Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra

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Abstract

In the present paper we consider the Volterra integration operator $V$ on the Wiener algebra $W(D)$ of analytic functions on the unit disc $D$ of the complex plane $\mathbb{C}$. A complex number $\lambda$ is called an extended eigenvalue of $V$ if there exists a nonzero operator $A$ satisfying the equation $AV = \lambda VA$. We prove that the set of all extended eigenvalues of $V$ is precisely the set $\mathbb{C}\setminus\{0\}$, and describe in terms of Duhamel operators and composition operators the set of corresponding extended eigenvectors of $V$. The similar result for some weighted shift operator on $\ell_p$ spaces is also obtained.

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1. Introduction and background

Denote by $\mathcal{B}(X)$ the algebra of all bounded linear operators on a Banach space $X$. Let $C \in \mathcal{B}(X)$ be a fixed operator. It can be happen that there are nonzero operators $A, B \in \mathcal{B}(X)$ such that

$$AC = CB.$$  \hspace{1cm} (1)
If we denote by $\mathcal{E}_C$ the set of all $A$ for which there exists an operator $B$ satisfying (1), then it is easy to see that $\mathcal{E}_C$ is an algebra. Furthermore, one can define the map $\Phi_C : \mathcal{E}_C \to \mathcal{B}(X)$ by $\Phi_C(A) = B$. One can easily see that $\Phi_C$ is an algebra homomorphism, and it can be verified that (see [1]) it is in fact a closed (generally unbounded) linear transformation.

When $B = \lambda A$, for some complex number $\lambda$, Eq. (1) becomes

$$AC = \lambda CA.$$  \hspace{1cm} (2)

Clearly, a pair $(A, \lambda)$ in $\mathcal{B}(X) \setminus \{0\} \times \mathbb{C}$ satisfies (2) if and only if $\lambda$ is an eigenvalue for $\Phi_C$ and $A$ is an eigenvector for $\Phi_C$. Following[1], an eigenvalue of $\Phi_C$ will be referred to as an extended eigenvalue of $C$.

One knows that, when $\lambda = 1$, Eq. (2) can be used to obtain information about the operator $A$ based on the properties of the operator $C$. In particular, a famous result of Lomonosov [8] asserts that if $C$ is compact then $A$ must have a nontrivial hyperinvariant subspace, that is the whole commutant $\{C\}'$ of $C$ has a common nontrivial invariant subspace. Later, it was shown independently by Brown [2] and Kim et al. [9] that if $C$ is compact and $A$ satisfies (2), for any number $\lambda \in \mathbb{C}$, then $A$ has a nontrivial hyperinvariant subspace. This extension naturally leads to the question as to whether there is an algebra $\mathcal{A}$ that properly contains $\{C\}'$ and which, under specific conditions, has an invariant subspace. Such an algebra has been introduced by Lambert and Petrovic [7] and it was shown that it contains not only those operators that commute with $C$ but also operators that satisfy (2) for some $|\lambda| \leq 1$. (For the related results see also [10,5].) Furthermore, if $C$ is a compact operator, then this algebra has a nontrivial invariant subspace. Certainly, if $\mathcal{A} = \{C\}'$ this is just Lomonosov’s theorem. Therefore, it is of interest to find out whether the inclusion

$$\{C\}' \subset \mathcal{A}$$  \hspace{1cm} (3)

is proper. It was established by Lambert and Petrovic [7] that this happens when the spectral radius of $A$ is positive. Thus, it remains to consider the case in which $C$ is compact and quasinilpotent.

A first step in this direction was made by Biswas et al. [1] by showing that inclusion (3) is proper when $C$ is a specific compact, quasinilpotent operator (i.e., Volterra operator). More precisely, for $X = L^2(0, 1)$ and $C = V$, where $V$ is the Volterra integration operator on $L^2(0, 1)$, defined by

$$(Vf)(x) = \int_0^x f(t) \, dt.$$ 

It was shown in [1] that the set of all extended eigenvalues of $V$ is precisely the set $(0, \infty)$ and for each such extended eigenvalue $\lambda$, the corresponding eigenvector can be found in the class of integral operators. It is easy to show that not all such extended eigenvectors $A$ commute with $V$. Independently, Karaev [4] has obtained the same result in somewhat strengthened form. Unfortunately, this line of attack is not universally available. Namely, Shkarin [11] has shown that there are Volterra operators on a separable Hilbert space with no extended eigenvalues except $\lambda = 1$. 


