Simultaneous approximation for Bézier variant of Szász–Mirakyan–Durrmeyer operators

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Abstract

We study the rate of convergence in simultaneous approximation for the Bézier variant of Szász–Mirakyan–Durrmeyer operators by using the decomposition technique of functions of bounded variation. © 2006 Elsevier Inc. All rights reserved.

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1. Introduction

To approximate Lebesgue integrable functions on the interval \([0, \infty)\), Szász–Mirakyan–Durrmeyer operators with the basis function \(p_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}\) (see [1,3]) are defined by

\[
S_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^\infty p_{n,k}(t) f(t) \, dt, \quad x \in [0, \infty).
\]

For \(\alpha \geq 1\), the Bézier variant of operators (1) is defined by

\[
S_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^\infty p_{n,k}(t) f(t) \, dt,
\]

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where
\[ Q_{n,k}^{(\alpha)}(x) = \left[ J_{n,k}(x) \right]^\alpha - \left[ J_{n,k+1}(x) \right]^\alpha, \quad J_{n,k}(x) = \sum_{j=k}^{\infty} p_{n,j}(x). \]

Srivastava and Zeng [6] and Gupta and Abel [2] established the rate of convergence for the Bézier variant of Szász–Mirakyan–Durrmeyer operators in ordinary approximation. Very recently Gupta et al. [6] estimated the rate of convergence in simultaneous approximation for the Szász–Mirakyan–Durrmeyer operators defined by (1). In [6] the authors were not able to study the rate of simultaneous approximation for the Bézier variant of Szász–Durrmeyer operators. We are now able to achieve this and in the present paper we extend and generalize the results of [4–6], here we establish the rate of convergence for the Bézier variant of Szász–Mirakyan–Durrmeyer operators in simultaneous approximation.

We denote the class \( B_{r,\beta} \) by \( B_{r,\beta} = \{ f : f^{(r-1)} \in C[0,\infty), \ f_\pm^{(r)}(x) \text{ exist everywhere and} \ \text{are bounded on every finite subinterval of} \ [0,\infty) \ \text{and} \ f_\pm^{(r)}(t) = O(e^{\beta t}) \ (t \to \infty), \ \text{for some} \ \beta > 0 \} \ (r = 0, 1, 2, \ldots). \ By \ f_\pm^{(0)}(x) \ \text{we mean} \ f(x \pm). \)

Our main result is stated as follows:

**Theorem.** Let \( f \in B_{r,\beta}, \ r \in N^0, \ \beta > 0. \ Then for every \( x \in (0, \infty), \ \alpha \geq 1 \ \text{and} \ n \geq \max\{r^2 + 3r + 2, 4\beta\}, \ \text{we have} \)

\[
\left| S_{n,\alpha}^{(r)}(f, x) - \frac{1}{\alpha + 1} \left\{ f_+^{(r)}(x) + \alpha f_-^{(r)}(x) \right\} \right| \\
\leq \frac{r + \alpha}{\sqrt{2enx}} \cdot \left[ f_+^{(r)}(x) - f_-^{(r)}(x) \right] \\
+ \frac{x^2 + 3\alpha(1 + 2x)}{nx^2} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}(g_{r,x}) + 2^{(r+1)/2} \alpha e^{2\beta x} \sqrt{\frac{2x + 1}{nx^2}}, \tag{3}
\]

where \( g_{r,x} \) is the auxiliary function defined by

\[
g_{r,x}(t) = \begin{cases} 
 f^{(r)}(t) - f_-^{(r)}(x), & 0 \leq t < x, \\
 0, & t = x, \\
 f^{(r)}(t) - f_+^{(r)}(x), & x < t < \infty,
\end{cases}
\]

\( V_a^b(g_{r,x}(t)) \) be the total variation of \( g_{r,x}(t) \) on \( [a, b] \). In particular \( g_{0,x}(t) \equiv g_x(t) \) (see e.g. [5]).

**2. Auxiliary results**

In the sequel, we shall need the following results.

