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On the orientable regular embeddings of complete multipartite graphs

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ABSTRACT

Let $K_{m[n]}$ be the complete multipartite graph with m parts, while each part contains n vertices. The regular embeddings of complete graphs $K_{m[1]}$ have been determined by Biggs (1971) [1], James and Jones (1985) [12] and Wilson (1989) [23]. During the past twenty years, several papers such as Du et al. (2007, 2010) [6,7], Jones et al. (2007, 2008) [14,15], Kwak and Kwon (2005, 2008) [16,17] and Nedela et al. (2002) [20] contributed to the regular embeddings of complete bipartite graphs $K_{2[n]}$ and the final classification was given by Jones [13] in 2010. Since then, the classification for general cases $m \geq 3$ and $n \geq 2$ has become an attractive topic in this area. In this paper, we deal with the orientable regular embeddings of $K_{m[n]}$ for $m \geq 3$. We in fact give a reduction theorem for the general classification, namely, we show that if $K_{m[n]}$ has an orientable regular embedding $\mathcal{M}$, then either $m = p$ and $n = p^e$ for some prime $p \geq 5$ or $m = 3$ and the normal subgroup $\mathrm{Aut}_0^+(\mathcal{M})$ of $\mathrm{Aut}^+(\mathcal{M})$ preserving each part setwise is a direct product of a 3-subgroup Q and an abelian $3'$-subgroup, where Q may be trivial. Moreover, we classify all the embeddings when $m = 3$ and $\mathrm{Aut}_0^+(\mathcal{M})$ is abelian. We hope that our reduction theorem might be the first necessary approach leading to the general classification.

1. Introduction

A *map* is a 2-cell embedding of a connected graph into a closed surface. The embedded graph is called the *underlying graph* of the map. An *automorphism* of a map is an automorphism of the underlying graph which can be extended to a self-homeomorphism of the supporting surface. It is

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well known that the automorphism group of a map acts freely on the set of flags (that is, triples of mutually incident i-cells, $0 \leq i \leq 2$). If it acts regularly, then the map is called *regular*.

For an embedding $\mathcal{M}$ on orientable surface, we use $\mathrm{Aut}^+(\mathcal{M})$ to denote the group of all orientation-preserving automorphisms of $\mathcal{M}$. If $\mathrm{Aut}^+(\mathcal{M})$ acts regularly on the arcs, then we call $\mathcal{M}$ an *orientable regular* map. Such maps fall into two classes: those that admit also orientation-reversing automorphisms, called *reflexible*, and those that do not, called *chiral*.

One of the central problems in topological graph theory is to classify all regular or orientable regular embeddings of a given class of graphs. In a general setting, the classification problem was treated by Gardiner et al. in [10]. However, for particular classes of graphs, it has been solved only in a few cases. Let $K_{m[n]}$ be the complete multipartite graph with m parts, while each part contains n vertices and two vertices are adjacent if and only if they belong to the different parts. All the regular embeddings of complete graphs $K_{m[1]}$ have been determined by Biggs, James and Jones [1,12] for orientable case and by Wilson [23] for nonorientable case. As for the complete bipartite graphs $K_{2[n]}$, the nonorientable regular embeddings of these graphs have recently been classified by Kwak and Kwon [18]; during the past twenty years, several papers [6,7,14–17,20] contributed to the orientable case, and the final classification was given by Jones [13] in 2010. Since then, the classification for general case $m \geq 3$ and $n \geq 2$ was started. The only known result is the determination of orientable regular embeddings of the graphs $K_{m[p]}$ (where p is a prime) given by Du et al. in [8].

In this paper, we shall focus on the orientable regular embeddings, which are simply called regular embeddings. In general, to classify the regular embeddings $\mathcal{M}$ of a given graph, one has to first analyze the possible group structure of $\mathrm{Aut}^+(\mathcal{M})$. We have noted that in the classification of regular embeddings of $K_{2[n]}$, the key point is a determination of the so called isobicyclic groups $H = \langle x\rangle\langle y\rangle$, where $|x| = |y| = n$, $\langle x\rangle \cap \langle y\rangle = 1$ and $x^\alpha = y$ for an involution $\alpha \in \mathrm{Aut}(H)$. Therefore, to classify the regular embeddings of $K_{m[n]}$ for $m \geq 3$ and $n \geq 2$, one should first analyze the structure of $\mathrm{Aut}^+(\mathcal{M})$ and then obtain a reduction theorem. Basing on the reduction, one may eventually give the final classification. Our following main theorem might be the first necessary approach leading to the general classification.

Theorem 1.1. *Let $\mathcal{M}$ be an orientable regular embedding of $K_{m[n]}$ where $m \geq 3$ and $n \geq 2$, and let $\mathrm{Aut}_0^+(\mathcal{M})$ be the normal subgroup of $\mathrm{Aut}^+(\mathcal{M})$ consisting of automorphisms preserving each part setwise. Then $\mathrm{Aut}_0^+(\mathcal{M})$ is an isobicyclic group. Moreover, we have*

(1) *if $m \geq 4$, then $m = p$ and $n = p^e$ for some prime p; or*
(2) *if $m = 3$, then $\mathrm{Aut}_0^+(\mathcal{M}) = Q \times K$, where Q is a 3-subgroup (may be trivial) and K is an abelian $3'$-subgroup. In particular, when $\mathrm{Aut}_0^+(\mathcal{M})$ is abelian, there is only one such map if $3 \nmid n$ and there are three if $3 \mid n$.*

The paper is organized as follows. After this introduction section, some notations, terminologies and preliminary results will be given in Section 2; some group theoretical results used later will be proved in Section 3; the cases $m \geq 4$ and $m = 3$ will be discussed in Sections 4 and 5, separately. Finally, the proof of Theorem 1.1 can be summarized immediately from Sections 4 and 5.

