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A property deducible from the generic initial ideal

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Abstract

Let S_d be the vector space of monomials of degree d in the variables $x_1,\dots,x_s$. For a subspace $V\subseteq S_d$ which is in general coordinates, consider the subspace $\mathrm{gin}(V)\subseteq S_d$ generated by initial monomials of polynomials in V for the revlex order. We address the question of what properties of V may be deduced from $\mathrm{gin}(V)$.

This is an approach for understanding what algebraic or geometric properties of a homogeneous ideal $I\subseteq k[x_1,\dots,x_s]$ that may be deduced from its generic initial ideal $\mathrm{gin}(I)$.

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0. Introduction

During the recent years the generic initial ideal of a homogeneous ideal has attracted some attention as an invariant. An intriguing problem is what algebraic or geometric properties of the original ideal can be deduced from the generic initial ideal.

In this paper we take perhaps the most elementary approach possible. Let $S=k[x_1,\dots,x_s]$ and let $>$ be the reverse lexicographic order of the monomials in S. Denote by S_d the graded piece of degree d in S. Suppose $V\subseteq S_d$ is a subspace. Denote by $\mathrm{gin}(V)$ the subspace of S_d generated by initial monomials of polynomials of the subspace of S_d obtained from V by performing a general change of coordinates. Then one may ask what properties of V may be deduced from $\mathrm{gin}(V)$? The following result gives an insight in this direction.

Let $W=(x_1,\dots,x_r)\subseteq S_1$ which is a linear space. Suppose that $s\geq r\geq 3$.

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Main Theorem. *Let $V \subseteq S_{n+m}$ be a linear space such that*

$$\mathrm{gin}(V) = W^n x_1^m \subseteq S_{n+m}.$$

Then there exists a polynomial $p \in S_m$ and a linear subspace $W_n \subseteq S_n$ such that $V = W_n p$.

Note that if $s=r$ then $W^n x_1^m$ are the largest monomials in S_{n+m} for the lexicographic order. Thus, if $>$ had been the lexicographic order and $\mathrm{gin}(V) = W^n x_1^m$ then we could deduce virtually nothing about V.

The general idea of the proof is inspired by Green [4] and worth attention because of its seeming naturality in dealing with problems of this kind.

The idea in its vaguest and most generally applicable form is the following. Suppose $\mathrm{gin}(V)$ has a given form, and suppose V is in general coordinates so $\mathrm{in}(V) = \mathrm{gin}(V)$. The given form of $\mathrm{in}(V)$ implies some algebraic or geometric property of V. Let now $g: S_1 \to S_1$ be a general change of coordinates. Then $\mathrm{in}(g^{-1}.V) = \mathrm{gin}(V)$ also. Thus, $g^{-1}.V$ will also have this property. Then this property may be translated back to a property of V. This gives a continuous set of properties that V will satisfy. From this one may proceed making deductions about what V may look like.

In this paper this is applied concretely as follows. In the case $r=s$ the given form of $\mathrm{in}(V) = \mathrm{gin}(V)$ implies that there is a p_1 in S_m such that $x_r^n.p_1 \in V$. The fact that $\mathrm{in}(g^{-1}.V) = \mathrm{gin}(V)$ also implies that there is a $p_{g^{-1}}$ in S_m such that $x_r^n.p_{g^{-1}} \in g^{-1}.V$. Translating this property back to V, we get

$$(g.x_r)^n \cdot g.p_{g^{-1}} \in V. \tag{1}$$

Now, for the family of linear forms $h = \sum t_i x_i$ one may choose a general family of g's depending on h such that $g.x_r = h$. Then Eq. (1) may be written as

$$h^n p \in V, \tag{2}$$

where p is a form of degree m depending on h.

The second technique, specifically suggested by Green [4], is to *differentiate* this equation with respect to the t_i's. All the derivatives will still be in V. (This is just the fact that when a vector varies in a vector space its derivative is also in the vector space.) Letting $V_{|h=0}$ be the image by the composition $V \to S \to S/(h)$ this enables us to show that the forms in $V_{|h=0}$ have a common factor of degree m.

The third basic ingredient is now Proposition 3.4 which says that if $V_{|h=0}$ have a common factor of degree m, then V has a common factor of degree m.

Having proven the case $s=r$, the case $s>r$ may now be proven by an induction process.

The organization of the paper is as follows. In the first three sections we develop general theory which does not presuppose *anything* about what $\mathrm{gin}(V)$ actually is.