**Lemma 1.** [7] For all \( x \in (0, \infty), \ \alpha \geq 1 \ \text{and} \ k \in N \cup \{0\}, \ \text{we have} \)

\[
p_{n,k}(x) \leq \frac{1}{\sqrt{2enx}} \quad \text{and} \quad Q_{n,k}^{(\alpha)}(x) \leq \frac{\alpha}{\sqrt{2enx}},
\]

where the constant \( 1/\sqrt{2e} \) and the estimation order \( n^{-1/2} \) (for \( n \to \infty \)) are best possible.
Lemma 2. [1] If \( f \in L_1[0, \infty) \), \( f^{(r-1)} \in A.C._{\text{loc}} \) and \( f^{(r)} \in L_1[0, \infty) \), then

\[
S_n(f,x) = n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k+r}(t) f^{(r)}(t) \, dt.
\]

Remark 1. For \( m \in N \cup \{0\} \), if \( V_{r,n,m}(x) = n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k+r}(t)(t-x)^m \, dt \), then for all \( x \in [0, \infty) \), we have \( V_{r,n,m}(x) = O(n^{-[(m+1)/2]}) \). Also if \( n \geq r^2 + 3r + 2 \), then from [4, Lemma 1], we have \( V_{r,n,2}(x) \leq \frac{(2x+1)}{n} \).

Lemma 3. Suppose \( x \in (0, \infty) \) and \( K_{r,n,\alpha}(x,t) = n \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k}(x)p_{n,k+r}(t) \). Then for \( n \geq r^2 + 3r + 2 \), there hold

\[
\int_0^{y} K_{r,n,\alpha}(x,t) \, dt \leq \frac{\alpha(2x+1)}{n(x-y)^2}, \quad 0 \leq y < x,
\]

\[
\int_{z}^{\infty} K_{r,n,\alpha}(x,t) \, dt \leq \frac{\alpha(2x+1)}{n(z-x)^2}, \quad x < z < \infty.
\]

The proof of the above lemma easily follows by using Remark 1.

3. Proof

Proof of Theorem. Obviously

\[
f^{(r)}(t) = \frac{f_+^{(r)}(x) + \alpha f_-^{(r)}(x)}{\alpha + 1} + g_{r,x}(t) + \frac{f^{(r)}(x) - f^{(r)}(x)}{\alpha + 1} \cdot \text{sign}_\alpha(t)
\]

\[
+ \delta_x(t) \left[ f^{(r)}(x) - \frac{f_+^{(r)}(x) + \alpha f_-^{(r)}(x)}{\alpha + 1} \right],
\]

where

\[
\text{sign}_\alpha(t) = \begin{cases} 
\alpha, & t > 0, \\
0, & t = 0, \\
-1, & t < 0,
\end{cases}
\]

and \( \delta_x(t) = \begin{cases} 
1, & t = x, \\
0, & t \neq x.
\end{cases} \)

Now as \( f \in B_{r,\beta} \), it is observed that \( S_{n,\alpha}(\delta_x, x) = 0 \). Applying Lemma 2, we have

\[
\left| S_n^{(r)}(f,x) - \frac{1}{\alpha + 1} \left\{ f_+^{(r)}(x) + \alpha f_-^{(r)}(x) \right\} \right|
\]

\[
\leq n \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k}(x) \int_0^{\infty} p_{n+r,k}(x) g_{r,x}(t) \, dt
\]

\[
+ \frac{1}{\alpha + 1} \left| f_+^{(r)}(x) - f_-^{(r)}(x) \right| \cdot \left| S_n^{(r)}(\text{sign}_\alpha(t-x), x) \right|.
\]