2. Preliminaries

Throughout this paper, all graphs are finite, simple and undirected. For a graph Γ, we use $\mathrm{V}(\Gamma)$ and $\mathrm{E}(\Gamma)$ to denote the vertex set and the edge set of Γ respectively. For any positive integer n, let $[n] = \{1, \ldots, n\}$. For two integers s and t, we use $\gcd\{s, t\}$ to denote the great common divisor of them. For a finite group G and a positive integer s, let $G^s = \langle g^s | g \in G\rangle$. For a ring S, we use S^* to denote the multiplicative group of S. The center of a group G will be denoted by $Z(G)$. The dihedral group of order n will be denoted by $\mathbb{D}_n$, and the cyclic group of order n as well as the integer residue ring modulo n will be denoted by $\mathbb{Z}_n$. When we denote the quotient group G/N by $\overline{G}$, we use the standard 'bar' convention, in which the overbar denotes the canonical homomorphism from G onto $\overline{G}$ (thus $\overline{g} = gN$ and $\overline{H} = H/N$ for every element $g \in G$ and every subgroup H with $N \leq H \leq G$). For the notions not defined here, please refer [2,11,5].

It is well known that the automorphism group $G = \mathrm{Aut}^+(\mathcal{M})$ of a regular map is generated by a generator a of the stabilizer (which is necessarily cyclic) of a vertex, say γ and by an involution b inverting the direction of an edge incident with γ, see [10]. Moreover, the embedding is determined by the group G and the choice of generators a and b [19,9]. A regular map given by $G = \langle a, b\rangle$, with $b^2 = 1$, is called an *algebraic map* $\mathcal{M}(G; a, b)$. Two algebraic maps $\mathcal{M}(G; a, b)$ and $\mathcal{M}(G; a', b')$ are isomorphic if and only if there is a group automorphism in $\mathrm{Aut}(G)$ taking $a \mapsto a'$ and $b \mapsto b'$. If the order of ab and a are s and t respectively, then $\mathcal{M}(G; a, b)$ has *type* $\{s, t\}$ in the notation of Coxeter and Moser [3], meaning that the faces are all s-gons and the vertices all have valency t.

Now we introduce some Propositions and Lemmas which will be used later.

Proposition 2.1 ([11, I.4.5]). *Let G be a group and $H \le G$. Then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of* $\mathrm{Aut}(H)$.

Proposition 2.2 ([4]). *Let p be an odd prime. Then*

(1) *the maximal subgroups of the projective special linear group* $\mathrm{PSL}(2, p)$ *are: one class of subgroups isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{p-1}{2}}$; one class isomorphic to $\mathbb{D}_{p-1}$, when $p \ge 13$; one class isomorphic to $\mathbb{D}_{p+1}$, when $p \ne 7$; two classes isomorphic to A_5 when $p \equiv \pm1 \pmod{10}$; two classes isomorphic to S_4, when $p \equiv \pm1 \pmod 8$; and one class isomorphic to A_4, when $p = 5$ or $p \equiv 3, 13, 27, 37 \pmod{40}$ and $p \ge 5$.*

(2) *the maximal subgroups of the projective general linear group* $\mathrm{PGL}(2, p)$ *are: one class of subgroups isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$; one class isomorphic to $\mathbb{D}_{2(p-1)}$, when $p \ge 7$; one class isomorphic to $\mathbb{D}_{2(p+1)}$; one class isomorphic to S_4, when $p = 5$ or $p \equiv 3, 13, 27, 37 \pmod{40}$ and $p \ge 5$; and one subgroup* $\mathrm{PSL}(2, p)$.

Lemma 2.3. *Let p be a prime. Then we have the following conclusions:*

(1) *If* $\mathrm{AGL}(1, m)$ *is isomorphic to a subgroup of* $\mathrm{GL}(2, p)$ *for a prime power $m \ge 3$, then either $m = 3$ or $m = p$;*

(2) *If $N \trianglelefteq H \trianglelefteq \mathrm{GL}(2, p)$ and $H/N \cong \mathrm{AGL}(1, q)$ for some prime q, then $p = q$ and $N = Z(H)$.*

Proof. Set $Z = Z(\mathrm{GL}(2, p))$. Then $\mathrm{GL}(2, p)/Z = \mathrm{PGL}(2, p)$

(1) Let K be a subgroup of $\mathrm{GL}(2, p)$ isomorphic to $\mathrm{AGL}(1, m)$. Then $Z(K) = 1$ and hence $K \bigcap Z = 1$. It follows that $K \cong KZ/Z \le G/Z$ and then $\mathrm{AGL}(1, m) \lesssim \mathrm{PGL}(2, p)$. By checking Proposition 2.2, we have $m = 3,\ 4,$ or p. Now we only need to show that $m \ne 4$. Suppose to the contrary that $m = 4$, then $K \cong A_4$. Clearly, K contains three elements of order 2, one of which is contained in $\mathrm{SL}(2, p)$. However, $\mathrm{SL}(2, p)$ contains only one involution, which is the center involution, a contradiction.

(2) Clearly,

$$NZ/Z \trianglelefteq HZ/Z \le \mathrm{PGL}(2, p)$$

and

$$(HZ/Z)/(NZ/Z) \cong H/N \cong \mathrm{AGL}(1, q).$$

Then by checking Proposition 2.2 again, we get $\mathrm{AGL}(1, q) \cong HZ/Z \cong \mathrm{AGL}(1, p)$. Therefore, $q = p$ and $Z(H) = N$. □

Lemma 2.4. *Let $G \cong \mathbb{Z}_{p^e} \times \mathbb{Z}_{p^d}$ where p is a prime and $e > d$. Then,*

(1) $\mathrm{Aut}(G)$ *is a* 2*-group when $p = 2$;*

(2) *every p'-subgroup of* $\mathrm{Aut}(G)$ *is abelian when p is odd.*

Proof. Suppose that $G = \langle a\rangle \times \langle b\rangle$ where $|\langle a\rangle| = p^e$ and $|\langle b\rangle| = p^d$. One can check that the mapping defined by $a \mapsto a^i b^j$ and $b \mapsto a^k b^l$ for some $i, k \in \mathbb{Z}_{p^e}$ and $j, l \in \mathbb{Z}_{p^d}$ is an automorphism of G if and only if $\gcd\{i, p\} = \gcd\{l, p\} = 1$ and $p^{e-d} \mid k$. Then by the choices of i, j, k and l, we have $|\mathrm{Aut}(G)| = p^{3d+e-2}(p-1)^2$.