In Section 1 we give some basic definitions and notions. In Section 2 we define the generical initial space of a subspace V of S by using a *generic* coordinate change on V. We also give some basic theory for this setting which will be used in Sections 4 and 5.

Section 3 presents the framework in which we will work. Instead of considering a continuously varying form $h=\sum_{i=1}^{s} t_i x_i$ in $k[x_1,\dots,x_s]$, we consider h as a linear form in $K[x_1,\dots,x_s]$ where $K=k(t_1,\dots,t_s)$, the field of rational functions of the t_i's.

If now $V\subseteq k[x_1,\dots,x_s]_d$ is a subspace let $V_K=V\otimes_k K\subseteq K[x_1,\dots,x_s]$. The main result here, Proposition 3.4, says that if the forms in $V_{K|h=0}$ have a common factor of degree m then the forms in V have a common factor of degree m. This is proven using differentiation of forms with respect to the t_i.

Only *from now on* do we assume that $\mathrm{gin}(V)$ has the special form given in the main theorem. In Section 4 we prove the case $s=r$ in the main theorem. Section 5 proves the case $s>r$ of the main theorem. In Section 6 we give an application of the main theorem. The example originated in discussions with Green and was what triggered this paper. Consider the complete intersection of three quadratic forms in $\mathbf{P}^3$. Let $I\subseteq k[x_1,x_2,x_3,x_4]$ be its homogeneous ideal. By standard theory one may deduce that there are two candidates for $\mathrm{gin}(I)$:

$$J^{(1)}=(x_1^2,x_1x_2,x_2^2,x_1x_3^2,x_2x_3^2,x_3^4),$$
$$J^{(2)}=(x_1^2,x_1x_2,x_1x_3,x_2^3,x_2^2x_3,x_2x_3^2,x_3^4).$$

By the main theorem, if $\mathrm{gin}(I)=J^{(2)}$ then the quadratic forms in $I_2\subseteq S_2$ would have to have a common factor. Impossible. Thus, $\mathrm{gin}(I)=J^{(1)}$.

Throughout the article all fields have characteristic zero.

1. Basic definitions and notions

1.1. Let $S=k[x_1,\dots,x_s]$. The graded piece of degree d is denoted by S_d. If $I=(i_1,i_2,\dots,i_s)$ we use the notation

$$\boldsymbol{x}^I=x_1^{i_1}\cdots x_s^{i_s}.$$

It has degree $|I|=\sum i_j$.

Suppose now we have given a total order on the monomials. For a homogeneous polynomial $f=\sum a_I\boldsymbol{x}^I$ in S (henceforth often referred to as a form) let the initial monomial be

$$\mathrm{in}(f)=\max\{\boldsymbol{x}^I \mid a_I\neq 0\}.$$

For a homogeneous vector subspace $V\subseteq S$ let the *initial subspace* be

$$\mathrm{in}(V)=(\{\mathrm{in}(f)\mid f\in V\})$$

the homogeneous vector subspace of S generated by the initial monomials of forms in V.

Sometimes we wish to consider another polynomial ring $R[x_1,\dots,x_r]$, where R is a commutative ring. Denote this by S_R. The initial monomials $\mathrm{in}(f)$ for $f \in V$ may equally well be considered as elements of S_R. We may thus speak of $\mathrm{in}(V)$ over R (when $V \subseteq S$) which is the free R-module in S_R generated by $\{\mathrm{in}(f) \mid f \in V\}$.

1.2. The monomial order we shall be concerned with in Sections 4 and 5 is the reverse lexicographic order. Then the monomials of a given degree is ordered by $x^I > x^J$ if $i_r < j_r$ where r is the greatest number with $i_r \neq j_r$. Intuitively x^J is "dragged down" by having a large "weight in the rear".

1.3. For a linear form $l \in S_1$ denote by $V|_{l=0}$ the image of the composition

$$V \to S \to S/(l).$$

The following basic fact for the revlex order, [2, Proposition 15.12a], will be used several times

$$\mathrm{in}(V_{|x_s=0}) = \mathrm{in}(V)_{|x_s=0}.$$

2. The generic initial space

The following section contains the definition of the generic initial space and some general theory related to it. The things presented here are certainly in the background knowledge of people but due to a lack of suitable references for a proper algebraic treatment we develop the theory here. The most important things are Proposition 2.9 and Paragraph 2.11.