In this article we consider the Volterra integration operator \( V \), \((Vf)(z) = \int_0^z f(t) \, dt\), on the Wiener algebra

\[
W(\mathbb{D}) := \left\{ f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in \operatorname{Hol}(\mathbb{D}) : \|f\| := \sum_{n=0}^{\infty} |\hat{f}(n)| < \infty \right\}
\]

over the unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), where \( \hat{f}(n) = f^{(n)}(0)/n! \) is the \( n \)th Taylor coefficient of \( f \). We prove that the set of extended eigenvalues of \( V \) is precisely the set \( \mathbb{C} \setminus \{0\} \), and describe in terms of Duhamel operators and composition operators the set of corresponding eigenvectors of \( V \).

Recall that for \( f, g \in \operatorname{Hol}(\mathbb{D}) \) their Duhamel product is defined by

\[
(f \circ g)(z) := \frac{d}{dz} \int_0^z f(z-t)g(t) \, dt = \int_0^z f'(z-t)g(t) \, dt + f(0)g(z),
\]

where the integrals are taken over the segment joining the points 0 and \( z \). It is easy to see that the Duhamel product satisfies all the axioms of multiplication, \( \operatorname{Hol}(\mathbb{D}) \) is an algebra with respect to \( \circ \) as well, and the function \( f(z) \equiv 1 \) is the unit element of the algebra \( (\operatorname{Hol}(\mathbb{D}), \circ) \) (see [12]). An operator \( \mathscr{D}_f, \mathscr{D}_f g := f \circ g \), is called as the Duhamel operator on \( W(\mathbb{D}) \).

2. Extended eigenvalues and extended eigenvectors of \( V \)

In this section we describe the sets of extended eigenvalues and extended eigenvectors of the Volterra integration operator \( V \) on the Wiener algebra \( W(\mathbb{D}) \).

Note that if \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in W(\mathbb{D}) \), then

\[
Vf(z) = \int_0^z f(t) \, dt = \hat{f}(0)z + \frac{\hat{f}(1)}{2} z^2 + \frac{\hat{f}(2)}{3} z^3 + \cdots.
\]

From this it is clear that if \( Vf(z) = 0 \), then

\[
\hat{f}(0) = \hat{f}(1) = \hat{f}(2) = \cdots = 0,
\]

that is \( f = 0 \), which shows that \( \ker V = \{0\} \). This shows that \( \lambda = 0 \) is not an extended eigenvalues of the operator \( V \). Therefore, the set of all extended eigenvalues of \( V \) lies in \( \mathbb{C} \setminus \{0\} \). The following our result shows that the set of extended eigenvalues of \( V \) is precisely the set \( \mathbb{C} \setminus \{0\} \). We also describe the corresponding extended eigenvectors of \( V \).

**Theorem 1.** Let \( \lambda \in \mathbb{C} \setminus \{0\} \), and let \( A \in \mathcal{B}(W(\mathbb{D})) \) be a nonzero operator.

(a) If \( |\lambda| \leq 1 \), then \( VA = \lambda AV \) if and only if an operator \( A \) has the form \( AC_\lambda = \mathscr{D}_A1 \), where \( \mathscr{D}_A1 \) is the Duhamel operator in \( W(\mathbb{D}) \) and \( (C_\lambda f)(z) = f(\lambda z) \) is a composition operator in \( W(\mathbb{D}) \).
(b) If $|\lambda| > 1$, then $VA = \lambda AV$ if and only if $A = D_{A1} C1_{\lambda}$; i.e.,

$$Af(z) = \frac{d}{dz} \int_{0}^{z} (A1)(z - t)f\left(\frac{t}{\lambda}\right) \, dt, \; f \in W(\mathbb{D}).$$

**Proof.**

(a) It is clear from (4) that

$$V^n f = \frac{z^n}{n!} \odot f, \; f \in W(\mathbb{D}), \quad (5)$$

$n = 0, 1, 2, \ldots$. Let $VA = \lambda AV$. Then

$$\lambda^n AV^n = V^n A$$

for each $n \geq 0$, that is

$$\lambda^n AV^n f = V^n Af$$

for all $f \in W(\mathbb{D})$, in particular,

$$\lambda^n AV^n 1 = V^n A1$$

for each $n \geq 0$. By considering (5), from this we have

$$A \left(\frac{(\lambda z)^n}{n!} \odot 1\right) = \left(A1 \odot \frac{z^n}{n!}\right)$$