Clearly, if $p = 2$, then $\mathrm{Aut}(G)$ is a 2-group. If p is odd, then $\mathrm{Aut}(G)$ has an abelian Hall p'-subgroup F which is contained in

$$\langle \alpha \in \mathrm{Aut}(G) \mid \alpha(a) = a^i, \alpha(b) = a^l, \gcd\{i, p\} = \gcd\{l, p\} = 1\rangle.$$

The theorem in [21, 9.1.10] tells us that every p'-subgroup of $\mathrm{Aut}(G)$ is contained in a conjugate of F and so it is abelian. □

3. Isobicycle groups

As mentioned before, if H is a group with cyclic subgroups $X = \langle x\rangle$ and $Y = \langle y\rangle$ of order n such that $H = XY$, $X \bigcap Y = 1$ and there is an automorphism α of H transposing x and y, then the group H or the triple (H, x, y) is said to be *n-isobicyclic* (or *isobicyclic* for brevity). In this section, by using some known results we shall deduce some properties of isobicyclic groups.

Lemma 3.1. *Let (H, x, y) be a n-isobicyclic triple. Then H has a characteristic series*

$$1 = H_0 < H_1 < \cdots < H_l = H$$

of subgroups $H_i = H^{s_i} = \langle x^{s_i}\rangle\langle y^{s_i}\rangle$ with $H_i/H_{i-1} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$ for all $i \in [l]$, where $p_1 \ge \cdots \ge p_l$ are the prime divisors of n and $s_i = n/(p_1 \cdots p_i)$.

Proof. We proceed the proof by induction on the order of H.

Suppose that p is the maximal prime divisor of n and P is a Sylow p-subgroup of H. A result of Wielandt [22] on products of nilpotent groups shows that $P \trianglelefteq H$ and hence P is a characteristic subgroup of H. Since P is the unique Sylow p-subgroup of H, we have $P = H^{n/p^d}$ where p^d is the highest power of p dividing n. Noting that $\langle x^{n/p^d}\rangle \bigcap \langle y^{n/p^d}\rangle = 1$ and $|\langle x^{n/p^d}\rangle\langle y^{n/p^d}\rangle| = p^{2d} = |P|$, we have $P = H^{n/p^d} = \langle x^{n/p^d}\rangle\langle y^{n/p^d}\rangle$. Clearly, $(P, x^{n/p^d}, y^{n/p^d})$ is a p^d-isobicyclic triple. By Lemma 3 in [14], P has a central series $1 = Z_0 < Z_1 < Z_{d-1} < Z_d = P$ of subgroups $Z_i = \langle (x^{n/p^d})^{p^{d-i}}\rangle\langle (y^{n/p^d})^{p^{d-i}}\rangle = P^{p^{d-i}}$ with $Z_i/Z_{i-1} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Now we consider the quotient group $\overline{H} = H/P$. By induction hypothesis, $\overline{H}$ has a characteristic series

$$\overline{1} = \overline{N_0} < \overline{N_1} < \cdots < \overline{N_j} = \overline{H}$$

of subgroups $\overline{N_i} = \overline{H}^{t_i} = \langle \overline{x}^{t_i}\rangle\langle \overline{y}^{t_i}\rangle$ with $\overline{H_i}/\overline{H_{i-1}} \cong \mathbb{Z}_{q_i} \times \mathbb{Z}_{q_i}$ for all $i \in [j]$, where $q_1 \ge \cdots \ge q_j$ are the prime divisors of n/p^d and $t_i = n/p^d(q_1 \cdots q_i)$. Set

$$p_i = \begin{cases} p, & 0 \le i \le d; \\ q_{i-d}, & d < i \le d+j \end{cases} \quad \text{and} \quad H_i = \begin{cases} Z_i, & 0 \le i \le d; \\ N_{i-d}, & d < i \le d+j. \end{cases}$$

Then $p_1 \ge \cdots \ge p_{d+j}$ are the prime divisors of n. Write $s_i = n/(p_1 \cdots p_i)$. Then $H_i = H^{s_i} = \langle x^{s_i}\rangle\langle y^{s_i}\rangle$ and $1 = H_0 < H_1 < \cdots < H_{d+j} = H$ is a characteristic series of H with $H_i/H_{i-1} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$ for all $i \in [d+j]$. □

Lemma 3.2. *Suppose that (H, x, y) is a n-isobicyclic triple and p is the maximal prime divisor of n. Let $L = H^{n/p}$. Then $H/C_H(L)$ is an isobicyclic group.*

Proof. By Lemma 3.1, $L = \langle x^{n/p}\rangle\langle y^{n/p}\rangle = \langle x^{n/p}\rangle \times \langle y^{n/p}\rangle$. Let t be the minimal positive integer such that $x^t y^{n/p} = y^{n/p} x^t$. Since there is an automorphism α of H transposing x and y, we also have $y^t x^{n/p} = x^{n/p} y^t$. Hence $\langle x^t, y^t\rangle \le C_H(L)$. On the other hand, taking any $x^i y^j \in C_H(L)$, from $x^i y^j x^{n/p} = x^{n/p} x^i y^j$, we obtain $y^j x^{n/p} = x^{n/p} y^j$. Let $d = \gcd\{t, j\}$. Then there exist two integers m and k such that $d = mt + kj$. Therefore

$$y^d x^{n/p} = y^{mt+kj} x^{n/p} = x^{n/p} y^{mt+kj} = x^{n/p} y^d.$$

By the minimality of t, we get $t = d$ and then $t|j$. Symmetrically, $t|i$ and hence $x^i y^j \in \langle x^t, y^t\rangle$. It follows that $C_H(L) = \langle x^t, y^t\rangle$.