2.1. We identify $S = k[x_1,\dots,x_s]$ as the affine coordinate ring of $\mathbf{A}^s$. Let $G = GL(S_1^\vee)$. There is a natural action

$$\mathbf{A}^s \times G \to \mathbf{A}^s$$

given by $(a,g) \mapsto g^{-1}.a$. This gives a k-algebra homomorphism

$$\gamma : k[x_1,\dots,x_s] \to k[x_1,\dots,x_s] \otimes_k k[G].$$

If R is a $k[G]$-algebra, we also by composition obtain a k-algebra homomorphism

$$\gamma_R : k[x_1,\dots,x_s] \to k[x_1,\dots,x_s] \otimes_k k[G] \to R[x_1,\dots,x_s].$$

Note that if $R = k(g)$ for a point $g \in G$, then $\gamma_{k(g)}$ is just the action of g on $k[x_1,\dots,x_s]$.

Let K_G be the function field of G. The image of a homogeneous subspace $V \subseteq S$ by γ_{K_G} generates a homogeneous subspace $(\gamma_{K_G}(V))$ of the same dimension as V. Suppose now a total monomial order is given. The initial monomials of $(\gamma_{K_G}(V))$ generate a linear subspace over k (or over K_G), which is called the *generic initial subspace* of V over k (or over K_G) and is denoted $\mathrm{gin}(V)$. Henceforth, we shall drop the outer parentheses of $(\gamma_{K_G}(V))$ and write this as $\gamma_{K_G}(V)$.

2.2. Let $\mathrm{gin}(V) = (m_1, \ldots, m_t)$ for some monomials m_i. Let $b_i \in \gamma_{K_G}(V)$ be such that

$$b_i = m_i + b_{i0},$$

where b_{i0} consists of monomials less than m_i for the given order. Now there is an open subset $U \subseteq G$ such that all the b_i lift to elements of $\mathcal{O}(U)[x_1, \ldots, x_s]$. Now we immediately get.

Proposition 2.3. *There is an open subset $U \subseteq G$ (take the one above) such that for $g \in U$, then*

$$\mathrm{in}(\gamma_{k(g)}(V)) = \mathrm{gin}(V) \text{ (over } k(g)).$$

(The original reference for this is [3].)

2.4. Now, choose a $g_0 \in G$ such that $k(g_0) = k$. There is then a diagram

$$\begin{tikzcd}
\mathbf{A}^s_{K_G} \arrow[r, "\alpha_{g_0}"] \arrow[d] & \mathbf{A}^s_{K_G} \arrow[d] \\
\mathbf{A}^s \times G \arrow[r, "\cdot g_0"] \arrow[d] & \mathbf{A}^s \times G. \arrow[d] \\
\mathbf{A}^s \arrow[r, "g_0^{-1}\cdot"] & \mathbf{A}^s
\end{tikzcd}$$

The lower horizontal map is the natural action. The middle map is given by $(a, g) \mapsto (a, gg_0)$ and the lower vertical maps are just the action of G. The upper horizontal map is the map induced by the middle map. From the commutativity of the diagram we see that

$$\gamma_{K_G}(g_0 . V) = \alpha^*_{g_0}(\gamma_{K_G}(V)),$$

where $\alpha^*_{g_0}$ is the automorphism of $K_G[x_1, \ldots, x_s]$ induced by α_{g_0}. Note that $\alpha^*_{g_0}$ comes from an automorphism of K_G. So it does not affect the variables x_i.

Thus, we see that the $\alpha^*_{g_0}(b_i) = m_i + \alpha^*_{g_0}(b_{i0})$ are a basis for $\gamma_{K_G}(g_0 . V)$, where the monomials in $\alpha^*_{g_0}(b_{i0})$ are less than m_i for the given order. Also note that the $\alpha^*_{g_0}(b_i)$ lift to the open subset $U . g_0^{-1} \subseteq G$. Thus, we have proven the following.

Lemma 2.5. *Given $g \in G$, by replacing the subspace V by $g_0.V$ and the open subset U by $U.g_0^{-1}$ for a suitable g_0, we may assume that g is in the open subset from Proposition* 2.3.

2.6. Now, let $\phi : X \to G$ be a morphism. We get a morphism

$$\mathbf{A}^s \times X \to \mathbf{A}^s$$

and, thus, a k-algebra morphism

$$\gamma_{K_X} : k[x_1, \dots, x_s] \to K_X[x_1, \dots, x_s],$$

where K_X is the function field of X. We get a homogeneous subspace $(\gamma_{K_X}(V))$ and also here we shall henceforth drop the outer parenthesis. By performing a suitable coordinate change of V we may assume (by Lemma 2.5) that $\phi(X) \cap U \neq \emptyset$. The following is now immediate from the results above.

