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A NEAR-SELECTION THEOREM

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A near-selection theorem is proven for carriers defined on spaces that are the countable union of finite dimensional compacta. As an application a new proof is given of the fact that the space of homeomorphisms on a compact piecewise linear manifold is locally contractible. In addition a new criterion is given to determine if the space of homeomorphisms on a compact n-manifold is an l_2-manifold.

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0. Introduction

The study of the space of homeomorphisms on a compact manifold can be reduced to the problem of canonically approximating homeomorphsisms by piecewise linear homeomorphism (cf. [9] and the last section of this paper). This problem is essentially a "near-selection" problem. In this paper we state and prove a general near-selection theorem, given an application to the space of piecewise linear homeomorphisms on a manifold and, in our final remarks, show that if for a given integer n, $H_\delta(B^n)$ is an ANR, then for any n-manifold, M^n, $H(M^n)$ is an ANR.

Our near-selection theorem is stated in terms of C-spaces which were originally defined in [12] and studied further in [1]. A metric space X is said to have Property C (be a C-space) if for each sequence of positive numbers $\{\varepsilon_i\}_{i=1}^{\infty}$, there exists a sequence of collection of open sets $\mathcal{U}_1, \mathcal{U}_2, \ldots$ such that

(a) if $U_i \in \mathcal{U}_i$, then diam $U_i < \varepsilon_i$;

(b) If U_i, $U'_i \in \mathcal{U}_i$ and $U_i \neq U'_i$, then $U_i \cap U'_i = \varphi$; and

(c) $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ is a cover of X.

In [12] it is shown that every countable-dimensional metric space is a C-space, so the following theorem can be considered as a theorem involving such spaces. Note that trivially every finite dimensional compactum is a C-space.

1. A near-selection theorem

Selection theorems have proven to be important tools in dealing with function spaces [cf. 5, 10, 16]. However, their use is limited by the hypothesis that the base space must be finite dimensional. In some cases obtaining a near-selection may be sufficient. Our theorem deals with obtaining near-selections when the base space is the countable union of compact C-spaces. Compare with Proposition 3 of [12] that holds only for compact C-spaces. A subset S of a metric space Y is CE if given $\varepsilon>0$ there is a $\delta>0$ so that $N_\delta(S)$ is null-homotopic in $N_\varepsilon(S)$. The function $\varphi: X\to 2^Y$ is a continuous carrier if for each $x\in X$ and $\varepsilon>0$ there is a $\delta>0$ so that if $d(x,x')<\delta$, then $\varphi(x)\subset N_\varepsilon(\varphi(x'))$ and $\varphi(x')\subset N_\varepsilon(\varphi(x))$. If $\mathscr{U}$ is an open cover of X, $\mathscr{N}(\mathscr{U})$ will denote the nerve of $\mathscr{U}$. When there is no possibility of confusion, we will not distinguish between a finite simplicial complex, K, and its underlying point set, $|K|$.

Theorem 1. *Let $X=\bigcup_{n=1}^\infty X_n$ be a metric space such that for each n, X_n is a compact space with property C. Let $\mathscr{S}$ be a collection of CE subspaces of Y and $\varphi: X\to\mathscr{S}\subset 2^Y$ be a continuous carrier. Then given a continuous function $\iota: X\to(0,1]$ there exists a map $f: X\to Y$ such that for all $x\in X$, $f(x)\in N_{\iota(x)}(\varphi(x))$.*

Proof. Without loss of generality we assume that $X_n\subset X_{n+1}$ for all n. We first define a decreasing sequence of positive numbers $\{\delta_i\}_{i=0}^\infty$, such that $\delta_i\leqslant 1/2^{i+3}$ for all i, and such that if $i>0$ and $x\in X_i$, then $N_{2\delta_i}(\varphi(x))$ is contractible in $N_{\delta_{i-1}}(\varphi(x))$. [The same number δ_i suffices for all $x\in X_i$ since X_i is compact.] Then let $\{\varepsilon_i\}_{i=1}^\infty$ be a decreasing sequence of positive numbers such that if $x\in X_i$ and $x'\in X$ with $d(x,x')<2\varepsilon_i$, then $\varphi(x)\subset N_{\delta_i}(\varphi(x'))$ and $\varphi(x')\subset N_{\delta_i}(\varphi(x))$.

