm{desc}(u_j)$ and $\bigcup_{j=1}^{q} \mathrm{desc}(h_j)$ identified by this isomorphism (since $B_l, \mathcal{T}_N \in \mathcal{D}$). We note that p_{l+1} has at least N common predecessors in B_{l+1} since $\{h_1, \ldots, h_q\}$ has N common predecessors in $\mathcal{T}_N$. Hence $B = B_t$ is a member of $\mathcal{D}$ as required. $\square$

Proposition 3.4. *Let $\mathcal{D} \subseteq \mathcal{C}$ be a $\leq$-amalgamation class and suppose that $\mathcal{T}_m \notin \mathcal{D}$ for some $m \geq 2$. Then $\mathcal{D} = \mathcal{C}_n$ where n is the least m such that $\mathcal{T}_m \notin \mathcal{D}$.*

Proof. Note that $\mathcal{D} \subseteq \mathcal{C}_n$ and $\mathcal{T}_{n-1} \in \mathcal{D}$. We shall show that $\mathcal{C}_n \subseteq \mathcal{D}$. Let $A \in \mathcal{C}_n$. We use induction on the number of generators of A to show that $A \in \mathcal{D}$. Let $a_1, \ldots, a_s$ be the (distinct) generators of A. If $s = 1$, or if A is the disjoint union of finitely many copies of T, then A embeds in T and therefore $A \in \mathcal{D}$, since $T \in \mathcal{D}$. Now let $s \geq 2$ and suppose that $E \in \mathcal{D}$ for all $E \in \mathcal{C}_n$ with at most $s - 1$ generators. Let $A_1 := \bigcup_{i=1}^{s-1} \mathrm{desc}(a_i)$ and let T be a copy of the q-valent tree with b its root. The digraph A is the free amalgam of A_1 and $\mathrm{desc}(a_s)$ ($\cong T$) over $A_1 \cap \mathrm{desc}(a_s)$ (which is finitely generated). So there are independent subsets $U = \{u_1, \ldots, u_k\}$ and $V = \{v_1, \ldots, v_k\}$ of A_1 and $T = \mathrm{desc}(b)$ respectively and an isomorphism f from $\mathrm{desc}(U)$ to $\mathrm{desc}(V)$ (taking u_i to v_i for all i), such that A is isomorphic to the free amalgam C of A_1 and T with $\mathrm{desc}(U)$ and $\mathrm{desc}(V)$ identified by f. See Fig. 1. To prove the result it then suffices to show that there is $D \in \mathcal{D}$ embedding C. We shall first 'expand' A_1 to a digraph $B \in \mathcal{D}$ (using Lemma 3.3) and then amalgamate B with a copy $T' \geq T$ of T over the descendant sets of some carefully chosen independent subsets. The resulting digraph is then the required digraph D.

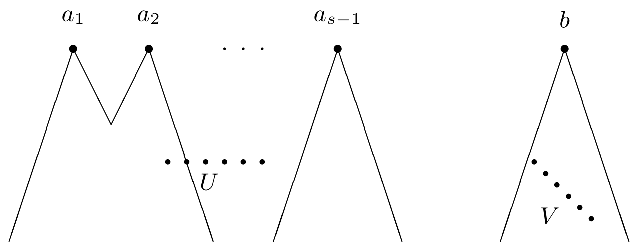

Fig. 1. The digraphs A_1 and $T = \mathrm{desc}(b)$.

By the induction hypothesis, $A_1 \in \mathcal{D}$ since $A_1 \leq A \in \mathcal{C}_n$ and A_1 has $s-1$ generators. Let P be the set of all q-element subsets of U.