Lemma 2.7. (1) *For x in the open subset $\phi^{-1}(U) \subseteq X$ we have*

$$\mathrm{in}(\gamma_{k(x)}(V)) = \mathrm{gin}(V)\,(\textit{over } k(x)).$$

(2) $\mathrm{in}(\gamma_{K_X}(V)) = \mathrm{gin}(V)$ (*over* K_X).

(3) *Given $x \in X$ then we may assume that $\phi(x) \in U$.*

2.8. By 1.3 we have

$$\mathrm{in}(V_{|x_s=0}) = \mathrm{in}(V)_{|x_s=0}.$$

We would like to have a suitable version of this for generic subspaces. The version we need is (2) in the following proposition. It is used most importantly in the proof of Lemma 5.2.

Proposition 2.9. (1) $\mathrm{gin}(\gamma_{K_X}(V)) = \mathrm{gin}(V)$ (*over* K_X).

(2) $\mathrm{gin}(\gamma_{K_X}(V)_{|x_s=0}) = \mathrm{gin}(V)_{|x_s=0}$ (*over* K_X).

Proof. We prove (2). The proof of (1) is analogous and easier. Besides we will not need (1). We just state it for completeness.

(a) Let $S_1^\circ = (x_1, \dots, x_{s-1})$ and $G^\circ = GL(S_1^{\circ\vee})$. Let $k \to K$ be a homomorphism of fields and let $G_K^\circ = GL(S_1^{\circ\vee} \otimes_k K)$. Due to the naturally split inclusion $S_1^\circ \subseteq S_1$, there is a diagram

$$\begin{tikzcd} \mathbf{A}_K^{s-1} \times G_K^\circ \arrow[r] \arrow[d] & \mathbf{A}_K^{s-1} \arrow[d] \\ \mathbf{A}_K^s \times G_K^\circ \arrow[r] & \mathbf{A}_K^s, \end{tikzcd}$$

where the upper action is given by $(a,g)\mapsto g^{-1}.a$. The lower map gives a K-algebra homomorphism

$$\gamma^\circ : K[x_1,\dots,x_s]\to K[G_K^\circ]\otimes_K K[x_1,\dots,x_s].$$

The upper map gives a K-algebra homomorphism

$$\gamma^\circ_{|x_s=0} : K[x_1,\dots,x_{s-1}]\to K[G_K^\circ]\otimes_K K[x_1,\dots,x_{s-1}].$$

For a homogeneous subspace $W\subseteq K[x_1,\dots,x_s]$ we now see that

$$\gamma^\circ_{|x_s=0}(W_{|x_s=0})=\gamma^\circ(W)_{|x_s=0}.$$

The initial space of the former is by definition $\mathrm{gin}(W_{|x_s=0})$. By applying 1.3 to the latter initial space we then get

$$\mathrm{gin}(W_{|x_s=0})=\mathrm{in}(\gamma^\circ(W))_{|x_s=0}. \tag{3}$$

(b) Now, there is a diagram

$$\begin{tikzcd} \mathbf{A}^s\times G^\circ\times G \arrow[r] \arrow[d] & \mathbf{A}^s\times G \arrow[d] \\ \mathbf{A}^s\times G \arrow[r] & \mathbf{A}^s. \end{tikzcd}$$

The upper horizontal map is given by $(a,h,g)\mapsto(h^{-1}.a,g)$. The lower horizontal map and the right vertical map are the actions. Lastly, the left vertical map is given by $(a,h,g)\mapsto(a,hg)$. It induces a diagram

$$\begin{tikzcd} \mathbf{A}^s\times G^\circ\times X \arrow[r] \arrow[d] & \mathbf{A}^s\times X \arrow[d] \\ \mathbf{A}^s\times G \arrow[r] & \mathbf{A}^s. \end{tikzcd} \tag{4}$$

Apply Lemma 2.7. Then $\gamma_{K_X}(V)$ has initial space $\mathrm{gin}(V)$. Also applying Lemma 2.7 to the composition $\mathbf{A}^s\times G^\circ\times X\to\mathbf{A}^s\times G\to\mathbf{A}^s$ (from the diagram), gives that $\gamma_{K_{G^\circ\times X}}(V)$ has initial ideal $\mathrm{gin}(V)$ over $K_{G^\circ\times X}$.