Then we can obtain a sequence of collections of open subsets of X, $\mathscr{U}_1,\mathscr{U}_2,\ldots$ such that $\mathscr{U}=\bigcup_{i=1}^\infty\mathscr{U}_i$ is a countable open cover of X and for each i if $U_i, U_i'\in\mathscr{U}_i$ then:

(i) if $U_i\neq U_i'$ then $U_i\cap U_i'=\emptyset$,
(ii) diam $U_i<\varepsilon_i$,
(iii) $U_i\cap X_i\neq\emptyset$,
(iv) if $U_i\cap\iota^{-1}\left(\left[\frac{1}{2^{j+1}},\frac{1}{2^j}\right]\right)\neq\emptyset$, then $i\geqslant j$.

[We construct the cover $\mathscr{U}$: First fix an integer j and for each positive integer n let

$$_jX_n=X_n\cap\iota^{-1}\left(\left[\frac{1}{2j},\frac{1}{2^{j-1}}\right]\right).$$

Partition the positive integers, A, into a countable number of infinite sets, A_n, in such a way that if $k\in A_n$, $k\geqslant n$. Now for each n consider $_jX_n$ and the sequence $\{\varepsilon_{j+2i_p}\}_{i_p\in A_n}$. Since $_jX_n$ is a compact C-space there exists a sequence of finite collections of open subsets of

$$\iota^{-1}\left(\left(\frac{1}{2^i}-\frac{1}{2^{i+2}},\frac{1}{2^{i-1}}+\frac{1}{2^{i+2}}\right)\right),\ \mathcal{U}_{i,j+2i_1},\ \mathcal{U}_{i,j+2i_2},\ \ldots,$$

such that

(a) if $U_{i,j+2i_p}\in\mathcal{U}_{i,j+2i_p}$, diam $U_{i,i+ti_p}<\varepsilon_{j+2i_p}$

(b) if $U_{i,j+2i_p}, U'_{i,j+2i_p}\in\mathcal{U}_{i,j+2i_p}$ and $U_{i,j+2i_p}\neq U'_{i,j+2i_p}$, then $U_{i,j+2i_p}\cap U'_{i,j+2i_p}=\Phi$ and

(c) $\bigcup_{i_p\in A_n} U_{i,j+2i_p}$ is a cover of ${}_jX_n$. Then for each j, the sequence of collections of open subsets of X, $\mathcal{U}_{i,j}, \mathcal{U}_{i,j+2},\ldots$ is such that $\bigcup_{k=0}^{\infty}\mathcal{U}_{i,j+2k}$ is an open cover of $\iota^{-1}([1/2^i, 1/2^{i-1}])$ and for each k if $U_{i,j+2k}, U'_{i,j+2k}\in\mathcal{U}_{i,j+2k}$ then:

(i′) if $U_{i,j+2k}\neq U'_{i,j+2k}$, $U_{i,j+2k}\cap U'_{i,j+2k}=\emptyset$,

(ii′) diam $U_{i,j+2k}<\varepsilon_{j+2k}$.

(iii′) $U_{i,j+2k}\cap X_{j+2k}\neq\emptyset$,

(iv′) $U_{i,j+2k}\subset\iota^{-1}([1/2^i-1/2^{i+2}, 1/2^{i-1}+1/2^{i+2}])$.

Then for each i, let $\mathcal{U}_i=\bigcup_{j=1}^{i}\mathcal{U}_{i,j}$. Obviously $\mathcal{U}=\bigcup_{i=1}^{\infty}\mathcal{U}_i$ is an open cover of $X=\bigcup_{i=1}^{\infty}\iota^{-1}([1/2^i, 1/2^{i-1}])$. Then property (i) follows from (i′) and (iv′); properties (ii), (iii) and (iv) follow from (ii′), (iii′) and (iv′) respectively.]