Claim 3.5. *There is $B \in \mathcal{D}$ containing A_1 such that every member of P with at most $n-2$ common predecessors in A_1 has at least one common predecessor in B which does not lie in A_1.*

Proof. Let M be the greatest number of common predecessors in A_1 of an element of P. Note that $M \leq n-1$ since $\mathcal{D} \subseteq \mathcal{C}_n$, and recall that $\mathcal{T}_{n-1} \in \mathcal{D}$. Now apply Lemma 3.3 with $N = n-1$ to obtain $B \in \mathcal{D}$ containing A_1 and such that every $p \in P$ has at least $n-1$ common predecessors in B. So for $p \in P$ with at most $n-2$ common predecessors in A_1, there is at least one common predecessor of p in B which does not lie in A_1. $\square$

Let $T' \geq T$ be a copy of T with root z such that (z, b) is a directed edge; let $b' \neq b$ be another successor of z.

We now find independent subsets $U' \cup I$ and $V' \cup J$ of B as given by the claim, and T' respectively, with $\mathrm{desc}(U) \subseteq \mathrm{desc}(U')$ and $\mathrm{desc}(V) \subseteq \mathrm{desc}(V')$, such that I is a complement to U in A_1 and $J \subseteq \mathrm{desc}(b')$, together with an isomorphism from $\mathrm{desc}(U' \cup I)$ to $\mathrm{desc}(V' \cup J)$ which takes I to J and extends f.

Indeed, if $n = 2$ we let $U' := U$ and $V' := V$. Now suppose $n \geq 3$ and let P' be the subset of P consisting of all q-element sets $p \subseteq U$ with at least one and at most $n-2$ common predecessors in A_1, and such that the image $f(p)$ in V has a common predecessor in T'. By Claim 3.5, p has a common predecessor w_p in $B \setminus A_1$. Let $w_{f(p)}$ be the common predecessor of $f(p)$ in T' and define

$$U' := \left(U \setminus \bigcup P'\right) \cup \{w_p \mid p \in P'\}$$

and

$$V' := \left(V \setminus \bigcup f(P')\right) \cup \{w_{f(p)} \mid p \in P'\}.$$

By Lemma 3.2, U' and V' are independent subsets of B and T' respectively and the natural extension F of f which takes w_p to $w_{f(p)}$ is an isomorphism from $\mathrm{desc}(U')$ to $\mathrm{desc}(V')$. In either case ($n = 2$ or $n \geq 3$), by Lemma 3.1, U has a complement I in A_1 and by Lemma 3.2, $U' \cup I$ is an independent set. Now let J be an independent subset of $\mathrm{desc}(b')$ with $|J| = |I|$. Since $\mathrm{desc}(b) \cap \mathrm{desc}(b') = \varnothing$, $V' \cup J$ is an independent subset of T'. So $U' \cup I$ and $V' \cup J$ are independent subsets of the same size and there is an isomorphism $\overline{F}$ from $\mathrm{desc}(U' \cup I)$ to $\mathrm{desc}(V' \cup J)$ extending f and taking I to J.

By $\leq$-amalgamation, there are $D \in \mathcal{D}$ and $\leq$-embeddings $g_1 : B \to D, g_2 : T' \to D$ such that $g_1(y) = g_2(\overline{F}(y))$ for all $y \in \mathrm{desc}(U' \cup I)$, where we may assume that g_1 is the identity map. As we now show, the point of the construction is that by extending *before* we amalgamate, we have ensured that in this amalgamation, unwanted identifications are avoided.

Claim 3.6. $A_1 \cap g_2(\mathrm{desc}(b)) = \mathrm{desc}(U)$.

Proof. We have $\mathrm{desc}(U) = g_2(\mathrm{desc}(V))$ since $\overline{F}_{|\mathrm{desc}(U)} = f$. As $\mathrm{desc}(V) \subseteq \mathrm{desc}(b)$, it follows that $\mathrm{desc}(U) \subseteq A_1 \cap g_2(\mathrm{desc}(b))$. Now suppose for a contradiction that there are vertices $\gamma \in A_1 \setminus \mathrm{desc}(U)$, $\gamma' \in \mathrm{desc}(b) \setminus \mathrm{desc}(V)$ such that $\gamma = g_2(\gamma')$.

