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Families of four-dimensional manifolds that become mutually diffeomorphic after one stabilization ☆

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Abstract

In this paper, we will introduce a cut and paste move, called a geometrically null log transform, and prove that any two manifolds related by a sequence of these moves become diffeomorphic after one stabilization. To motivate the cut and paste move, we will use the symplectic fiber sum, and a construction of Fintushel and Stern to construct several large families of 4-manifolds. We will then proceed to prove that the members of any one of these families become diffeomorphic after one stabilization. Finally, we will compute the Seiberg–Witten invariants of each member of each of the families.

MSC: 57R57

Keywords: Stabilization of 4-manifolds; Seiberg–Witten invariants

It is well known that two homotopy equivalent, simply connected 4-manifolds become diffeomorphic after taking the connected sum with enough copies of $S^2 \tilde{\times} S^2$ [21]. The same result is true with $S^2 \tilde{\times} S^2$ replaced by $S^2 \times S^2$, and similar results are known for special families of 4-manifolds when $S^2 \tilde{\times} S^2$ is replaced by other manifolds. Taking the connected sum with one of these specific manifolds is called stabilization. For this paper, we will only consider connected sums with $S^2 \tilde{\times} S^2$, and stabilization will refer to taking the connected sum with this specific manifold. Most of the arguments in this paper can be easily modified to address other summands as well. Many families of distinct homotopy equivalent simply connected 4-manifolds that become mutually diffeomorphic after one stabilization are known [15]. There is, in fact, no known pair of homotopy equivalent simply connected 4-manifolds which are not diffeomorphic after one stabilization.

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In this paper, we will introduce a cut and paste move, called a geometrically null log transform, and prove that any two manifolds related by a sequence of these moves become diffeomorphic after one stabilization. To motivate the cut and paste move, we will use the symplectic fiber sum, and a construction of Fintushel and Stern to construct several large families of 4-manifolds. We will then proceed to prove that the members of any one of these families become diffeomorphic after one stabilization. Finally, we will compute the Seiberg–Witten invariants of each member of each of the families.

Even though the Donaldson and Seiberg–Witten invariants can distinguish some homotopy equivalent four-manifolds, these invariants cannot directly distinguish manifolds of the form $X \# S^2 \tilde{\times} S^2$. This is because both invariants are trivial on 4-manifolds with an $S^2 \tilde{\times} S^2$ summand, provided that the second positive Betti number of the remaining summand is positive [19]. A priori, it is possible that $X \# S^2 \tilde{\times} S^2 \cong Y \# S^2 \tilde{\times} S^2$ implies some relation between the Seiberg–Witten invariants of X and the Seiberg–Witten invariants of Y. The first reason for considering a specific set of families in this paper is to show that no simple relation between Seiberg–Witten invariants is implied by equivalence after one stabilization.

If it was known that any pair of homotopy equivalent simply connected 4-manifolds are related by a sequence of geometrically null log transforms, it would follow that any two such manifolds become equivalent after one stabilization. It is known that any manifold homotopy equivalent to a simply connected 4-manifold may be constructed by removing a contractible 4-manifold and reglueing it via an involution [3,14]. This motivates the question: Is it possible to modify the proof of the decomposition theorem to find a finite set of moves which could be used to pass between any two homotopy equivalent 4-manifolds? The contractible piece is known as a cork. A second reason for constructing specific families is to study the effect that applying a geometrically null log transform to one manifold of a pair of homotopy equivalent simply connected 4-manifolds has on the cork.

1. Families of 4-manifolds

All of the 4-manifolds explicitly considered in this paper are formed by applying a cut and paste operation, the fiber sum, to copies of a standard building block, called the K3 surface. This section begins with a short description of the K3 surface. (See the book by Harer, Kas, and Kirby for more information about the K3 surface [9].) This section will end with explicit handle decompositions of the 4-manifolds contained in the specific families considered in this paper. S. Akbulut gave handle decompositions for the result of fiber summing a nucleus with $S^1 \times S^3$ among a S^1 times a knot [2].