Let $\mathcal{V}$ be a countable star finite refinement of $\mathcal{U}$ and $b:X\to N(\mathcal{V})$ be the canonical barycentric map to the nerve of the cover $\mathcal{V}$. For each $V\in\mathcal{V}$ pick an element U of $\mathcal{U}$ that contains V. Set $\mu(\langle V\rangle)=U$ and, by extending linearly, define a map $\mu:N(\mathcal{V})\to N(\mathcal{U})$. Let Q denote the (not necessarily locally finite) polyhedron $\mu(N(\mathcal{V}))$. We wish to define $g:Q\to y$ so that $d(g\mu b(x),\varphi(x))<\iota(x)$ for all $x\in X$.

For each vertex $U_i\in\mathcal{U}_i$ of $N(\mathcal{U})$, pick $x_{U_i}\in U_i\cap X_i$ and $y_{U_i}\in\varphi(x_{U_i})$. We define $\tilde g$ from the 0-skelton of $N(\mathcal{U})$ into Y by $\tilde g(\langle U_i\rangle)=y_{U_i}$. Since $x_{U_i}\in X_i$ we can define the map

$$\Psi_{U_i}:N_{2\delta_i}(\varphi(x_{U_i}))\times I\to N_{\delta_{i-1}}(\varphi(x_{U_i})),$$

with $\Psi_{U_i}(w,0)=w$ and $\Psi_{U_i}(w,1)=y_{U_i}$ for all $w\in N_{2\delta_i}(\varphi(x_{U_i}))$. (It is critical for the following that such a map Ψ_{U_i} be fixed at this point.)

We need now some specialized terminology. With the convention that $U_{m_s}\in\mathcal{U}_m$ for each s, let

$$K_j=\{\sigma<Q\mid\sigma=\langle U_{m_1},\ldots,U_{m_t}\rangle\quad\text{with } m_s\leq j\quad\text{for } 1\leq s\leq t\},$$

$$n_{K_j}=\{\sigma<K_j\mid\sigma=\langle U_{m_1},U_{m_2},\ldots,U_{m_t}\quad\text{with } m_s\geq n\quad\text{for } 1\leq s\leq t.\}$$

We first define $g_1:K_1\to Y$ by $g_1(\langle U_1\rangle)=\tilde g(\langle U_1\rangle)$ for all $U_1\in K_1$. Assume inductively that we have defined $g_j:K_j\to Y$ for all $j<i$ such that g_j extends $g_{j-1}|K_{j-1}$ and if $\sigma=\langle U_{m_1},\ldots,U_{m_t}\rangle\subset K_j$ with $m_1<\cdots<m_t$ and w is a point of σ with $w=s\langle U_{m_1}\rangle+(1-s)w'$, where w' is a point of $\langle U_{m_2},\ldots,U_{m_t}\rangle$, then $g_i(w)=\Psi_{U_{m_1}}(g_i(w'),s)$.

We will now define $g_i: K_i \to Y$ satisfying the inductive hypothesis. Assume (subinductively) that we have defined ${}^j g_i: {}^j K_i \to Y$ for all j with $i \geq j > n$, such that for $i > j > n$, ${}^j g_i$ extends ${}^{j+1} g_i$ and for $i \geq j > n$, ${}^j g_i$ extends $g_{i-1} | K_{i-1} \cap {}^j K_i$ and if $\sigma = \langle U_{m_1}, \ldots, U_{m_t}\rangle \in {}^j K_i$ with $m_1 < \cdots < m_t$ and w is a point of σ with $w = s(U_{m_1}) + (1-s)w'$ (where w' is a point of $\langle U_{m_2}, \ldots, U_{m_t}\rangle$) then ${}^j g_i(w) = \psi_{U_{m_1}}({}^j g_i(w'), s)$. For the following note that this condition implies that ${}^j g_i(\sigma) \subset N_{\delta_{m_1-1}}(\varphi(x_{U_{m_1}}))$ and recall that we are now subinducting downward.