Now go back to part (a) of this proof and put $K=K_X$ and $W=\gamma_{K_X}(V)$. By the commutativity of the diagram (4) we see that

$$\gamma^\circ(W)=\gamma_{K_{G^\circ\times X}}(V).$$

Thus,

$$\mathrm{in}(\gamma^\circ(W))=\mathrm{in}(\gamma_{K_{G^\circ\times X}}(V))=\mathrm{gin}(V)\,(\text{over } K_{G^\circ\times X}).$$

Putting this together with (3) we get

$$\mathrm{gin}(\gamma_{K_X}(V)_{|x_s=0}) = \mathrm{gin}(V)_{|x_s=0}\,(\mathrm{over}\,K_X). \qquad \square$$

2.10. Now, there is of course also a natural action

$$G \times \mathbf{A}^s \to \mathbf{A}^s$$

given by

$$(g,a) \mapsto g.a.$$

The morphism

$$\rho : G \times \mathbf{A}^s \to \mathbf{A}^s \times G$$

given by

$$(g,a) \mapsto (g.a, g)$$

is an isomorphism and its inverse ρ^{-1} is given by

$$(b,g) \mapsto (g, g^{-1}.b).$$

The morphism ρ induces a $k[G]$-algebra isomorphism

$$\Gamma : k[x_1,\dots,x_s] \otimes_k k[G] \to k[G] \otimes_k k[x_1,\dots,x_s].$$

Note that Γ^{-1} is the $k[G]$-algebra isomorphism induced by ρ^{-1}. For any $k[G]$-algebra R we get an R-algebra isomorphism

$$\Gamma_R : R[x_1,\dots,x_s] \to R[x_1,\dots,x_s].$$

The homogeneous subspace $V \subseteq S$ induces an R-submodule

$$V_R = V \otimes_k R \subseteq R[x_1,\dots,x_s]$$

and so we get a free R-module

$$\Gamma_R^{-1}(V_R) \subseteq R[x_1,\dots,x_s],$$

which is in fact just $\gamma_R(V)$.

2.11. For a morphism $\phi : X \to G$ with $\phi(X) \cap U \neq 0$ we now see that

$$\mathrm{in}(\Gamma_{K_X}^{-1}(V_{K_X})) = \mathrm{in}(\gamma_{K_X}(V)) = \mathrm{gin}(V)\,(\mathrm{over}\,K).$$

Now consider $G = GL(S_1^\vee)$ to be an open subset of $\mathbf{A}^{s^2}$ with coordinate functions u_{ij} for $i,j = 1,\dots,r$. Let the u_{ij} take general values of k for $i<r$ and let $u_{rj} = t_j$. Let D be the determinant of the matrix thus obtained and let $T = k[t_1,\dots,t_s]_D$. The situation to which we will apply the above is to the situation where $X =$ Spec T. For the rest of the paper let $K = K_X = k(t_1,\dots,t_s)$, the field of rational functions in the t_i.

Finally, if we let $h = \sum_{i=1}^{s} t_i x_i$, note the following which will be used repeatedly in Sections 4 and 5: $\Gamma_K(x_s) = h$.

3. Derivatives of forms

Given a form p in $S_K = K[x_1,\dots,x_s]$. One may then differentiate it with respect to the t_i and obtain partial derivatives $\partial^{|I|}p/\partial t^I$ where $I=(i_1,\dots,i_s)$. More generally, for a homogeneous form $s(\boldsymbol{t}) = \sum \alpha_I t^I$ of degree d we get the directional derivative $\partial^d p/\partial s(\boldsymbol{t}) = \sum \alpha_I \partial^d p/\partial t^I$.

For a form f in S_K let $\bar{f}$ be its image in $S_K/(h)$. Now consider a specific form p. Let $l(\boldsymbol{t}) = \sum \alpha_i t_i$ be such that $\overline{l(\boldsymbol{x})} = \sum \alpha_i \bar{x_i}$ is not a factor of $\bar{p}$.

