STRONGLY HARMONIZABLE APPROXIMATIONS OF BOUNDED CONTINUOUS RANDOM FIELDS

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Every continuous and bounded random field on $\mathbb{R}^k$ is the limit of a sequence of strongly harmonizable random fields, uniformly on compact subsets of $\mathbb{R}^k$. These harmonizable fields are obtained from the given random field by nonstationary linear filterings.

continuous and bounded random fields * harmonizable random fields * approximation * filtering

1. Harmonizability

A random field $X$ on $\mathbb{R}^k$ is a function $X : \mathbb{R}^k \to L^2_c(S, \mathcal{F}, P)$ where $(S, \mathcal{F}, P)$ is any probability space. Such a function is called a random process if $k = 1$.

The field is said to be weakly harmonizable if there exists a $\sigma$-additive set function $\mu : \mathcal{B}(\mathbb{R}^k) \to L^2_c(S, \mathcal{F}, P)$, $\mathcal{B}(\mathbb{R}^k)$ being the Borelian $\sigma$-field of $\mathbb{R}^k$, such that

$$ \forall t \in \mathbb{R}^k, \quad X(t) = \int \exp(i(t, x)) \mu(dx). $$

$\mu$ may be called a stochastic measure.

In particular, if there exists a complex measure $M$ on $\mathbb{R}^k \times \mathbb{R}^k$ such that

$$ \forall A, B \in \mathcal{B}(\mathbb{R}^k), \quad E(\mu(A) \cdot \overline{\mu(B)}) = M(A \times B), $$

we have

$$ \forall t, s \in \mathbb{R}^k, \quad E(X(t) \cdot \overline{X(s)}) = \iint \exp i((t, x) - (s, y)) M(dx, dy), $$

and the weakly harmonizable random field under consideration is said to be strongly harmonizable, which admits $M$ as spectral measure.

2. Our result

There exist weakly harmonizable random fields which are not strongly harmonizable [see 3]. H. Niemi [5, Corollary 3.4.2] proved that each weakly harmonizable
process can be approximated by a sequence of strongly harmonizable processes uniformly on compact subsets of \( \mathbb{R} \).

All weakly harmonizable random fields on \( \mathbb{R}^k \) are continuous and bounded, but there exist continuous and bounded random fields which are not weakly harmonizable [see 3]. We prove below the following extension of Niemi's result.

**Theorem.** Each continuous and bounded random field \( X \) on \( \mathbb{R}^k \) is the limit, uniformly on compact subsets of \( \mathbb{R}^k \), of a sequence of strongly harmonizable random fields, each of them admitting an absolutely continuous spectral measure.

3. Literature

The equivalence of Niemi's definition of harmonizability with the definition used here is to be found in [1, p. 41]. General properties of integrals and harmonizable processes are collected in [4]. One can also see [6] for harmonizable random fields and [2, Chapter IV] for integration with respect to vector-valued measures.

4. Proof of the Theorem. Let \( X : \mathbb{R}^k \rightarrow L^2_c(S, \mathcal{F}, P) \) be a continuous and bounded random field.

All the \( L^2_c(S, \mathcal{F}, P) \)-valued functions and measures introduced below will take values in the separable Hilbert subspace \( L^2(X) \) generated by \( (X(t), t \in \mathbb{R}^k) \).

In the sequel, we use the following notation:

- \( \lambda \) is the Borel measure on \( \mathbb{R}^k \)
- \( C = \sup(\|X(t)\|, t \in \mathbb{R}^k) \)
- \( t = (t_1, \ldots, t_k) \in \mathbb{R}^k \)
- \( I = \{1, 2, \ldots, k\} \)
- \( \sigma > 0 \)
- \( f_\sigma(t) = \sigma^{-2k} (2\pi)^{-k/2} \exp(-t^2/(2\sigma^2)) \)

(a) Let \( \sigma > 0 \). \( \forall x \in \mathbb{R}^k \), the continuous function \( s \mapsto X(s) \exp(-i(s, x) - |s|^2/\sigma) \) is \( \lambda \)-integrable since

\[
\int \left\| X(s) \exp\left(-i(s, x) - \frac{|s|^2}{\sigma}\right) \right\| ds \leq 2^k \sigma^k C < +\infty.
\]