To estimate \( S_{n,\alpha}(\text{sign}(t-x), x) \), using the identity \( n \int_0^x p_{n,k}(t) \, dt = 1 - \sum_{j=0}^{k} p_{n,j}(x) \) and Lemma 1, we have
\( S_n^{(r)}(\text{sign}(t - x), x) \)
\[
= n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \left( \int_{0}^{\infty} \alpha p_{n,k+r}(t) dt - (1 + \alpha) \int_{0}^{x} p_{n,k+r}(t) dt \right)
\]
\[
= \alpha - (1 + \alpha) \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \left( 1 - \sum_{j=0}^{k+r} p_{n,j}(x) \right)
\]
\[
= (1 + \alpha) \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \sum_{j=0}^{k+r} p_{n,j}(x) - 1
\]
\[
= (1 + \alpha) \left[ \sum_{j=0}^{\infty} p_{n,j}(x) \sum_{k=j}^{\infty} Q_{n,k}^{(\alpha)}(x) + \frac{r}{\sqrt{2enx}} \right] - 1
\]
\[
= (1 + \alpha) \left[ \sum_{j=0}^{\infty} p_{n,j}(x) \left[ J_{n,j}(x) \right]^\alpha + \frac{r}{\sqrt{2enx}} \right] - \sum_{j=0}^{\infty} Q_{n,j}^{(\alpha)}(x).
\]

By mean value theorem, we find that
\[
Q_{n,j}^{(\alpha+1)}(x) = \left[ J_{n,j}(x) \right]^\alpha - \left[ J_{n,j+1}(x) \right]^\alpha = (\alpha + 1) p_{n,j}(x) \left[ \gamma_{n,j}(x) \right]^\alpha,
\]
where \( J_{n,j+1}(x) < \gamma_{n,j}(x) < J_{n,j}(x) \). Again using Lemma 1, we have
\[
|S_n^{(\alpha)}(\text{sign}_\alpha(t - x), x)| \leq \left( 1 + \alpha \right) \sum_{j=0}^{\infty} p_{n,j}(x) \left[ \left[ J_{n,j}(x) \right]^\alpha - \left[ \gamma_{n,j}(x) \right]^\alpha \right] + \frac{(1 + \alpha)r}{\sqrt{2enx}}
\]
\[
\leq (1 + \alpha) \left[ \frac{\alpha}{\sqrt{2enx}} + \frac{r}{\sqrt{2enx}} \right] = \frac{(1 + \alpha)(\alpha + r)}{\sqrt{2enx}}. \tag{5}
\]

Next
\[
n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{0}^{\infty} p_{n+r,k}(x) \left| g_{r,x}(t) \right| dt
\]
\[
= \int_{0}^{\infty} g_{r,x}(t) K_{r,n,\alpha}(x,t) dt
\]
\[
= \left( \int_{0}^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{2x} + \int_{2x}^{\infty} \right) K_{r,n,\alpha}(x,t) g_{r,x}(t) dt
\]
\[
=: E_1 + E_2 + E_3 + E_4. \tag{6}
\]

Applying Lemma 3 and proceeding along the lines of [4,6], the estimates of \( E_1 - E_3 \) are given by:
\[ |E_1| \leq \frac{2\alpha(2x+1)}{nx^2} \sum_{k=1}^{n} V_x^{x-x/\sqrt{k}}(gr,x), \]  
\[ |E_3| \leq \frac{3\alpha(2x+1)}{nx^2} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(gr,x), \]  
\[ |E_2| \leq V_x^{x+x/\sqrt{n}}(gr,x) \leq \frac{1}{n} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(gr,x). \]  

Finally by Remark 1, we have
\[ |E_4| = \left| \int_{2x}^{\infty} K_{r,n,\alpha}(x,t)gr,x(t)dt \right| \leq n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} p_{n,k+r}(t)e^{\beta t} dt \leq x^{-1} \alpha \left( n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{0}^{\infty} p_{n,k+r}(t)(t-x)^2 dt \right)^{1/2} \times \left( n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{0}^{\infty} p_{n,k+r}(t)e^{2\beta t} dt \right)^{1/2} \leq \alpha x^{-1/2}2^{(r+1)/2} \frac{2x+1}{n} e^{2\beta x} \sqrt{2x+1}, \] for \( n > 4\beta. \)  

Collecting the estimates of (4)–(10), we get (3). This completes the proof of the theorem. \( \square \)

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References