Lemma 3.1. *Suppose $\partial^k p/\partial l^k = \alpha_k p$ for $k \geq 0$. If f is a form such that $\overline{\partial^k f/\partial l^k}$ has $\bar{p}$ as a factor for all $k \geq 0$, then f has p as a factor.*

Proof. We have

$$f = u_1 p + h a_1 \tag{5}$$

for some u_1 and a_1. Differentiating this gives

$$\partial f/\partial l = \partial u_1/\partial l \cdot p + u_1 \partial p/\partial l + l(\boldsymbol{x}) a_1 + h \partial a_1/\partial l.$$

Thus, $\bar{p}$ divides $\bar{a_1}$. So $a_1 = v_1 p + h a_2$ for some v_1 and a_2. Inserting this in (5) gives

$$f = u_2 p + h^2 a_2,$$

where $u_2 = u_1 + h v_1$. Now differentiate twice with respect to l. We may conclude that

$$a_2 = v_2 p + h a_3$$

for some v_2 and a_3. Continuing we get in the end that $f = up$. □

The following result is [1, Proposition 10] and is due to Green. It is assumed there that the field $k = \mathbf{C}$ but the proof is readily seen to work for any field of characteristic zero. Given a form $\bar{p}$ in $S_K/(h)$ it gives a criterion for it to lift to a form in S_K which is essentially a form in S.

Proposition 3.2. *Let $p \in S_K$ be a form such that*

$$x_i \overline{\partial p/\partial t_j} \equiv x_j \overline{\partial p/\partial t_i} \pmod{\bar{p}}$$

for all i and j. Then $p = \alpha p_0 + hR$, where $p_0 \in S$ and $\alpha \in K$.

Consider now a form $f \in S \subseteq S_K$. It gives a hypersurface in $\mathbf{P}^{s-1}$. The following says that if all hyperplane sections of this hypersurface are reducible with a component of a given degree then the same is true for the hypersurface defined by f.

Corollary 3.3. *Suppose $\bar{f} = \bar{u}.\bar{p}$ in $S_K/(h)$, where $\bar{u}$ and $\bar{p}$ do not have a common factor. Then $\bar{p}$ lifts to a form αp_0 where $p_0 \in S$. Furthermore, p_0 is a factor of f.*

Proof. Let u and p in S_K be liftings of $\bar{u}$ and $\bar{p}$. We get

$$f = up + hR.$$

Differentiating with respect to $\partial/\partial t_i$ gives

$$0 = \partial u/\partial t_i . p + u\partial p/\partial t_i + x_i R + h\partial R/\partial t_i.$$

Thus, we get

$$\bar{u}(x_j\overline{\partial p/\partial t_i} - x_i\overline{\partial p/\partial t_j}) \equiv 0 \pmod{\bar{p}}.$$

Then by Proposition 3.2 we conclude that $\bar{p}$ has a lifting αp_0 where $p_0 \in S$. By Lemma 3.1 we conclude that p_0 is a factor of f since the $\partial^k f/\partial l^k = 0$ for $k \geq 1$. □

Now, suppose $V \subseteq S_{n+m}$ is a subspace so we get a subspace $V_K = V \otimes_k K \subseteq S_{K,n+m}$ and $V_{K|h=0} \subseteq S_K/(h)$.

Proposition 3.4. *Suppose the forms of $V_{K|h=0}$ have a common factor $\bar{p}$, where $\bar{p}$ is a common factor of maximal degree m. Then V has a common factor p_0 of degree m such that $\bar{p} = \alpha\overline{p_0}$ for some $\alpha \in K$.*

Proof. We may choose an $f_0 \in V$ such that

$$\bar{f}_0 = \bar{u}_0 \bar{p},$$

where $\bar{u}_0$ and $\bar{p}$ are relatively prime. This is seen as follows. Let $\bar{p} = \bar{a}_1^{e_1} \cdots \bar{a}_r^{e_r}$ be a factorization where the $\bar{a}_i$ are distinct irreducible factors. It is easily seen that the set of f in V where $\bar{f}$ has $\bar{a}_i^{e_i+1}$ as a factor, is a linear subspace V_i of V. On the other hand, if f varies all over V the restrictions $\bar{f}$ generate $V_{K|h=0}$. Thus, we cannot have $V_i = V$ for any i. But since $\operatorname{char} k = 0$ the field k is infinite, so there must be an f_0 in $V - \bigcup V_i$.

By Corollary 3.3, $\bar{p}$ lifts to αp_0 where $p_0 \in S$. Choose now any f in $V \subseteq V_K$. Then

$$\bar{f} = \bar{u} \cdot \overline{\alpha p_0}.$$

By Lemma 3.1 we may conclude that p_0 is a factor of f and, thus, a common factor of V. □

4. The case when $s = r$

Now, we are ready for the specific work in proving the Main theorem. Consider $S = k[x_1, \ldots, x_r]$. Let $W = (x_1, \ldots, x_r) = S_1$ which is a linear space. Use the notation $W^n = S_n$. (This will make our statements more unified in form.) Let the monomial order be the revlex order. In this section we prove the following (which is the case $s=r$ of the Main theorem).