Moreover, by the dominated convergence theorem for the strong integral, the function \( F_\sigma \):

\[
x \mapsto \sigma^2 (2\pi)^{-k/2} f_\sigma(x) \int X(s) \exp\left(-i(s, x) - \frac{|s|^2}{\sigma}\right) ds
\]

is also continuous and \( \lambda \)-integrable since we have

\[
\int \| F_\sigma(x) \| dx \leq 2^k \sigma^{2k} (2\pi)^{-k/2} C < +\infty.
\]

Let \( \mu_\sigma : \mathcal{B}(\mathbb{R}^k) \rightarrow L^2(X) \) be the stochastic measure defined by

\[
\forall B \in \mathcal{B}(\mathbb{R}^k), \quad \mu_\sigma(B) = \int 1_B(x) F_\sigma(x) dx,
\]
and $X_\sigma$ be the weakly harmonizable random field associated to $\mu_\sigma$ by

$$\forall t \in \mathbb{R}^k, \quad X_\sigma(t) = \int \exp(i(t, x))\mu_\sigma(dx).$$

$X_\sigma$ is a strongly harmonizable random field since, by using twice the Pettis property of the strong integral, we obtain

$$\forall A, B \in \mathcal{B}(\mathbb{R}^k), \quad E(\mu_\sigma(A) \cdot \mu_\sigma(B)) = E\left(\int A(x)F_\sigma(x)dx \cdot \int B(y)F_\sigma(y)dy\right)$$

$$= \int \int A(x)1_B(y)E(F_\sigma(x) \cdot \overline{F_\sigma(y)})dx dy$$

$$= M_\sigma(A \times B),$$

where $M_\sigma$ is the complex measure on $\mathbb{R}^k \times \mathbb{R}^k$ admitting the density

$$(x, y) \mapsto E(F_\sigma(x) \cdot \overline{F_\sigma(y)}).$$

(b) We now have to show that $X_\sigma$ is obtained from $X$ by filtering:

$\forall Z \in L^2(X)$, let $\mu_{\sigma, Z}$ be the complex measure on $\mathbb{R}^k$ defined by

$$\forall B \in \mathcal{B}(\mathbb{R}^k), \quad \mu_{\sigma, Z}(B) = E(\mu_\sigma(B) \cdot Z).$$

$\mu_{\sigma, Z}$ admits the density $x \mapsto E(F_\sigma(x) \cdot \overline{Z}).$

Consequently, by the Pettis property and Fubini's theorem we have, $\forall t \in \mathbb{R}^k$:

$$E(X_\sigma(t) \cdot \overline{Z}) = \int \exp(i(t, x))\mu_{\sigma, Z}(dx) = \int \exp(i(t, x))E(F_\sigma(x) \cdot \overline{Z})dx$$

$$= \sigma^k(2\pi)^{-k/2} \int \exp(i(t, x))f_\sigma(x)$$

$$\times \left(\int E(X(s) \cdot \overline{Z}) \exp\left(-i(s, x) - \frac{|s|}{\sigma}\right)ds\right)dx$$

$$= \sigma^k(2\pi)^{-k/2} \int E(X(s) \cdot \overline{Z}) \exp\left(-\frac{|s|}{\sigma}\right)$$

$$\times \left(\int \exp(i(t-s, x))f_\sigma(x)dx\right)ds$$

$$= \sigma^k(2\pi)^{-k/2} \int E(X(s) \cdot \overline{Z}) \exp\left(-\frac{|s|}{\sigma} - \frac{\sigma^2}{2}(t-s)^2\right)ds$$

$$= E\left(\sigma^k(2\pi)^{-k/2} \int X(s) \exp\left(-\frac{|s|}{\sigma} - \frac{\sigma^2}{2}(t-s)^2\right)ds \cdot \overline{Z}\right).$$

From this we deduce that

$$X_\sigma(t) = \sigma^k(2\pi)^{-k/2} \int X(s) \exp\left(-\frac{|s|}{\sigma} - \frac{\sigma^2}{2}(t-s)^2\right)ds,$$

(1)
i.e. \( X_\sigma \) is obtained by letting \( X \) pass through the integral-type linear filter \( F_{\sigma} \), the kernel of which is

\[
(s, t) \mapsto \sigma^k (2\pi)^{-k/2} \exp \left( -\frac{|s|}{\sigma} - \frac{\sigma^2}{2} (t-s)^2 \right).
\]

Obviously, \( F_{\sigma} \) is not stationary.