Recall that the K3 surface is essentially the quotient of a 4-torus by an involution. The group, $\mathbb{Z}_2$ acts on T^4 via the map:

$$\varepsilon : T^4 = \frac{\mathbb{C}^2}{\mathbb{Z}[i]^2} \to T^4, \quad \varepsilon\big([x, y]\big) = [-x, -y].$$

It also acts on $\mathbb{C}P^2$ via

$$\eta : \mathbb{C}P^2 \to \mathbb{C}P^2, \quad \eta\big([x : y : z]\big) = [-x : -y : z].$$

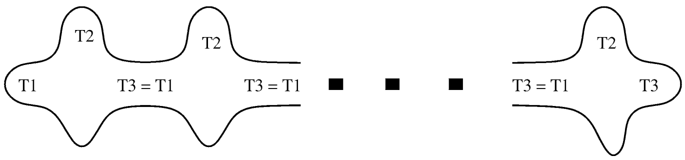

Fig. 1. X_N.

There are 16 fixed points on T^4, namely, $(1/2\mathbb{Z}[i])^2/(\mathbb{Z}[i])^2$, and the fixed point set in $\mathbb{C}P^2$ is $\{[0:0:1]\} \cup \{[x:y:0]\}$. We may cut invariant neighborhoods of the 16 fixed points out of T^4 and glue in 16 copies of the complement of an invariant neighborhood of $\{[0:0:1]\} \subseteq \mathbb{C}P^2$, to get a $\mathbb{Z}_2$ action on $T^4 \# \overline{\mathbb{C}P^2}^{\#16}$. The bar refers to the fact that $\mathbb{C}P^2$ is taken with the opposite orientation. The quotient of $T^4 \# \overline{\mathbb{C}P^2}^{\#16}$ by $\mathbb{Z}_2$ is the $K3$ surface. It is manifold essentially because the quotient of the disk,

$$D_{[x_0:y_0]} = \bigl\{[x:y:z] | [x:y] = [x_0:y_0] \ \& \ |z|^2 \leqslant |x|^2 + |y|^2\bigr\},$$

in $\mathbb{C}P^2$ is also a disk.

All of the examples that we construct will be obtained by cut and paste along three tori in the $K3$ surface. Let

$$T_1 = \bigl\{\bigl[(x, 1/3 + 1/3i)\bigr] \in K3 \mid x \in \mathbb{C}\bigr\},$$
$$T_2 = \bigl\{\bigl[(x, y)\bigr] \in K3 \mid \operatorname{Im} x = \operatorname{Im} y = 1/4\bigr\} \quad \text{and}$$
$$T_3 = \bigl\{\bigl[(x, y)\bigr] \in K3 \mid \operatorname{Im} x = \operatorname{Re} y = 1/5\bigr\}.$$

Let X_N be the manifold obtained by fiber summing N copies of the $K3$ surface together along T_3 in one copy and T_1 in the next copy. (See Fig. 1.)
The fiber sum of (X, S) and (Y, T) is $(X - \mathring{N} N(S)) \cup_{\partial N(T) = \partial N(S)} (Y - \mathring{N}(T))$. It will be denoted by $(X, S) \# (Y, T)$. If S and T are symplectic submanifolds with opposite self-intersection numbers, the fiber sum will also be symplectic [9]. The definition of the fiber sum requires an orientation reversing glueing map from the boundary of a tubular neighborhood of S to the boundary of a tubular neighborhood of T. Every thing that we will assert about the manifolds, X_N will be independent of the glueing maps. To be definite one could choose $\varphi : \partial N(S) \to \partial N(T)$ given by, $\varphi(x_1 + 1/5i + 10^{-2} i \cos(\theta), 1/5 + 10^{-2} \sin(\theta) + x_2 i) = (x_1 + x_2 i, 1/3 + 10^{-2} \cos(\theta) + 1/3i - 10^{-2} i \sin(\theta))$. The manifold X_N has $N + 2$ of tori the T_i remaining. Copies of $S^1 \times S^3$ may be fiber summed onto these remaining tori, each along S^1 cross a knot (take the glueing map which identifies the 0-framed longitude of the knot with a meridian or the torus). Fintushel and Stern proved a remarkable formula relating the Alexander polynomial of a knot to the change in the Seiberg–Witten invariant of a manifold after fiber summing with $S^1 \times S^3$ along S^1 cross the knot. [6]. This formula will be used to compute the Seiberg–Witten invariants at the conclusion of this paper. All of the manifolds obtained from a fixed X_N, by fiber summing with $S^3 \times S^1$ as above are homotopy equivalent. We will show that all members of the family of manifolds obtained from a fixed X_N become diffeomorphic after one stabilization. The last section of the paper describes the Seiberg–Witten invariants of these manifolds.