Theorem 4.1. *Let $V \subseteq S_{n+m}$ be a linear space such that*

$$\operatorname{gin}(V) = W^n x_1^m \subseteq S_{n+m}.$$

Then there exists a polynomial $p \in S_m$ such that $V = W^n p$.

We assume V to be in general coordinates so Lemma 2.7 applies.

Lemma 4.2. *There is a form p in $S_{K,m}$ such that*

$$h^n p \in V_K.$$

Proof. From 2.11 we have $\operatorname{in}(\Gamma_K^{-1}(V_K)) = \operatorname{gin}(V)$ over K. Thus, there exists a q_0 in $\Gamma_K^{-1}(V_K)$ such that

$$q_0 = x_r^n x_1^m + \text{terms with smaller monomials}.$$

By the property of the revlex order, x_r^n will divide all terms of q_0 so there exists a $p_0 \in S_{K,m}$ such that

$$q_0 = x_r^n p_0.$$

Let $p = \Gamma_K(p_0)$. Then, we get

$$h^n p = \Gamma_K(x_r)^n \Gamma_K(p_0) = \Gamma_K(q_0) \in V_K. \qquad \square$$

From $V_K \subseteq S_K$ we obtain the subspace

$$V_{K|h=0} \subseteq S_K/(h).$$

Let $\bar{p}$ be the image of p in $V_{K|h=0}$.

Lemma 4.3. *The elements in $V_{K|h=0}$ have $\bar{p}$ as a common factor. Furthermore, it is a common factor of maximal degree.*

Proof. We first find the dimension of the space $V_{K|h=0}$. The map Γ_K gives an isomorphism

$$\overline{\Gamma}_K : K[x_1,\ldots,x_r]/(x_r) \to K[x_1,\ldots,x_r]/(h).$$

Thus, $\overline{\Gamma}_K{}^{-1}(V_{K|h=0}) = \Gamma_K^{-1}(V_K)_{|x_r=0}$. Since $\Gamma_K^{-1}(V_K)$ has initial space

$$(x_1,\ldots,x_r)^n \cdot x_1^m,$$

we get by 1.3 that $\Gamma_K^{-1}(V_K)_{|x_r=0}$ has initial space

$$(x_1,\ldots,x_{r-1})^n \cdot x_1^m.$$

Hence, the dimension of $V_{K|h=0}$ is equal to the dimension of this space.

Now differentiate the equation

$$h^n p \in V_K$$

with respect to $\partial^{|I|}/\partial t^I$ where $I=(i_1,\dots,i_{r-1})$ and $|I|=n$. The derivative will also be in V_K. This is essentially the fact that when a vector varies in a vector space the derivatives will also be in that vector space. We thus get

$$x^I p + hR_I \in V_K$$

for some R_I. Thus,

$$x^I \bar{p} \in V_{K|h=0}. \tag{6}$$

But when I varies, all these forms are linearly independent since h does not divide any linear combination of the x^I. By our statement about the dimension of $V_{K|h=0}$, the forms (6) must generate $V_{K|h=0}$, thus proving the lemma. □

By Corollary 3.4 we may now conclude that V has a maximal common factor p_0 of degree m. Thus proving Theorem 4.1.

5. The case when $s>r$

Now, we assume $S=k[x_1,\dots,x_s]$. As before $W=(x_1,\dots,x_r)\subseteq S_1$, a linear subspace and assume $s>r$. The monomial order is revlex. In this section we prove the following by induction on s.

Theorem 5.1. *Let $V\subseteq S_{n+m}$ be a linear space such that*

$$\mathrm{gin}(V)=W^n x_1^m \subseteq S_{n+m}.$$

Then there exists a polynomial $p\in S_m$ and a linear subspace $W_n\subseteq S_n$ such that $V=W_n p$.

Assume V to be in general coordinates. Let $g:S_1\to S_1$ be a general coordinate change. Since $\mathrm{in}(g^{-1}.V)=(x_1,\dots,x_r)^n\cdot x_1^m$, by 1.3 it follows that $\mathrm{in}(g^{-1}.V_{|x_s=0})=(x_1,\dots,x_r)^n\cdot x_1^m$ also. By induction $g^{-1}.V_{|x_s=0}$ has a common factor. By translating back, $V_{|g.x_s=0}$ also has a common factor (depending on g). The following expresses this in the algebraic language we use.