We now wish to define ${}^n g_i: {}^n K_i \to Y$ satisfying the subinductive hypothesis (we start trivially by letting ${}^i g_i(\langle U\rangle) = g(\langle U\rangle)$ for each simplex $\langle U\rangle$ of ${}^i K_i$). Let $\sigma = \langle U_{m_1}, \ldots, U_{m_t}\rangle$ be an element of ${}^n K_i$ with $m_1 < m_2 < \cdots < m_t$. (Of course the entire argument relies on the strict inequality of the integers $m_1, m_2, \ldots, m_t$.) If $m_1 > n$, $\sigma < {}^{n+1} K_i$ and we let ${}^n g_i | \sigma = {}^{n+1} g_i | \sigma$. If $m_1 = n$ we note by the subinductive hypothesis that

$$ {}^{n+1} g_i(\langle U_{m_2}, \ldots, U_{m_t}\rangle) \subset N_{\delta_{m_2-1}}(\varphi(x_{U_{m_2}})) \subset N_{\delta_{m_1}}(\varphi(x_{U_{m_2}})). $$

We note that $U_{m_1} \cap U_{m_2} \neq \varphi$ and hence that

$$ d(x_{U_{m_1}}, x_{U_{m_2}}) < \operatorname{diam} U_{m_1} + \operatorname{diam} U_{m_2} < \varepsilon_{m_1} + \varepsilon_{m_2} < 2\varepsilon_{m_1} $$

and hence that $\varphi(x_{U_{m_2}}) \subset N_{\delta_{m_1}}(\varphi(x_{U_{m_1}}))$. Therefore

$$ {}^{n+1} g_i(\langle U_{m_2}, \ldots, U_{m_t}\rangle) \subset N_{2\delta_{m_1}}(\varphi(x_{U_{m_1}})). $$

We can therefore define ${}^n g_i | \sigma$ as desired: If w is a point of σ with $w = s(U_{m_1}) + (1-s)w'$ (where w' is a point of $\langle U_{m_2}, \ldots, U_{m_t}\rangle$) let ${}^n g_i(w) = \Psi_{U_{m_1}}({}^{n+1} g_i(w'), S)$. Obviously ${}^n g_i | \sigma$ extends ${}^{n+1} g_i | \langle U_{m_2}, \ldots, U_{m_t}\rangle$; it also extends $g_{i-1} | \sigma \cap K_{i-1}$ since in both cases we made use of the map $\psi_{U_{m_1}}$. We repeat the argument for each $\sigma < {}^n K_i$ to obtain ${}^n g_i: {}^n K_i \to Y$. This completes the subinductive argument. Let $g_i = {}^1 g_i: {}^1 K_i = K_i \to Y$.

Finally let $g: Q \to Y$ be defined by $g = \lim_{i\to\infty} g_i$; $g\mu$ is obviously well-defined and continuous. We need only check that for each $x \in X$, $d(g\mu b(x), \varphi(x)) < \iota(x)$. Suppose $\mu b(x) \in \langle U_{m_1}, \ldots, U_{m_t}\rangle$ with $m_1 < \cdots < m_t$. Now there is a non-negative integer j with $1/2^{j+1} \leq \iota(x) \leq 1/2^j$. We note that since $x \in U_{m_1}$, condition (iv) implies that $m_1 \geq j$ and hence that $1/2^{m_1+1} \leq \iota(x)$. But

$$ g\mu b(x) \in N_{\delta_{m_1-1}}(\varphi(x_{U_{m_1}})) \subset N_{\delta_{m_1-1}}(N_{\delta_{m_1}}(\varphi(x))) \subset N_{2\delta_{m_1-1}}(\varphi(x)). $$

Hence

$$ d(g\mu b(x), \varphi(x)) < 2\delta_{m_1-1} \leq 2\left(\frac{1}{2^{m_1-1+3}}\right) = \frac{1}{2^{m_1+1}} \leq \iota(x). $$

2. An application

Selection theorems have been particularly effective in dealing with finite dimensional spaces of homeomorphisms [5, 10, 16]. In [8] Geoghegan showed that the space of piecewise linear homeomorphisms, PLH(M), on a compact piecewise linear manifold, M, is the countable union of finite dimensional compacta. It is therefore not surprising that the near-selection theorem is useful in dealing with PLH(M). We give here another proof of the fact that PLH(M) is locally contractible. This was first shown by R. Edwards ([11], see also [7]). These proofs involve refining the arguments of Chernavskii [3] or Edwards–Kirby [6], while the present proof demonstrates that the local contractibility of PLH(M) follows from general principles and information concerning H(M), the space of all topological homeomorphisms on M. Our proof is valid for piecewise linear manifolds M of dimension not equal to 4.