or

$$\frac{1}{n!} A(\lambda z)^n = \frac{1}{n!} A1 \odot z^n,$$

which shows that

$$Ap(\lambda z) = A1 \odot p(z)$$

for all polynomials $p \in \mathcal{P}$. Since the set $\mathcal{P}$ is dense in $W(\mathbb{D})$ and $(W(\mathbb{D}), \odot)$ is a Banach algebra (see, for instance, [3,6]), from the last equality we obtain that

$$Af(\lambda z) = A1 \odot f(z)$$

for all $f \in W(\mathbb{D})$. Therefore, $AC_{\lambda} f = D_{A1} f$ for all $f \in W(\mathbb{D})$, and hence $AC_{\lambda} = D_{A1}$.

Conversely if $AC_{\lambda} = D_{A1}$, then we have for each polynomial $p \in \mathcal{P}$ that

$$VAp(z) = VAC_{\lambda} p\left(\frac{z}{\lambda}\right) = V D_{A1} p\left(\frac{z}{\lambda}\right) = D_{A1} Vp\left(\frac{z}{\lambda}\right) = AC_{\lambda} Vp\left(\frac{z}{\lambda}\right)$$

$$= AC_{\lambda} \left(z \odot p\left(\frac{z}{\lambda}\right)\right) = \lambda AC_{\lambda} \left(\frac{z}{\lambda} \odot p\left(\frac{z}{\lambda}\right)\right) = \lambda AC_{\lambda} (Vp)\left(\frac{z}{\lambda}\right) = \lambda AVp(z)$$

thus

$$VAp(z) = \lambda AVp(z)$$
for all polynomials $p$, and hence
\[ VAf = \lambda AVf \]
for all $f \in W(D)$. Therefore,
\[ VA = \lambda AV, \]
which proves (a).

(b) Suppose that $\lambda AV = VA$. Then,
\[ \frac{1}{\lambda} VA = AV \]
and hence
\[ \frac{1}{\lambda^n} V^n A = AV^n \tag{6} \]
for all $n \geq 0$. By the same arguments, using (6) we can prove that (see the proof of (a))
\[ Af(z) = A1 \otimes f \left( \frac{z}{\lambda} \right) \]
for all $f \in W(D)$, which implies that
\[ A = \mathcal{D}_{A1} C_{1/\lambda}, \]
that is
\[ Af(z) = \frac{d}{dz} \int_0^z (A1)(z - t)f \left( \frac{t}{\lambda} \right) dt \]
as desired.

On the other hand, let us now show that an operator $A$ of the form $A = \mathcal{D}_{A1} C_{1/\lambda}$ satisfies the equation
\[ \lambda AV = VA. \]
Indeed, for every $f \in W(D)$ we have that
\[
(AVf)(z) = (\mathcal{D}_{A1} C_{1/\lambda} Vf)(z) = \mathcal{D}_{A1}(Vf) \left( \frac{z}{\lambda} \right) \\
= A1 \otimes (Vf) \left( \frac{z}{\lambda} \right) = A1 \otimes \left( \frac{z}{\lambda} \otimes f \left( \frac{z}{\lambda} \right) \right) \\
= \frac{z}{\lambda} \otimes \left( A1 \otimes f \left( \frac{z}{\lambda} \right) \right) = \frac{z}{\lambda} \otimes \mathcal{D}_{A1} C_{1/\lambda} f(z) \\
= \frac{1}{\lambda} V \mathcal{D}_{A1} C_{1/\lambda} f(z) = \frac{1}{\lambda} VAf(z),
\]
which completes the proof of (b). Theorem 1 is proved. \[ \Box \]
Corollary 1. \( \{V\}' = \{\mathcal{D}_f : f \in W(\mathbb{D})\} \), i.e., the commutant of the Volterra integration operator \( V \in \mathcal{B}(W(\mathbb{D})) \) is the set of all Duhamel operators on \( W(\mathbb{D}) \).

Taking into account that the Duhamel product is commutative, via Corollary 1 we have that
\[
\{V\}'' = \{V\}',
\]
where \( \{V\}'' \) denotes the bicommutant of \( V \).

Recall that a composition operator \( C_\theta \), acting in the Wiener algebra \( W(\mathbb{D}) \) (which is a subalgebra of the disc algebra \( C_A(\mathbb{D}) \)), is defined as
\[
(C_\theta f)(z) = (f \circ \theta)(z) = f(\theta(z)),
\]
where \( \theta : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}} \) be a suitable analytic function. In the following corollary we are interested in determining whether a composition operator can be an extended eigenvector of \( V \). (Obviously \( C_\lambda = C_{\lambda z} \) and \( C_{1/\lambda} = C_{z/\lambda} \).)