Lemma 5.2. *There is a form p in $S_{K,m}$ such that $\bar{p}$ in $S_{K|h=0}$ is a common factor of $V_{K|h=0}$. Furthermore, it is a common factor of maximal degree.*

Proof. By 2.9.2 the generic initial ideal of $\Gamma_K^{-1}(V_K)_{|x_s=0}=\gamma_K(V)_{|x_s=0}$ is $\mathrm{gin}(V)_{|x_s=0}$ (over K). The latter is, by 1.3, seen to be

$$(x_1,\dots,x_r)^n\cdot x_1^m.$$

By induction there is a form $\bar{p}_1$ in $S_{K,m|x_s=0}$ which is a common factor of $\Gamma_K^{-1}(V_K)_{|x_s=0}$. Now x_1^m is a common factor of $\mathrm{in}(\Gamma_K^{-1}(V_K)_{|x_s=0})$ of maximal degree. Then $\bar{p}_1$ must also have maximal degree as a common factor of $\Gamma_K^{-1}(V_K)_{|x_s=0}$. Lift this to a form p_1 in $S_{K,m}$. Then $p=\Gamma_K(p_1)$ is the required form. □

By Corollary 3.4 we may now conclude that V has a maximal common factor p_0 of degree m. Thus proving Theorem 5.1.

6. An example

Consider the complete intersection of three quadratic forms in $\mathbf{P}^3$. Let $I \subseteq k[x_1,x_2,x_3,x_4]$ be its homogeneous ideal. We have the following facts.

(1) I and $\mathrm{gin}(I)$ have the same Hilbert functions.

(2) $\mathrm{gin}(I)$ is Borel-fixed. (See [2, Proposition 15.20].)

(3) Since I is saturated, by [4, Proposition 2.21] we have $\mathrm{gin}(I):x_4=\mathrm{gin}(I)$. This is really just the fact that $\mathrm{in}(I:x_4)=\mathrm{in}(I):x_4$ for the revlex order ([2, Proposition 15.12b]), and that if I is in general coordinates and saturated then $I:x_4=I$.

These three facts imply that there are two possible candidates for $\mathrm{gin}(I)$:

$$J^{(1)}=(x_1^2,x_1x_2,x_2^2,x_1x_3^2,x_2x_3^2,x_3^4),$$

$$J^{(2)}=(x_1^2,x_1x_2,x_1x_3,x_2^3,x_2^2x_3,x_2x_3^2,x_3^4).$$

However, by the theorem above if $\mathrm{gin}(I)=J^{(2)}$ then the quadratic forms in $I_2 \subseteq S_2$ would have to have a common factor. Impossible. Thus, $\mathrm{gin}(I)=J^{(1)}$. On the other hand, if I is an ideal with $\mathrm{gin}(I)=J^{(2)}$ then since the quadratic forms in I_2 would have a common factor it must be the ideal of seven points in a plane plus one extra point not in the plane.

Note also the following. Let $>_1$ be the ordering of the monomials which is lexicographic in the three first variables, and then refined with the reverse lexicographic order with respect to the last variable, i.e.

$$x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4} > x_1^{b_1}x_2^{b_2}x_3^{b_3}x_4^{b_4},$$

if $a_4<b_4$, or $a_4=b_4$ and

$$x_1^{a_1}x_2^{a_2}x_3^{a_3} > x_1^{b_1}x_2^{b_2}x_3^{b_3}$$

for the lexicographic order. Then, if the three forms are general it is easily seen that $\mathrm{gin}(I)=J^{(2)}$. In fact, it is not difficult to argue that one will always have $\mathrm{gin}(I)=J^{(2)}$ if you have a complete intersection of three forms and this order. Thus, both $J^{(1)}$ and $J^{(2)}$ are in fact specialisations of I.

Furthermore, it is not difficult to give an example of a complete intersection of three forms such that $\mathrm{in}(I)=J^{(2)}$ for the reverse lexicographic order. Thus, the fact that one can read some interesting algebraic or geometric information from the initial ideal depends on the fact that you are looking at the *generic initial ideal.*

To sum up, $J^{(2)}$ is a specialisation of the ideal I of a complete intersection of three quadratic forms in general coordinates through the order $>_1$ given above. It is also the specialisation of an ideal I of a complete intersection of three quadratic forms through the revlex order, but it is *never* a specialisation of the ideal I of a complete intersection of three quadratic forms through the revlex order when the forms are in general coordinates.

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