Set $\overline{H} = H/C_H(L)$. Noting that

$$|C_H(L)| = |\langle x^t, y^t\rangle| \geq |\langle x^t\rangle||\langle y^t\rangle| = (n/t)^2,$$

we have

$$|\overline{H}| = |H|/|C_H(L)| \leq t^2.$$

If $\bar{x}^i = \bar{y}^j$, then $x^i y^{-j} \in C_H(L) = \langle x^t, y^t\rangle$. It follows that $t|i$ and $t|j$. Hence $\bar{x}^i = \bar{y}^j = \bar{1}$ and then $\langle\bar{x}\rangle \bigcap \langle\bar{y}\rangle = \bar{1}$. Since $|\langle\bar{x}\rangle| = |\langle\bar{y}\rangle| = t$, we have

$$|\overline{H}| \geq |\langle\bar{x}\rangle||\langle\bar{y}\rangle| = t^2.$$

Thus $|\overline{H}| = t^2$ and $\overline{H} = \langle\bar{x}\rangle\langle\bar{y}\rangle$. Since

$$\alpha(C_H(L)) = \alpha(\langle x^t, y^t\rangle) = \langle y^t, x^t\rangle = C_H(L),$$

we have that α induces an automorphism of $\overline{H}$ transporting $\bar{x}$ and $\bar{y}$. Therefore, $\overline{H}$ is an isobicyclic group. □

Lemma 3.3. GL$(2, p)$ *does not contain any subgroup* M *which is an extension of a nontrivial isobicyclic group by* S_3 *and has trivial center.*

Proof. The Lemma is clear for $p = 2$. Now we assume p is an odd prime. Suppose to the contrary that GL$(2, p)$ contains such a subgroup M. Since $Z(M) = 1$, we have $M \lesssim$ PGL$(2, p)$. By checking Proposition 2.2, $M \cong S_4$. However, GL$(2, p)$ does not contain any subgroup isomorphic to S_4, a contradiction. □

An abelian group is said to be *homogeneous* if it is a direct product of some isomorphic cyclic groups, otherwise it is called *inhomogeneous*. The following result can be extracted from [14,6,7].

Proposition 3.4. *Let* (H, x, y) *be a non-abelian* p^e*-isobicyclic triple. Then* H/H' *is an inhomogeneous abelian group of rank* 2.

4. Case $m \geq 4$

The main result of this section is the following theorem.

Theorem 4.1. *Let* $\mathcal{M}$ *be a regular embedding of* $K_{m[n]}$*, where* $m \geq 4$ *and* $n \geq 2$*, and let* $\mathrm{Aut}_0^+(\mathcal{M})$ *be the kernel of* $\mathrm{Aut}^+(\mathcal{M})$ *on the set of* m *parts. Then* $m = p$ *and* $n = p^e$ *for some prime* $p \geq 5$*. Moreover,* $Z(\mathrm{Aut}^+(\mathcal{M})) = 1$ *and* $\mathrm{Aut}_0^+(\mathcal{M})$ *is a* n*-isobicyclic group.*

Proof. To prove the theorem, set $\Gamma = K_{m[n]}$, with the vertex set.

$$V(\Gamma) = \bigcup_{i=1}^{m} \Delta_i, \quad \text{where } \Delta_i = \{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}\}$$

and the edges are all pairs $\{\gamma_{ij}, \gamma_{kl}\}$ of vertices with $i \neq k$. Then $\mathrm{Aut}(\Gamma) = S_n \wr S_m$, which has blocks Δ_i for $1 \leq i \leq m$.

Set $H = \mathrm{Aut}_0^+(\mathcal{M})$ and $G = \mathrm{Aut}^+(\mathcal{M}) = \langle a, b\rangle$, where $\langle a\rangle = G_{\gamma_{11}}$ and b reverses the arc $(\gamma_{11}, \gamma_{21})$. Let $x = a^{m-1}$ and $y = x^b$. Then $H = \langle x, y\rangle$. Write $\overline{G} = G/H$ and we use $\overline{\Gamma}$ to denote the quotient (block) graph of Γ induced by H. Clearly, $\overline{\Gamma} \cong K_m$.

Then we prove the theorem by the following four steps:

Step 1. Show that m is a prime power, $\overline{G} \cong$ AGL$(1, m)$ and H is a n-isobicyclic group.

By considering the order of G, we know that $|H| = n^2$ and $\overline{G}$ acts arc-regularly on $\overline{\Gamma}$. From the classification of regular embeddings of K_m, m is a prime power and $\overline{G} \cong$ AGL$(1, m)$ (see [1,12]).

Since $\langle x\rangle \leq H_{\gamma_{11}}$ and $\langle y\rangle \leq H_{\gamma_{21}}$, we have $\langle x\rangle \bigcap \langle y\rangle = H_{(\gamma_{11}, \gamma_{21})} = 1$. Noting that $x^b = y, y^b = x$ and $|\langle x\rangle||\langle y\rangle| = n^2 = |H|$, we have $H = \langle x\rangle\langle y\rangle$ is a n-isobicyclic group.

Step 2. Show that $C_G(H_i) = C_H(H_i)$ and $C_{G/H_i}(H/H_i) = Z(H/H_i)$, where $H_i = H^{s_i}$ for $s_i = n/(p_1 \cdots p_i)$ and $n = p_1 \cdots p_l$ where $p_1 \geq \cdots \geq p_l$ are primes.

Taking any $g \in G \setminus H$, there exists $k \in [m]$ such that $\Delta_k^g \neq \Delta_k$. Write $H_{\gamma_{k1}} = \langle z \rangle$. Clearly, (H, z, z^g) is a n-isobicyclic triple. By Lemma 3.1,

$$1 = H_0 < H_1 < \cdots < H_l = H$$

is a series of characteristic subgroups of H and $H_i = \langle z^{s_i} \rangle \langle (z^g)^{s_i} \rangle$ with $H_i/H_{i-1} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$ for all $i \in [l]$. Therefore $z^{s_i} \neq (z^{s_i})^g$ and $zH_i \neq z^g H_i$ for all $i \in [l]$. It follows that $g \notin C_G(H_i)$ and $gH_i \notin C_{G/H_i}(H/H_i)$, from which we have $C_G(H_i) = C_H(H_i)$ and $C_{G/H_i}(H/H_i) = Z(H/H_i)$.

Step 3. Show that $m = p$ and $Z(G) = 1$ where p is the minimal prime divisor of n.

Let p is the minimal prime divisor of n. Set $N = H^p$. By Lemma 3.1, $N = \langle x^p \rangle \langle y^p \rangle$ is a characteristic subgroup of H with $H/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and so $N \trianglelefteq G$. As shown in *Step* 2, $C_{G/N}(H/N) = Z(H/N) = H/N$. Then from Proposition 2.1, we have

$$\mathrm{AGL}(1, m) \cong \overline{G} \cong (G/N)/(H/N) \lesssim \mathrm{Aut}(H/N) \cong \mathrm{GL}(2, p).$$

Now by Lemma 2.3.(1), we get $m = p$ and then $\overline{G} \cong \mathrm{AGL}(1, p)$.