Corollary 2. The composition operator \( C_\theta \) is the solution of the equation
\[
VA = \lambda AV,
\]
where \( |\lambda| \geq 1 \), if and only if \( \theta(z) = z/\lambda \).

Proof. Obviously \( C_\theta 1 = 1 \). Then according to assertion (b) of Theorem 1, we have that
\[
VC_\theta = \lambda C_\theta V
\]
if and only if
\[
C_\theta f(z) = \frac{d}{dz} \int_0^z f\left(\frac{t}{\lambda}\right) dt = f\left(\frac{z}{\lambda}\right) = C_{1/\lambda} f(z)
\]
for all \( f \in W(\mathbb{D}) \), which implies that \( C_\theta = C_{1/\lambda} \), that is \( \theta(z) = z/\lambda \). The proof of the corollary is completed. \( \square \)

It turns out that Corollary 2 describes the only situation in which a composition operator \( C_\theta \) can satisfy
\[
VC_\theta = \lambda C_\theta V.
\]
In fact, suppose that \( VC_\theta = \lambda C_\theta V \) for some \( \lambda \), \( 0 < |\lambda| < 1 \). Then, according to assertion (a) of Theorem 1, we have that
\[
C_\theta C_\lambda = \mathcal{D}_\theta 1 = \mathcal{D}_1 = I,
\]
where \( I \) is an identity operator in \( W(\mathbb{D}) \), which implies that \( C_\theta C_\lambda z = z \), that is \( C_\theta(\lambda z) = z \), or \( \lambda \theta(z) = z \), and hence \( \theta(z) = z/\lambda \). Therefore,
\[
|\theta(1)| = \left|\frac{1}{\lambda}\right| > 1,
\]
which contradicts \( \theta(1) \in \overline{\mathbb{D}} \).
In conclusion note that the method used above for the Volterra operator acting on the Wiener algebra applies also to other classes of operators, like some weighted shifts on $\ell_p$ spaces, for instance. Indeed, let us consider the weighted shift operator $T e_n = 1/(n+1)e_{n+1}$, $n \geq 0$, on the sequence space $\ell_p$ ($1 \leq p < \infty$), where $\{e_n\}$ is the standard basis of $\ell_p$. For the arbitrarily chosen elements $x = \sum_{n=0}^{\infty} x_n e_n$ and $y = \sum_{n=0}^{\infty} y_n e_n$ of the space $\ell_p$, let us define the so-called Duhamel product (see, [12,4]) by the following formula:

$$x \otimes y := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{n!m!}{(n+m)!} y_n x_m e_{n+m}. \tag{7}$$

It is easy to see that formula (7) is correctly defined.

It is also easy to verify that the Duhamel product (7) satisfies all the axioms of multiplication, $\ell_p$ is the commutative algebra with respect to $\otimes$, an element $e_0 = (1, 0, \ldots)$ is the unit element of the algebra $(\ell_p, \otimes)$, and $T x = e_1 \otimes x$ for every $x \in \ell_p$, where $e_1 = (0, 1, 0, \ldots)$. An operator $D_y$, $D_y x := y \otimes x$, is called the Duhamel operator on $\ell_p$. The diagonal operator on $\ell_p$ with diagonal elements $a_n \in \mathbb{C}$, $n \geq 0$, is denoted by $D_y$, $D_y e_n = a_n e_n$.

Now, by the same method, as in the proof of Theorem 1, it can be proved (the proof is omitted) the following theorem which shows that the set of extended eigenvalues of $T$ is the set $\mathbb{C} \setminus \{0\}$.

**Theorem 2.** Let $\lambda \in \mathbb{C} \setminus \{0\}$ and let $X \in \mathcal{B}(\ell_p)$ be a nonzero operator. Then we have

(i) if $\lambda \in \mathbb{D}$, then $T X = \lambda X T$ if and only if an operator $X$ satisfies $X D_{\lambda} = D_{X e_0}$, where $D_{\lambda}$, $D_{\lambda} e_n = \lambda^n e_n$, $n \geq 0$, is the diagonal operator on $\ell_p$ and $D_{X e_0}$ is the Duhamel operator in $\ell_p$;

(ii) if $\lambda \notin \mathbb{D}$, then $T X = \lambda X T$ if and only if $X = D_{X e_0} D_{1/\lambda}$.

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