Let $H_\partial(M)$ denote the subspace of all homeomorphisms which equal the identity on the boundary of M and $PLH_\partial(M) = H_\partial(M) \cap PLH(M)$. The various function spaces on compact manifolds will be assumed to have the supremum metric, ρ, i.e., if $f, g \in H(M)$, with d the metric on M, then $\rho(f, g) = \sup_{x \in M} \{d(f(x), g(x))\}$. If $A \in X$ let 1_A denote the inclusion of A in X; for positive δ let $N_\delta(1_M) = \{h \in H(M) \mid \rho(h, 1_M) < \delta\}$.

To begin we state two lemmas. The first is from Edwards–Kirby [6, p. 79], and the second will follow easily from Theorem 1.

Lemma 2.1 (Edwards–Kirby). *Let M^n be a compact n-manifold (not necessarily p.l.) and $\{B_1, \dots, B_p\}$ be an open cover of M^n with $\bar{B}_i$ a closed n-ball for each i. Then there exists a $\delta > 0$ and a map $\varphi : N_\delta(1_M) \to H_\partial(\bar{B}_1) \times \cdots \times H_\partial(\bar{B}_p)$ such that for each $h \in N_\delta(1_M)$, $h = [\pi_p(\varphi(h))]' \circ \cdots \circ [\pi_1(\varphi(h))]'$ and $[\pi_i(\varphi(1_M))]' = 1_M$ for $i = 1, \dots, p$.*

Here for each i, $[\pi_i(\varphi(h))]' : M \to M$ is the homeomorphism defined by

$$[\pi_i(\varphi(h))]'(x) = \begin{cases} \pi_i(\varphi(h))(x), & x \in B_i, \\ x, & x \notin B_i. \end{cases}$$

Lemma 2.2. *Let X be the countable union of compact C-spaces. Given continuous $\lambda : X \to (0, 1]$ and $\mu : X \to H_\partial(B^n)$ there exists a map $f : X \to PLH_\partial(B^n)$ with $\rho(f(x), \mu(x)) < \lambda(x)$ for all $x \in X$.*

Proof. For $h \in H_\partial(B^n)$ and $\alpha > 0$, let $S_{h,\alpha} = \{g \in PLH_\partial(B^n) \mid \rho(g, h) < \alpha\}$ and let $\mathcal{S} = \{S_{h,\alpha} \mid h \in H_\partial(B^n) \text{ and } \alpha > 0\}$. Then define $\varphi : X \to \mathcal{S}$ by $\varphi(x) = S_{\mu(x),\lambda(x)}$ for all $x \in X$. An application of the triangle inequality shows that φ is a continuous carrier, so to apply Theorem 1 we need only show that for $h \in H_\partial(B^n)$ and $\alpha > 0$, $S_{h,\alpha}$ is CE. Suppose $\varepsilon > 0$ is given. Let $\delta = \min(\alpha/4, \varepsilon/4)$ and let $h' \in PLH_\partial(B^n)$ be such that $\rho(h', h) < \delta$. Such an h' exists since $PLH_\partial(B^n)$ is dense in $H_\partial(B^n)$, $n \neq 4$. (See [9, 14]. This is the only place the requirement that $n \neq 4$ is necessary.) Then note that $S_{h',\alpha+2\delta}$ is contractible within itself by the piecewise linear Alexander isotopy (cf. [17]). But

$$N_\delta(S_{h,\alpha}) = S_{h,\alpha+\delta} \subset S_{h',\alpha+2\delta} \subset S_{h,\alpha+\varepsilon} = N_\varepsilon(S_{h,\alpha}),$$

so $N_\delta(S_{h,\alpha})$ is contractible in $N_\varepsilon(S_{h,\alpha})$.