Since $Z(\overline{G}) \cong Z(\mathrm{AGL}(1, p)) = 1$, we have $Z(G) \leq H$. Suppose that $x^i y^j \in Z(G)$ for some $i, j \in [n]$. From $x = a^{m-1}$, we have $[y^j, a] = [x^i y^j, a] = 1$ and hence $(y^a)^j = (y^j)^a = y^j$. Noting that $\langle y \rangle \bigcap \langle y^a \rangle = 1$, we get $j = n$ and then $x^i = x^i y^j \in Z(G)$. Since $x^{-i} y^i = [x^i, b] = 1$, we get $x^i = y^i \in \langle x \rangle \bigcap \langle y \rangle = 1$ and then $i = n$. Therefore $Z(G) = 1$.

Step 4. Show that $n = p^e$ for some e.

By *Step* 3, $m = p$ is the minimal prime divisor of n. Let q be the maximal prime divisor of n and set $J = H^{n/q}$. By Lemma 3.1, $J \cong \mathbb{Z}_q \times \mathbb{Z}_q$ is a characteristic subgroup of H and then $J \trianglelefteq G$. Set $L = C_G(J)$. Then the *Step* 2 implies that $L = C_H(J)$. By Proposition 2.1, $G/L \lesssim \mathrm{Aut}(J) \cong \mathrm{GL}(2, q)$. On the other hand, $(G/L)/(H/L) \cong G/H \cong \mathrm{AGL}(1, p)$. By Lemma 2.3.(2), we have $q = p$, that is, $n = p^e$ for some integer e. $\square$

Remark 4.2. It is easy to see from the proof that the conclusions in Steps 1–3 of Theorem 4.1 hold for $m = 3$ as well.

Proposition 4.3. *For each pair of admissible parameters* (p, p^e)*, there exists at least one regular embedding of* $K_{p[p^e]}$.

Proof. We prove the proposition by constructing a family of regular embeddings of $K_{p[p^e]}$ as follows.

Suppose that $p \geq 5$ is a prime. We identify $\mathbb{Z}_{p^{e+1}}$ with the set

$$\{0, 1, 2, \ldots, p^{e+1} - 1\}.$$

Let

$$\Delta_i = \{jp + i | j = 0, 1, 2, \ldots, p^e - 1\} \quad \text{for } i = 0, 1, \ldots, p - 1.$$

Then we have $\mathbb{Z}_{p^{e+1}} = \Delta_0 \bigcup \Delta_1 \bigcup \cdots \bigcup \Delta_{p-1}$. Now we identify the vertex set of $K_{p[p^e]}$ with $\mathbb{Z}_{p^{e+1}}$ and its edge set with $\mathrm{E} = \{\{\alpha, \beta\} \mid \alpha, \beta \in \mathbb{Z}_{p^{e+1}}, p \nmid \alpha - \beta\}$. Clearly, $\Delta_0, \Delta_1, \ldots, \Delta_{p-1}$ are the p parts of $K_{p[p^e]}$.

Let

$$G = \mathbb{Z}_{p^{e+1}} \rtimes \mathbb{Z}_{p^{e+1}}^* = \{(\pi, \tau) \mid \pi \in \mathbb{Z}_{p^{e+1}}, \tau \in \mathbb{Z}_{p^{e+1}}^*\},$$

and $(\pi, \tau)(\mu, \nu) = (\pi\nu + \mu, \tau\nu)$, for all $(\pi, \tau), (\mu, \nu) \in G$. Then define an action of G on $\mathbb{Z}_{p^{e+1}}$ by $\alpha^{(\pi,\tau)} = \alpha\tau + \pi$ for all $\alpha \in \mathbb{Z}_{p^{e+1}}$ and $(\pi, \tau) \in G$. It is easy to verify that this is indeed an faithful action of G on $\mathbb{Z}_{p^{e+1}}$. Noting that

$$p \nmid \alpha - \beta \Leftrightarrow p \nmid (\alpha\tau + \pi) - (\beta\tau + \pi) \Leftrightarrow p \nmid \alpha^{(\pi,\tau)} - \beta^{(\pi,\tau)}$$

for all $(\pi, \tau) \in G$, we have

$$\{\alpha, \beta\} \in \mathrm{E} \Leftrightarrow \{\alpha^{(\pi,\tau)}, \beta^{(\pi,\tau)}\} \in \mathrm{E},$$

and hence G is a subgroup of $\mathrm{Aut}(K_{p[p^e]})$.

It is well known that $\mathbb{Z}^*_{p^{e+1}}$ is a cyclic group of order $p^e(p-1)$. Set $\mathbb{Z}^*_{p^{e+1}} = \langle\theta\rangle$ for some $\theta \in \mathbb{Z}^*_{p^{e+1}}$. Let $a = (0, \theta)$, $b = (1, -1)$. Clearly $0^a = 0$, $0^b = 1$ and $1^b = 0$, moreover $\langle a\rangle$ cyclically permutes the elements of $\mathbb{Z}^*_{p^{e+1}}$. Noting that $\{0, 1\} \in \mathrm{E}$ and $\mathbb{Z}^*_{p^{e+1}}$ is the neighborhood of 0, we have $\langle a, b\rangle$ is an arc-transitive subgroup of $\mathrm{Aut}(K_{p[p^e]})$. Since the number of arcs of $K_{p[p^e]}$ is $p^{2e+1}(p-1)$, we have $|\langle a, b\rangle| \geq p^{2e+1}(p-1)$. Clearly $|G| = p^{2e+1}(p-1)$, and hence we have $G = \langle a, b\rangle$ is an arc-regular subgroup of $\mathrm{Aut}(K_{p[p^e]})$ with cyclic vertex stabilizer $G_0 = \langle a\rangle$.