We first show that $\mathrm{desc}(\gamma) \setminus \mathrm{desc}(U)$ is finite. Indeed, suppose $a \in A_1$ is such that $\mathrm{desc}(a) \cap \mathrm{desc}(I) \neq \varnothing$. Then $a \neq g_2(\gamma'')$ for any $\gamma'' \in \mathrm{desc}(b) \setminus \mathrm{desc}(V)$ since $\mathrm{desc}(I) = g_2(\mathrm{desc}(J)) \subseteq g_2(\mathrm{desc}(b'))$ and $\mathrm{desc}(b') \cap \mathrm{desc}(b) = \varnothing$. So $\mathrm{desc}(\gamma) \cap \mathrm{desc}(I) = \varnothing$, and $\mathrm{desc}(\gamma) \setminus \mathrm{desc}(U) = \mathrm{desc}(\gamma) \setminus \mathrm{desc}(U \cup I)$ is finite.

Now we show that there is a q-element subset p of $U \cap \mathrm{desc}(\gamma)$ with a common predecessor in $\mathrm{desc}(\gamma)$. Choose $u \in U \cap \mathrm{desc}(\gamma)$ at maximal distance from γ, and let y be the predecessor of u in $\mathrm{desc}(\gamma)$ (note that $y \in A_1$). Since $\mathrm{desc}(\gamma) \setminus \mathrm{desc}(U)$ is finite, $\mathrm{desc}(y) \setminus \mathrm{desc}(U)$ is finite. So if u' is another successor of y, $\mathrm{desc}(u') \setminus \mathrm{desc}(U)$ is finite and our choice of u implies that $u' \in U \cap \mathrm{desc}(\gamma)$. Thus we can take p to be the set of successors of y.

Now we finish off the proof of Claim 3.6. Since $\gamma = g_2(\gamma')$, the q-element subset $f(p)$ of $\mathrm{desc}(\gamma') \cap V$ has a common predecessor, y' say, in $\mathrm{desc}(\gamma')$ and $y = g_2(y')$. If p has $n-1$ predecessors in A_1, then there is a copy of $\mathcal{T}_n$ in A because A is the free amalgam of A_1 and T over $\mathrm{desc}(U) = \mathrm{desc}(V)$. This is a contradiction. Therefore p has at most $n-2$ common predecessors in A_1 and this means that $p \in P'$. It follows that $y' = w_{f(p)}$ since a q-element set of vertices of T' has at most one common predecessor in T' as T' is a tree. Now as $w_p = g_2(w_{f(p)})$, we have $y = g_2(y') = g_2(w_{f(p)}) = w_p$. This is a contradiction since $w_p \in B \setminus A_1$ and $y \in A_1$. $\square$

We have therefore shown that, $A_1 \cup g_2(\mathrm{desc}(b))$ as a subdigraph of D is isomorphic to A. So A embeds in D and therefore $A \in \mathcal{D}$. This completes the proof that $\mathcal{D} = \mathcal{C}_n$. $\square$

Finally suppose $\mathcal{D} \subseteq \mathcal{C}$ is a $\leq$-amalgamation class and $\mathcal{T}_n \in \mathcal{D}$ for all $n \geq 1$. A similar argument as in the above proof can be used to show that any $A \in \mathcal{C}$ lies in $\mathcal{D}$. The two important points in that proof which we need to modify slightly, are the choice of the digraph B and of the subset P' of P. We want a digraph $B \in \mathcal{D}$ containing A_1 such that every q-element set of vertices of U has at least $M + 1$ common predecessors in B, where M is the greatest number of common predecessors of p in A_1 as p ranges over P. For this we apply Lemma 3.3 to A_1 with $N := M + 1$. In this case it will follow that for *every* p in P, there is at least one common predecessor of p in B which does not lie in A_1. We then take P' to be the subset of P consisting of all q-element sets p which have at least one common predecessor in A_1 and such that $f(p)$ has a common predecessor in T'. The remainder of the argument follows similarly, except that when showing that $A_1 \cap g_2(\mathrm{desc}(b)) = \mathrm{desc}(U)$, there is only one case to consider since for every p in P' there is a vertex $w_p \in B \setminus A_1$. We deduce the following.