We are now in a position to prove that PLH(M) is locally contractible. In the following if B is homeomorphic to the standard n-ball, B^n, let ${}_BA: \mathrm{H}_\delta(B) \times I \to \mathrm{H}_\delta(B)$ be the induced Alexander isotopy (cf. [6, 16]) on B where ${}_BA(h, 0) = {}_BA_0(h) = h$ and ${}_BA(h, 1) = {}_BA_1(h) = 1_B$ for all $H \in \mathrm{H}_\delta(B)$.

Theorem 2.3. *Let M^n be a compact piecewise linear manifold, $n \neq 4$. Then PLH(M^n) is locally contractible.*

Proof. Since H(M^n) is a topological group, it suffices to show local contractibility at 1_M. Let $\{B_1, \ldots, B_p\}$ be an open cover of M^n with $\bar{B}_i$ a closed piecewise linear n-ball for each i. Then by Lemma 2.1 there exists a $\delta > 0$ and a map

$$\varphi: N_\delta(1_M) \to \mathrm{H}_\delta(\bar{B}_1) \times \cdots \times \mathrm{H}_\delta(\bar{B}_p),$$

such that for each $h \in N_\delta(1_M)$, $h = [\pi_p(\varphi(h))]' \circ \cdots \circ [\pi_1(\varphi(h))]'$ and $[\pi_i(\varphi(1_M))]' = 1_M$ for $i = 1, \ldots, p$.

To show that PLH(M) is locally contractible at 1_M, it suffices to define a map

$$\psi: [N_\delta(1_M) \cap \mathrm{PLH}(M)] \times [0, 1] \to \mathrm{PLH}(M),$$

with $\psi(h, 0) = h$ and $\psi(h, 1) = 1_M$ for all $h \in N_\delta(1_M)$ and $\psi(1_M, t) = 1_M$ for each $t \in [0, 1]$. To obtain this map we will make p applications of Lemma 2.2. First recall that PLH(M) is the countable union of finite dimensional compacta [8] and hence that Lemma 2.2 is applicable with $X = \mathrm{PLH}(M) \times (0, 1)$. Let $\lambda_i: \mathrm{PLH}(M) \times (0, 1) \to (0, 1]$ be given by $\lambda_i(h, t) = \min(t, 1 - t)$ and $\mu_i: \mathrm{PLH}(M) \times (0, 1) \to \mathrm{H}_\delta(\bar{B}_i)$ be given by $\mu_i(h, t) = {}_{B_i}A(\pi_i\varphi(h), t)$. Then by Lemma 2.2 there exists a map $f_i: \mathrm{PLH}(M) \times (0, 1) \to \mathrm{PLH}_\delta(\bar{B}_i)$ with $\rho(f_i(h, t), \mu_i(h, t)) < \min(t, 1 - t)$. Define

$${}_i\psi: [N_\delta(1_M) \cap \mathrm{PLH}(M)] \times I \to \mathrm{PLH}_\delta(\bar{B}_i)$$

by

$${}_i\psi(h, t) = \begin{cases} f_i(h, t) \circ (f_i(1_M, t))^{-1}, & t \in (0, 1), \\ \pi_i\varphi(h), & t = 0, \\ 1_{B_i^n}, & t = 1. \end{cases}$$

We note that ${}_i\psi$ canonically assigns to each $h \in N_\delta(1_M) \cap \mathrm{PLH}(M)$ a path in $\mathrm{PLH}_\delta(\bar{B}_i)$ taking $\pi_i\varphi(h)$ to $1_{\bar{B}_i}$ [of course $\pi_i\varphi(h) \in \mathrm{H}_\delta(\bar{B}_i)$ may not itself be in $\mathrm{PLH}_\delta(\bar{B}_i)$].

Finally define $\psi: [N_\delta(1_M) \cap \mathrm{PLH}(M)] \times [0, 1] \to \mathrm{PLH}(M)$ by $\psi(h, t) = [{}_p\psi(h, t)]' \circ \cdots \circ [{}_1\psi(h, t)]'$, where $[{}_i\psi(h, t)]'$ is the extension to all of M by the identity of ${}_i\psi(h, t)$.