Set $\mathcal{M}_\theta = \mathcal{M}(G; a, b)$. Then $\mathcal{M}_\theta$ is a regular embedding of $K_{p[p^e]}$. $\square$

Proposition 4.4. *The genus of $\mathcal{M}_\theta$ in Proposition 4.3 is*

$$g(\mathcal{M}_\theta) = \begin{cases} 1 + \dfrac{p^{e+1}(p^{e+1} - p^e - 4)}{4}, & p \equiv 1 \pmod 4; \\ 1 + \dfrac{p^{e+1}(p^{e+1} - p^e - 6)}{4}, & p \equiv 3 \pmod 4. \end{cases}$$

Proof. Since $ab = (0, \theta)(1, -1) = (1, -\theta)$ and the identity of G is $(0, 1)$, we get

$$(ab)^n = (1, -\theta)^n = (1 - \theta + \cdots + (-\theta)^{n-1}, (-\theta)^n) = \left(\frac{1 - (-\theta)^n}{1 + \theta}, (-\theta)^n\right).$$

Therefore, $(ab)^n = (0, 1) \Leftrightarrow (-\theta)^n = 1$, which implies that ab and $-\theta$ have the same order. Noting that

$$(-\theta)^{\frac{p^e(p-1)}{2}} = (-1)^{\frac{p^e(p-1)}{2}} \theta^{\frac{p^e(p-1)}{2}} = -(-1)^{\frac{p^e(p-1)}{2}},$$

the order of ab is $p^e(p-1)$ for $p \equiv 1 \pmod 4$; and $\frac{p^e(p-1)}{2}$ for $p \equiv 3 \pmod 4$. It follows that the number of faces of $\mathcal{M}$ is p^{e+1} for $p \equiv 1 \pmod 4$; and $2p^{e+1}$ for $p \equiv 3 \pmod 4$. Thus we get the desired formula for $g(\mathcal{M}_\theta)$. $\square$

5. Case $m = 3$

In this section we study the regular embeddings of $K_{3[n]}$. As before, set $\Gamma = K_{3[n]}$, with the vertex set $V(\Gamma) = \Delta_1 \bigcup \Delta_2 \bigcup \Delta_3$ where $\Delta_i = \{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}\}$ for $1 \leq i \leq 3$ and the edges are all pairs $\{\gamma_{ij}, \gamma_{kl}\}$ of vertices with $i \neq k$.

Suppose that $\mathcal{M}$ is a regular embedding of $K_{3[n]}$, where $n \geq 2$. Let $\mathrm{Aut}_0^+(\mathcal{M})$ be the kernel of $\mathrm{Aut}^+(\mathcal{M})$ on the set of three parts. As before, write $G = \mathrm{Aut}^+(\mathcal{M})$ and $H = \mathrm{Aut}_0^+(\mathcal{M})$. Let $G = \langle a, b\rangle$, where $\langle a\rangle = G_{\gamma_{11}}$ and b reverses the arc $(\gamma_{11}, \gamma_{21})$. Set $x = a^2$ and $y = x^b$. Then by Remark 4.2, $H = \langle x, y\rangle$ is an n-isobicyclic group. Moreover, we have the following theorem.

Theorem 5.1. *If $n = 3^e k$ with $e \geq 0$ and $3 \nmid k$, then $H = Q \times K$, where Q is a 3^e-isobicyclic group and K is an abelian k-isobicyclic group.*

Proof. We prove the theorem by the following two steps:

Step 1. Show that H is nilpotent.

Set $n = p_1 \cdots p_l$ where $p_1 \geq \cdots \geq p_l$ are the prime divisors of n. Let H_i and s_i be defined as in Lemma 3.1. Then $1 = H_0 < H_1 < \cdots < H_{l-1} < H_l = H$ is a series of characteristic subgroups of H and hence $H_i \trianglelefteq G$ for all $i \in [l]$. Consider the quotient graphs $(K_{3[n]})_{H_{i-1}}$ as well as the quotient maps $\mathcal{M}_{H_{i-1}}$ induced by the normal subgroups H_{i-1} for all $i \in [l]$. Then $(K_{3[n]})_{H_{i-1}} \cong K_{3[s_{i-1}]}$ (here we set $s_0 = n$) and $\mathrm{Aut}(\mathcal{M}_{H_{i-1}}) \cong G/H_{i-1}$. From Remark 4.2, we know

$$Z(G/H_{i-1}) = 1 \quad \text{and} \quad C_{G/H_{i-1}}(H_i/H_{i-1}) = C_{H/H_{i-1}}(H_i/H_{i-1}).$$

Then by Proposition 2.1, we have

$$\begin{aligned}(G/H_{i-1})/C_{H/H_{i-1}}(H_i/H_{i-1}) &= (G/H_{i-1})/C_{G/H_{i-1}}(H_i/H_{i-1}) \\ &\lesssim \mathrm{Aut}(H_i/H_{i-1}) \cong \mathrm{GL}(2, p_i).\end{aligned}$$

If $C_{H/H_{i-1}}(H_i/H_{i-1}) < H/H_{i-1}$, then by Lemma 3.2, $(H/H_{i-1})/C_{H/H_{i-1}}(H_i/H_{i-1})$ is a nontrivial isobicyclic group. Since $(G/H_{i-1})/(H_i/H_{i-1}) \cong G/H \cong S_3$, we have $(G/H_{i-1})/C_{H/H_{i-1}}(H_i/H_{i-1})$ is an extension of a nontrivial isobicyclic group by S_3, which contradicts to Lemma 3.3. Therefore, $C_{H/H_{i-1}}(H_i/H_{i-1}) = H/H_{i-1}$ and then we get a central series of H, namely the series

$$1 = H_0 < H_1 < \cdots < H_{l-1} < H_l = H.$$

It follows that H is a nilpotent group.

Step 2. Show that the Hall $3'$-subgroup of H is abelian.

Write $H = Q \times K$ where Q and K are the Sylow 3-subgroup and Hall $3'$-subgroup of H respectively. Suppose to the contrary that K is nonabelian. Then there exists a prime divisor p of n such that the Sylow p-subgroup P of H is nonabelian. Clearly, both P and P' are normal subgroups of G. Consider the quotient group $\overline{G} = G/P'$. Since H is a nilpotent group and $\overline{P}$ is an abelian Sylow p-subgroup of $\overline{H}$, we get $\overline{H} \le C_{\overline{G}}(\overline{P})$. Taking any element $c \in G \setminus H$, there exists $1 \le i \le 3$ such that $\Delta_i^c \ne \Delta_i$. Set $H_{i1} = \langle z\rangle$, we have (H, z, z^c) is a n-isobicyclic triple. Let $n = sp^d$ where $\gcd\{s, p\} = 1$. Then $P = \langle z^s, (z^s)^c\rangle$ is a p^d-isobicyclic group. By Lemma 3.3, $\overline{P}$ is an inhomogeneous abelian group generated by two elements. Hence we have $\overline{z}^s \ne (\overline{z}^s)^{\overline{c}}$, which implies that $\overline{c} \notin C_{\overline{G}}(\overline{P})$. It follows that $C_{\overline{G}}(\overline{P}) \le \overline{H}$ and hence $\overline{H} = C_{\overline{G}}(\overline{P})$. By Proposition 2.1, we have

$$S_3 \cong G/H \cong (\overline{G})/(\overline{H}) \lesssim \mathrm{Aut}(\overline{P}),$$

which contradicts to Lemma 2.4. $\square$

If H is abelian, then we have the following lemma.