(c) We now study the convergence of \( X_\sigma(t) \) towards \( X(t) \). \( \forall \alpha > 0 \), let \( A(\alpha) = \{ t, t \in \mathbb{R}^k, |t| \leq \alpha \}; \forall \sigma \in \mathbb{R}^k \) and \( \forall \sigma > 0 \), from (1) we have

\[
X_\sigma(t) = (2\pi)^{-k/2} \int X \left( t + \frac{u}{\sigma} \right) \exp \left( -\frac{1}{\sigma} \left| t + \frac{u}{\sigma} - \frac{u^2}{2} \right| \right) \, du,
\]

\[
(2\pi)^{k/2} (X_\sigma(t) - X(t)) = \int f(\sigma, t, u) \, du + \int g(\sigma, t, u) \, du,
\]

where

\[
f(\sigma, t, u) = \left( X \left( t + \frac{u}{\sigma} \right) - X(t) \right) \exp \left( -\frac{1}{\sigma} \left| t + \frac{u}{\sigma} - \frac{u^2}{2} \right| \right),
\]

\[
g(\sigma, t, u) = X(t) \exp \left( -\frac{u^2}{2} \right) \left( 1 + \exp \left( -\frac{1}{\sigma} \left| t + \frac{u}{\sigma} \right| \right) \right).
\]

We have to prove that

\[
\forall \alpha > 0, \quad \sup_{t \in A(\alpha)} \| X_\sigma(t) - X(t) \| \to 0,
\]

or equivalently that

\[
\sup_{t \in A(\alpha)} \left\| \int f(\sigma, t, u) \, du \right\| \to 0 \quad \text{(I)}
\]

and

\[
\sup_{t \in A(\alpha)} \left\| \int g(\sigma, t, u) \, du \right\| \to 0. \quad \text{(II)}
\]

Let \( \alpha > 0, \varepsilon > 0 \).

Proof of (I): We choose \( \beta > \alpha \) such that

\[
\int_{\mathbb{R}^k - A(\beta)} \exp \left( -\frac{u^2}{2} \right) \, du \leq \varepsilon.
\]

Then the random field \( X \) being uniformly continuous on \( A(\beta) \), there exists \( \eta \in ]0, \beta - \alpha[ \) such that

\[
u, v \in A(\beta), \quad |u - v| < \eta \Rightarrow \| X(u) - X(v) \| \leq \varepsilon.
\]

From the decomposition

\[
\int f(\sigma, t, u) \, du = \int_{A(\beta)} f(\sigma, t, u) \, du + \int_{\mathbb{R}^k - A(\beta)} f(\sigma, t, u) \, du,
\]
we obtain, $\forall \sigma \geq \beta / \eta$:
\[
\sup \left( \left\| \int f(\sigma, t, u) \, du \right\| ; \, t \in A(\alpha) \right) \leq \varepsilon \int_{A(\beta)} \exp \left( -\frac{u^2}{2} \right) \, du \\
+ 2C \int_{\mathbb{R}^k - A(\beta)} \exp \left( -\frac{u^2}{2} \right) \, du \\
\leq (2C + (2\pi)^{k/2}) \varepsilon,
\]
and this inequality proves the convergence property (I).

Proof of (II): There exists $\gamma > 0$ such that
\[
\forall x \in \mathbb{R}, \quad |x| \leq \gamma \Rightarrow |e^x - 1| \leq \varepsilon.
\]
From the decomposition
\[
\int g(\sigma, t, u) \, du = \int_{A(\beta)} g(\sigma, t, u) \, du + \int_{\mathbb{R}^k - A(\beta)} g(\sigma, t, u) \, du,
\]
we obtain, $\forall \sigma > 0$:
\[
\frac{1}{\sigma} \left( \alpha + \frac{\beta}{\sigma} \right) \leq \gamma \Rightarrow \sup \left( \left\| \int g(\sigma, t, u) \, du \right\| ; \, t \in A(\alpha) \right) \\
\leq \varepsilon C \int_{A(\beta)} \exp \left( -\frac{u^2}{2} \right) \, du + C \int_{\mathbb{R}^k - A(\beta)} \exp \left( -\frac{u^2}{2} \right) \, du \\
\leq C((2\pi)^{k/2} + 1) \varepsilon,
\]
and this inequality proves the convergence property (II).

References