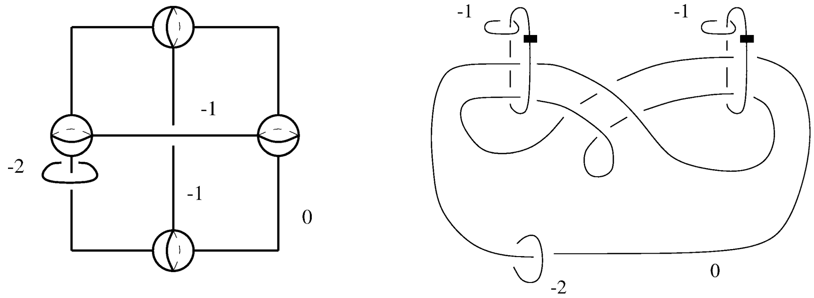

Fig. 2. N_2.

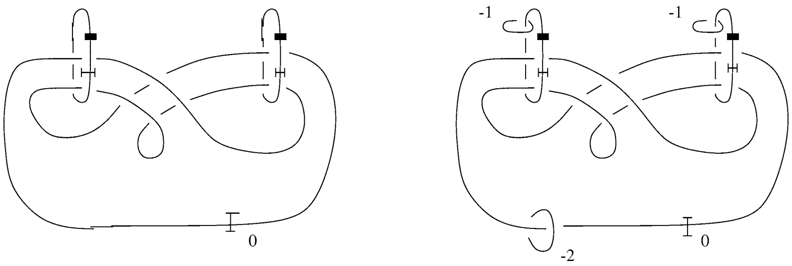

Fig. 3. $N(\partial(T^2 \times D^2))$, $N_2 - \mathring{N}(T^2)$.

A well-known handle decomposition of the K3 surface is given in the book by Harer et al. [9]. This handle decomposition has 24 handles, the minimal number of handles in a handle decomposition of the K3 surface. Other 4-manifolds will require even more handles. Because of this complexity, it is useful to decompose 4-manifolds into a union of compact pieces and then describe handle decompositions of the pieces. One important piece of the K3 surface is the Gompf Nucleus. By definition, this is a neighborhood of the union of a cusp fiber and a section [6]. The nucleus of K3 will be denoted by N_2. It may be constructed by attaching three two-handles to $T^2 \times D^2$ (see Fig. 2).

There are three disjoint copies of the nucleus in the K3 surface. Each one contains one of the, T_i, tori described above as $T^2 \times \{0\}$ in Fig. 2. Given a handle decomposition of a 4-manifold with boundary, it will be useful to denote a collar of the boundary by putting an I on each handle. For example, Fig. 3 displays handle decompositions of $N(\partial(T^2 \times D^2))$ and $N_2 - \mathring{N}(T^2)$.

To construct a handle decomposition of the fiber sum of a pair of nuclei, we will turn a copy of $N_2 - \mathring{N}(T^2)$ upside down and glue it to a second copy of $N_2 - \mathring{N}(T^2)$. To turn a handle decomposition upside down, first reverse the orientation (reverse every crossing and framing), then double. Assuming that the original manifold has no 3-handles, attach one 0-framed 2-handle to the co-core of each original 2-handle, then delete the original manifold (add I's to all of the original components). Fig. 4 displays $N_2 - \mathring{N}(T^2)$ turned upside down and a fiber sum of a pair of nuclei constructed by glueing the $N_2 - \mathring{N}(T^2)$ from Fig. 3 to the $N_2 - \mathring{N}(T^2)$ from Fig. 4.

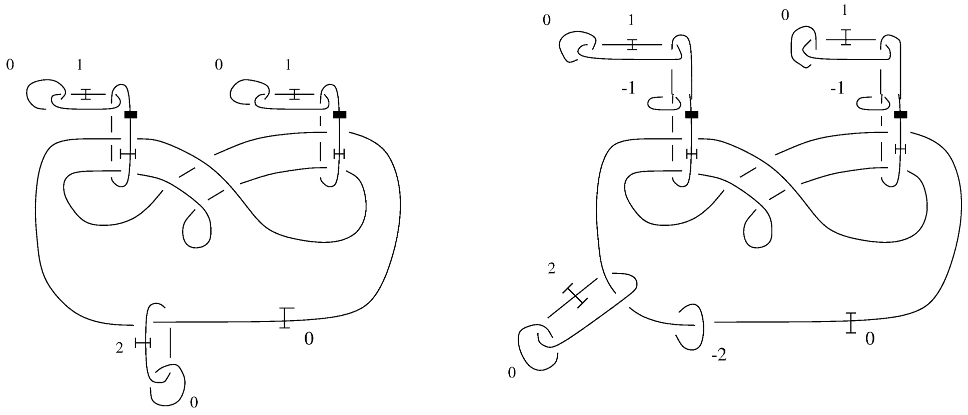

Fig. 4. $N_2 - \mathring{N}(T^2)$, $(N_2, T^2)\,\#\,(N_2, T^2)$.