Lemma 5.2. *Suppose that H is abelian. Then G has one of the following presentations*

$$G = G(n, k) = \langle a, b \mid a^{2n} = b^2 = 1, a^2 = x, x^b = y, [x, y] = 1,$$
$$y^a = x^{-1}y^{-1}, (ab)^3 = x^{\frac{kn}{3}} y^{-\frac{kn}{3}}\rangle,$$

where $k = 0$ if $3 \nmid n$ and $k = 0$ or 1 if $3 \mid n$, and $\mathcal{M}$ is isomorphic to one of the maps

$$\mathcal{M}(n, k, j) = \mathcal{M}(G(n, k); a^j, b),$$

where $(k, j) = (0, 1)$ for $3 \nmid n$ and $(k, j) = (0, 1), (1, 1)$ or $(1, -1)$ for $3 \mid n$. Moreover, $\mathcal{M}(n, k, j)$ has the type $\{3, 2n\}$ if $k = 0$ and $\{9, 2n\}$ if $k = 1$.

Proof. We prove the lemma by the following two steps:

Step 1. Determine the presentation of G.

Write $c = ab$. Since $H \trianglelefteq G$ and $c^3 \in H$, we can set

$$y^a = x^s y^t \quad \text{and} \quad c^3 = x^u y^v$$

where s, t, u and v are integers to be determined. Since

$$x^c = x^{ab} = x^b = y \quad \text{and} \quad y^c = y^{ab} = (x^s y^t)^b = x^t y^s,$$

we have

$$x^{c^3} = y^{c^2} = (x^t y^s)^c = y^t (x^t y^s)^s = x^{st} y^{t+s^2}.$$

On the other hand, $c^3 = x^u y^v$ implies that $x^{c^3} = x^{x^u y^v} = x$. Therefore,

$$st \equiv 1 \pmod{n} \quad \text{and} \quad t + s^2 \equiv 0 \pmod{n}.$$

Then from

$$x^t y^s = y^c = x^{c^2} = x^{c^{-1}x^u y^v} = x^{c^{-1}} = x^{ba^{-1}} = y^{a^{-1}} = y^{x^{-1}a} = y^a = x^s y^t,$$

we have $s \equiv t \equiv -1 \pmod{n}$ and hence $y^a = x^{-1}y^{-1}$.

Noting that $x^u y^v = (x^u y^v)^c = y^u (x^{-1}y^{-1})^v = x^{-v} y^{u-v}$, we get

$$u \equiv -v \pmod{n} \quad \text{and} \quad v \equiv u - v \pmod{n}.$$

Then $3u \equiv -3v \equiv 0 \pmod{n}$, that is, $u \equiv v \equiv 0 \pmod{n}$ if $3 \nmid n$ and $u \equiv -v \equiv \frac{kn}{3} \pmod{n}$ where $k = 0, 1, 2$ if $3 \mid n$.

Now we set

$$G(n,k) = \langle a, b \mid a^{2n} = b^2 = 1, a^2 = x, x^b = y, [x,y] = 1, y^a = x^{-1}y^{-1}, (ab)^3 = x^{\frac{kn}{3}} y^{-\frac{kn}{3}} \rangle,$$

where $k = 0$ if $3 \nmid n$ and $k = 0, 1, 2$ if $3 \mid n$. Then $G(n,k) \le G$. It is straightforward to check that $|G(n,k)| = 6n^2 = |G|$. Thus we have $G = G(n,k)$.

If $3 \mid n$ and $(ab)^3 = x^{\frac{2n}{3}} y^{-\frac{2n}{3}}$, then

$$(a^{-1}b)^3 = b(ab)^{-3}b = b(x^{\frac{2n}{3}} y^{-\frac{2n}{3}})^{-1} b = x^{\frac{2n}{3}} y^{-\frac{2n}{3}} = (x^{-1})^{\frac{n}{3}} (y^{-1})^{-\frac{n}{3}}.$$

It follows that

$$\begin{aligned} G(n,2) = \langle a^{-1}, b \mid & (a^{-1})^{2n} = b^2 = 1, (a^{-1})^2 = x^{-1}, (x^{-1})^b = y^{-1}, [x^{-1}, y^{-1}] = 1, \\ & (y^{-1})^a = (x^{-1})^{-1}(y^{-1})^{-1}, (a^{-1}b)^3 = (x^{-1})^{\frac{n}{3}} (y^{-1})^{-\frac{n}{3}} \rangle, \end{aligned}$$

from which we have $G(n,2) \cong G(n,1)$. Therefore, we get the desired presentation of G.

Step 2. Determine $\mathcal{M}$.

Recalling that $G_{\gamma_{11}} = \langle a \rangle$ and b reverses the arc $(\gamma_{11}, \gamma_{21})$, we know $\mathcal{M} = \mathcal{M}(G, a^j, b)$ for some $j \in [2n]$ with $\gcd\{j, 2n\} = 1$. Write $i = (j-1)/2$. Then $a^j = (a^2)^{(j-1)/2} a = x^i a$. It follows that

$$(y^j)^{a^j} = (y^{a^j})^j = (y^{x^i a})^j = (y^a)^j = (x^j)^{-1} (y^j)^{-1}$$

and

$$(a^j b)^3 = (x^i c)^3 = c(x^i)^c x^i c x^i c = c y^i x^i c x^i c = c^2 (y^i x^i)^c x^i c = c^2 x^{-i} y^{-i} y^i x^i c = c^3.$$

If $3|n$, then the equality $\gcd\{j, 2n\} = 1$ implies that $j \equiv 1$ or $5 \pmod{6}$. It follows that

$$n/3 \equiv jn/3 \pmod{n} \quad \text{or} \quad n/3 \equiv -jn/3 \pmod{n}.$$

Therefore we have

$$\begin{cases} (a^j b)^3 = 0, & k = 0; \\ (a^j b)^3 = (x^j)^{n/3} (y^j)^{-n/3}, & k = 1 \quad \text{and} \quad j \equiv 1 \pmod{6}; \\ (a^j b)^3 = (x^j)^{-n/3} (y^j)^{n/3}, & k = 1 \quad \text{and} \quad j \equiv 5 \pmod{6}. \end{cases}$$

Basing on the above paragraph, one may check that the following two arguments hold.