For $t \in (0, 1]$, $\psi(h, t) \in \mathrm{PLH}(M)$ since for each i, $[{}_i\psi(h, t)]' \in \mathrm{PLH}(M)$. For $t = 0$, $\psi(h, 0) = h \in \mathrm{PLH}(M)$. It is trivial to check that ψ is continuous and that $\psi(1_M, t) = 1_M$ for $t \in [0, 1]$.

3. Remarks

One reason for studying the relationship between PLH(M) and H(M) is to gain insight into the question of whether H(M) is an ANR or, equivalently, an l_2-manifold. (Here l_2 denotes the Hilbert space of square summable sequences and l_2^f the subspace of sequences having only finitely many non-zero entries.) Using the fact that PLH(M) is locally contractible and the countable union of compact C-spaces it follows that PLH(M) is an ANR [11] and hence an l_2^f-manifold [13, 18].

It is known that for any compact 2-manifold, M^2, H(M^2) is an ANR [15, 16] and hence an l_2-manifold [18]; however, for $n \geqslant 3$ the question is unresolved. Using the lemma from Edwards–Kirby we make the following observation:

Proposition 3.1. *For a given integer n, if* $\mathrm{H}_\delta(B^n)$ *is an* ANR, *then for any n-manifold* M^n, $\mathrm{H}(M^n)$ *is an* ANR.

Proof. Let M^n be an n-manifold. As in the proof of Theorem 2.3, let $\{B_1, \ldots, B_p\}$ be an open cover of M^n with $\bar{B}_i$ a closed n-ball for each i. Again by Lemma 2.1, there exists a $\delta > 0$ and a map

$$\varphi : N_\delta(1_M) \to \mathrm{H}_\delta(\bar{B}_1) \times \cdots \times \mathrm{H}_\delta(\bar{B}_p),$$

with $h = [\pi_p(\varphi(h))]' \circ \cdots \circ [\pi_1(\varphi(h))]'$ for all $h \in N_\delta(1_M)$. Define $\Phi : \mathrm{H}_\delta(\bar{B}_1) \times \cdots \times \mathrm{H}_\delta(\bar{B}_p) \to \mathrm{H}(M)$ by $\Phi(f_1, \ldots, f_p) = f_p' \circ \cdots \circ f_1'$. Then $\Phi \mid \Phi^{-1}(N_\delta(1_M)) : \Phi^{-1}(N_\delta(1_M)) \to N_\delta(1_M)$ is an r-map; i.e., there exists a map

$$\varphi : N_\delta(1_M) \to \Phi^{-1}(N_\delta(1_M)),$$

such that $[\Phi \mid \Phi^{-1}(N_\delta(1_M))] \circ \varphi : N_\delta(1_M) \to N_\delta(1_M)$ is equal to $1_{N_\delta(1_M)}$. But $\Phi^{-1}(N_\delta(1_M))$ is an open subset of $\mathrm{H}_\delta(\bar{B}_1) \times \cdots \times \mathrm{H}_\delta(\bar{B}_p)$ and hence, by assumption, is an ANR. Therefore, being the r-image of an ANR [2], $N_\delta(1_M) \subset \mathrm{H}(M^n)$ is an ANR. But then since H(M) is a topological group, each point has an open ANR neighborhood and hence $\mathrm{H}(M^n)$ is an ANR.

Finally we note that since $\mathrm{PLH}_\delta(B^n)$ is an ANR, an immediate consequence of the preceding observation is that an affirmative answer to the following near-selection question would show that for every n-manifold n, $\mathrm{H}(M^n)$ is an ANR (and hence an l_2-manifold):

Question 3.2. Given $\varepsilon : \mathrm{H}_\delta(B^n) \to (0, 1]$, does there exist a map $\varphi : \mathrm{H}_\delta(B^n) \to \mathrm{PLH}_\delta(B^n)$ with $\rho(\varphi(h), h) < \varepsilon(h)$ for all $h \in \mathrm{H}_\delta(B^n)$?

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