Proposition 3.7. *Let $\mathcal{D} \subseteq \mathcal{C}$ be a $\leq$-amalgamation class with $\mathcal{T}_n \in \mathcal{D}$ for all $n \geq 1$. Then $\mathcal{D} = \mathcal{C}$.*

We have therefore shown that

Theorem 3.8. *Any $\leq$-amalgamation class $\mathcal{D} \subseteq \mathcal{C}$ is equal to $\mathcal{C}$ or to $\mathcal{C}_n$ for some $n \geq 2$.*

This means that if D is a countable descendant-homogeneous digraph whose descendant set is isomorphic to T, then $D \cong D_n$ for some $n \in \{2, \ldots, \infty\}$.

4. A general construction

4.1. Descendant sets

In this subsection we prove the following.

Theorem 4.1. *Suppose Γ is a countable digraph. Then there is a countable, vertex transitive, descendant-homogeneous digraph M in which all descendant sets are isomorphic to Γ if and only if the following conditions hold:*

(C1) $\mathrm{desc}(u) \cong \Gamma$ *for all $u \in \Gamma$;*
(C2) *If X is a finitely generated subdigraph of Γ then the subgroup of automorphisms of X which extend to automorphisms of Γ is of countable index in* $\mathrm{Aut}(\Gamma)$.

For one direction of this, suppose M is a vertex transitive, descendant-homogeneous digraph. The descendant sets of vertices in M are all isomorphic to a fixed digraph Γ, so (C1) holds. Condition (C2) is a special case of (4′) in Remark 2, so follows from Theorem 2.1 and Remark 2.

We now prove the converse. So for the rest of this subsection, suppose that Γ is a countable digraph which satisfies conditions (C1) and (C2). Let $\mathcal{C}_\Gamma$ be the class of digraphs A satisfying the following conditions:

(D1) $\mathrm{desc}(a)$ is isomorphic to Γ, for all $a \in A$;
(D2) A is finitely generated;
(D3) for $a, b \in A$, the intersection $\mathrm{desc}(a) \cap \mathrm{desc}(b)$ is finitely generated.

Then $\mathcal{C}_\Gamma$ is closed under isomorphism and taking finitely generated descendant-closed substructures. Moreover, it is easy to see that if $A \leq B_1, B_2 \in \mathcal{C}_\Gamma$ and A is finitely generated, then the free amalgam of B_1 and B_2 over A is in $\mathcal{C}_\Gamma$. Thus, Theorem 4.1 will follow once we verify that the countability conditions in (1) and (4) of Theorem 2.1 hold for $\mathcal{C}_\Gamma$. The following lemma will suffice.

Lemma 4.2. *Suppose $A \in \mathcal{C}_\Gamma$. Then there are only countably many isomorphism types of $\leq$-embeddings $f : A \to B$ with $B \in \mathcal{C}_\Gamma$.*

Once we have this, taking $A = \emptyset$ (or $A = \Gamma$) gives that $\mathcal{C}_\Gamma$ contains only countably many isomorphism types; for fixed $A, B \in \mathcal{C}_\Gamma$, the lemma gives condition (4) of Theorem 2.1.

Proof. We say that a $\leq$-embedding $f : A \to B$ with $A, B \in \mathcal{C}_\Gamma$ is an *n-extension* if B can be generated by $f(A)$ and at most n extra elements. We prove by induction on n that for every $A \in \mathcal{C}_\Gamma$ there are only countably many isomorphism types of n-extensions of A.

Suppose $f : A \to B$ is a 1-extension (with $A, B \in \mathcal{C}_\Gamma$). Let $b \in B$ be such that B is generated by $f(A)$ and b and let $C = f(A) \cap \mathrm{desc}(b)$. It follows from property (D3) in B and finite generation of A, that C is finitely generated. Moreover, as each of $f(A)$ and $\mathrm{desc}(b)$ is descendant-closed in B, we have that B is the free amalgam of $f(A)$ and $\mathrm{desc}(b)$ over C. Choose an isomorphism from $\mathrm{desc}(b)$ to Γ and let h be the restriction of this to C and $g : f^{-1}(C) \to \Gamma$ be given by $g = h \circ f$. Then, in the notation of Remark 1, we have an isomorphism from B to $A *_g \Gamma$ and therefore f is isomorphic to a 1-extension $A \to A *_g \Gamma$ for some finitely generated $D \leq A$ and $\leq$-embedding $g : D \to \Gamma$.