1. The mapping $a^j \mapsto a, b \mapsto b$ can be extended to an automorphism of G when $G = G(n,0)$ or $G = G(n,1)$ and $j \equiv 1 \pmod{6}$;
2. The mapping $a^j \mapsto a^{-1}, b \mapsto b$ can be extended to an automorphism of G when $G = G(n,1)$ and $j \equiv 5 \pmod{6}$.

Therefore $\mathcal{M}$ is isomorphic to one of the maps

$$\mathcal{M}(n,k,j) = \mathcal{M}(G(n,k); a^j, b),$$

where $(k,j) = (0,1)$ for $3 \nmid n$ and $(k,j) = (0,1), (1,1)$ or $(1,-1)$ for $3 \mid n$. Clearly, the type of $\mathcal{M}(n,k,j)$ is $\{3, 2n\}$ if $k = 0$ and $\{9, 2n\}$ if $k = 1$. Thus the map $\mathcal{M}(n,0,1)$ is different from $\mathcal{M}(n,1,1)$ and $\mathcal{M}(n,1,-1)$ up to map isomorphism. Now we prove that $\mathcal{M}(n,1,1)$ is not isomorphic to $\mathcal{M}(n,1,-1)$. Suppose to the contrary that $\mathcal{M}(n,1,1) \cong \mathcal{M}(n,1,-1)$. Then there exists $\phi \in \mathrm{Aut}(G(n,1))$ such that $\phi(a) = a^{-1}$ and $\phi(b) = b$ and hence

$$\phi(c^3) = [\phi(c)]^3 = [\phi(a)\phi(b)]^3 = (a^{-1}b)^3 = c^3 = x^{n/3} y^{-n/3}.$$

On the other hand, since

$$\phi(x) = \phi(a^2) = a^{-2} = x^{-1} \quad \text{and} \quad \phi(y) = \phi(x^b) = (x^{-1})^b = y^{-1},$$

we have

$$\phi(c^3) = \phi(x^{n/3}y^{-n/3}) = x^{-n/3}y^{n/3}.$$

Therefore, $x^{n/3}y^{-n/3} = x^{-n/3}y^{n/3}$. It follows that $n/3 \equiv -n/3 \pmod{n}$, a contradiction. Thus we have $\mathcal{M}(n, 1, 1)$ is not isomorphic to $\mathcal{M}(n, 1, -1)$. $\square$

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References

[1] N.L. Biggs, Classification of complete maps on orientable surfaces, Rend. Math. 4 (6) (1971) 132–138.
[2] N.L. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, Cambridge, 1993.
[3] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Discrete Groups, fourth ed., Springer, Berlin, 1984.
[4] L.E. Dickson, Linear Groups with an Exposition of the Galois Field Theory, Dover Publ., Leipzig, 1901,1958.
[5] J.D. Dixon, B. Mortimer, Permutation Groups, in: Graduate Texts in Mathematics, vol. 163, Springer, New York, 1996.
[6] S.F. Du, G.A. Jones, J.H. Kwak, R. Nedela, M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. I: metacyclic case, European J. Combin. 28 (2007) 1595–1609.
[7] S.F. Du, G.A. Jones, J.H. Kwak, R. Nedela, M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. II: the non-metacyclic case, European J. Combin. 31 (2010) 1946–1956.
[8] S.F. Du, J.H. Kwak, R. Nedela, Regular embeddings of complete multipartite graphs, European J. Combin. 26 (2005) 505–519.
[9] S.F. Du, J.H. Kwak, R. Nedela, Regular maps with pq vertices, J. Algebraic Combin. 19 (2004) 123–141.
[10] A. Gardiner, R. Nedela, J. Širán, M. Škoviera, Characterization of graphs which underlie regular maps on closed surfaces, J. Lond. Math. Soc. 59 (1999) 100–108.
[11] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin, 1967.
[12] L.D. James, G.A. Jones, Regular orientable imbeddings of complete graphs, J. Combin. Theory Ser. B 39 (1985) 353–367.
[13] G.A. Jones, Regular embeddings of complete bipartite graphs: classification and enumeration, Proc. Lond. Math. Soc. 101 (2010) 427–453.
[14] G.A. Jones, R. Nedela, M. Škoviera, Regular embeddings of $K_{n,n}$ where n is an odd prime power, European J. Combin. 28 (2007) 1863–1875.
[15] G.A. Jones, R. Nedela, M. Škoviera, Complete bipartite graphs with a unique regular embedding, J. Combin. Theory Ser. B 98 (2008) 241–248.
[16] J.H. Kwak, Y.S. Kwon, Regular orientable embeddings of complete bipartite graphs, J. Graph Theory 50 (2005) 105–122.
[17] J.H. Kwak, Y.S. Kwon, Classification of reflexible regular embeddings and self-Petrie dual regular embeddings of complete bipartite graphs, Discrete Math. 308 (2008) 2156–2166.
[18] J.H. Kwak, Y.S. Kwon, Classification of nonorientable regular embeddings of complete bipartite graphs, J. Combin. Theory Ser. B 101 (2011) 191–205.
[19] R. Nedela, M. Škoviera, Exponents of orientable maps, Proc. London Math. Soc. 75 (1997) 1–31.
[20] R. Nedela, M. Škoviera, A. Zlatoš, Regular embeddings of complete bipartite graphs, Discrete Math. 258 (2002) 379–381.
[21] D.J.S. Robinson, A course in the theory of group, second ed., in: Graduate Texts in Mathematics, vol. 80, Springer, New York, 2003.
[22] H. Wielandt, Uber das produkt von paarweise abelschen gruppen, Math. Z. 62 (1955) 1–7.
[23] S.E. Wilson, Cantankerous maps and rotary embeddings of K_n, J. Combin. Theory Ser. B 47 (1989) 262–273.