There are countably many possibilities for D and the image $g(D)$ here (as D is finitely generated), so it will suffice to show that there are only countably many isomorphism types of $A *_g \Gamma$ with $g : D \to \Gamma$ having fixed domain D and image $E \leq \Gamma$. If $g_1, g_2 : D \to \Gamma$ have image E then $g_1 \circ g_2^{-1}$ gives an automorphism of E. This extends to an automorphism of Γ if and only if there is an isomorphism between the extensions $g_i : A \to A *_{g_i} \Gamma$. Thus, the isomorphism types here are in one-to-one correspondence with the cosets in $\mathrm{Aut}(E)$ of the subgroup of automorphisms which extend to automorphisms of Γ. So there are only countably many isomorphism types, by (C2).

This proves that there are countably many isomorphism types of 1-extensions of A. For the inductive step, we can take countably many representatives $f_j : A \to B'_j$ (for $j \in \mathbb{N}$) of the isomorphism types of $(n-1)$-extensions of A, and representatives $h_{jk} : B'_j \to B'_{jk}$ of the 1-extensions of B'_j (for $j, k \in \mathbb{N}$). We claim that any n-extension $f : A \to B$ is isomorphic to some $h_{jk} \circ f_j : A \to B'_{jk}$. Indeed, let $f(A) \le B_1 \le B$ be such that B_1 is generated by $f(A)$ and $n-1$ elements, and B is generated by B_1 and one extra element. So we can write $f = i \circ g$ where $g : A \to B_1$ is an $(n-1)$-extension and $i : B_1 \to B$ is a 1-extension. There is $j \in \mathbb{N}$ and an isomorphism $h : B'_j \to B_1$ with $h \circ f_j = g$. We can then find $k \in \mathbb{N}$ and an isomorphism $p : B'_{jk} \to B$ with $i \circ h = p \circ h_{jk}$. Then $p \circ h_{jk} \circ f_j = i \circ g = f$, as required. □

It then follows by Theorem 2.1 that the Fraïssé limit D_Γ of $(\mathcal{C}_\Gamma, \le)$ is a countable descendant-homogeneous digraph with $\mathcal{C}_\Gamma$ as its class of finitely generated $\le$-subdigraphs. Vertex transitivity follows from (C1).

4.2. Examples and further remarks

In this subsection we show that a class of digraphs Γ arising in [1] in the context of highly arc transitive digraphs satisfy the conditions in Theorem 4.1 and therefore arise as the descendant sets in descendant-homogeneous digraphs. We begin by reviewing some of the results of [1] and related papers.

The paper [1] studies highly arc transitive digraphs of finite out-valency and gives conditions which the descendant set Γ of a vertex in such a digraph must satisfy. In particular:

Theorem 4.3. *Suppose Γ is the descendant set of a vertex α in an infinite highly arc transitive digraph D of finite out-valency. Then the following properties hold:*

(T1) $\Gamma = \mathrm{desc}(\alpha)$ *is a rooted digraph with finite out-valency and* $\mathrm{desc}^s(\alpha) \cap \mathrm{desc}^t(\alpha) = \varnothing$ *whenever* $s \ne t$.
(T2) $\mathrm{desc}(u) \cong \Gamma$ *for all* $u \in \Gamma$.
(T3) $\mathrm{Aut}(\Gamma)$ *is transitive on* $\mathrm{desc}^s(\alpha)$*, for all* s.
(T4) *There is a natural number* $N = N_\Gamma$ *such that for* $l > N$ *and* $x, a \in \Gamma$*, if* $b \in \mathrm{desc}^l(x) \cap \mathrm{desc}^1(a)$*, then* $a \in \mathrm{desc}(x)$.

Proof. Properties (T2) and (T3) follow immediately from high arc transitivity of D. Property (T1) is proved in Lemma 3.1 of [1] and (T4) is deduced from (T1, T2, T3) in [1, Propositionp 4.7(a)]. □

Remark 4. The paper [2] shows that there are only countably many isomorphism types of digraphs Γ satisfying properties (T1, T2, T3). In fact, the same is true with (T3) replaced by the weaker:

(G3) There is a natural number k such that if $\ell \ge k$ and $x \in \mathrm{desc}^\ell(\alpha)$ and $\beta \in \mathrm{desc}^1(\alpha)$, then $\mathrm{desc}(\beta) \cap \mathrm{desc}(x) \ne \emptyset$ implies $x \in \mathrm{desc}(\beta)$.

Moreover, these (T1, T2, G3) imply (T4). See Corollary 1.5 and Lemma 2.1 of [2] for proofs.

Explicit examples $\Gamma(\Sigma, k)$ of digraphs satisfying (T1, T2, T3) (and which are not trees) are constructed in Section 5 of [1] and constructions of highly arc transitive, but not descendant-homogeneous, digraphs with these as descendant sets are given in [1,3]. The construction we give here (using Theorem 4.1) gives a highly arc transitive, descendant-homogeneous digraph with descendant set $\Gamma(\Sigma, k)$ (and which does not have property Z). Indeed, it is a slightly curious corollary of the results of this section that if Γ is a digraph of finite out-valency which is the descendant set of a vertex in an infinite, highly arc transitive digraph, then there is a descendant-homogeneous, highly arc transitive digraph which has Γ as its descendant set.

Corollary 4.4. *Suppose Γ is a digraph of finite out-valency which satisfies conditions* (T1, T2, T4)*. Then there is a countable, vertex transitive, descendant-homogeneous digraph in which all descendant sets are isomorphic to Γ.*

Proof. We use Theorem 4.1. The digraph Γ satisfies condition (C1) of this, by assumption (T2). So it remains to show that Γ satisfies (C2).

Let X be a finitely generated subdigraph of Γ with minimal generating set $\{x_1, \ldots, x_k\}$. Let $N = N_\Gamma$ as in (T4) and $Y := \bigcup_{i=1}^k B^N(x_i)$. Note that by (T1) Y is finite, and it is clearly invariant under $\mathrm{Aut}(X)$. We will show that any automorphism of X fixing Y pointwise extends to an automorphism of Γ. Such automorphisms form the kernel of the restriction of $\mathrm{Aut}(X)$ to Y, and so form a subgroup of finite index in $\mathrm{Aut}(X)$ (the quotient group is just the group of permutations induced on the finite set Y by elements of $\mathrm{Aut}(X)$). Thus condition (C2) will follow.

Let $a, b \in \Gamma$. We first observe that if $b \in X \setminus Y$ and a is a predecessor of b in Γ, then $a \in X$. Indeed, $b \in \mathrm{desc}^l(x_i)$ for some $l > N$ and $i \in \{1, \ldots, k\}$. Then by definition of N, $a \in \mathrm{desc}(x_i)$. Since $\mathrm{desc}(x_i) \subseteq X$, it follows that $a \in X$.

Let γ be an automorphism of X which fixes Y pointwise. Define $\theta = \gamma \cup id_{\Gamma \setminus X}$. To prove θ is an automorphism of Γ we must show that θ preserves edges and non-edges. For $u \in (\Gamma \setminus X) \cup Y, \theta u = u$ and for $u \in X, \theta u = \gamma u$. So for $a, b \in (\Gamma \setminus X) \cup Y$, we have $\theta(a, b) = (\theta a, \theta b) = (a, b)$. Similarly, θ preserves edges and non-edges when $a, b \in X$ as in this case, $\theta(a, b) = \gamma(a, b)$. Now suppose $a \in \Gamma \setminus X$ and $b \in X \setminus Y$. Then $\theta(a, b) = (\theta a, \theta b) = (a, \gamma b)$. Since γ preserves $Y, \gamma b \in X \setminus Y$. Then by the observation above, (a, b) and $(a, \gamma b)$ are non-edges. □

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