Turn now to the construction of handle decompositions for manifolds of the form $M^3 \times S^1$. Restrict the boundary of M to be a disjoint union of tori. If M is described by surgery to the complement of a link is S^3, there will be two approaches for constructing handle decompositions for $M^3 \times S^1$. Both methods begin by constructing a handle decomposition for $M^3 \times I$. The first method is to pick a tunnel system for the link L when M is obtained by Dehn filling on L. This tunnel system may be used to construct a handle decomposition of $S^3 - \mathring{N}(L)$. This is easily translated into a handle decomposition of M^3 and then $M^3 \times I$ (see [19, p. 250] for this process applied to the Poincaré homology sphere).

The second approach is based on the observation that proves that $K \# - K$ is slice for any knot, K [1,3]. Namely, $(K - \mathring{N}(pt)) \times I$ is a slice disk for $K \# - K$. For any link, L, $I \times (S^3 - L)$ may be described as the exterior of a surface, F, in D^4. The surface, F, is constructed in the same way as the slice disk for $K \# - K$. If M^3 is surgery on L, a handle decomposition of $I \times (S^3 - L)$ may easily be converted into a decomposition of $M^3 \times I$. To begin the description of $D^4 - \mathring{N}(F)$, notice that

$$I \times S^3 - \mathring{N}(I \times L) = I \times S^3 - \mathring{N}(I \times pt) - \mathring{N}\big(I \times \big(L - \mathring{N}(pt)\big)\big) = D^4 - \mathring{N}(F).$$

Fig. 5 shows a typical link and the frames of the movie obtained by intersecting $D^3 \times \{t\}$ with the canonical cobordism in $D^3 \times I = D^4$.
Note that

$$\begin{aligned}
&\big(D^4 - \mathring{N}(F)\big) \cap \big(D^3 \times [0.6, 1]\big) \\
&\quad = D^4 - \mathring{N}\big(\big(D^2\big)^{\amalg k}\big) = \big(D^2 - \mathring{N}(k\,pts)\big) \times D^2 \\
&\quad = \Big(D^2 \underset{k(\partial D^2) \times D^1}{\cup} kD^1 \times D^1\Big) \times D^2 \\
&\quad = D^4 \underset{k(\partial D^1) \times D^3}{\cup} kD^1 \times D^3.
\end{aligned}$$

This will allow us to describe a handle decomposition for $D^4 - \mathring{N}(F)$ in terms of a handle decomposition for F. So far we see that 0-handles in F correspond to 1-handles

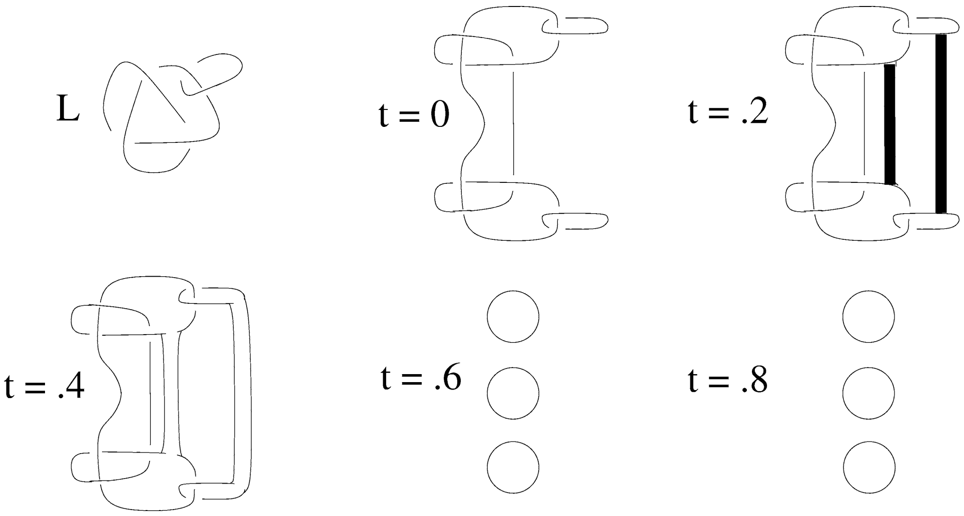

Fig. 5. L and F.

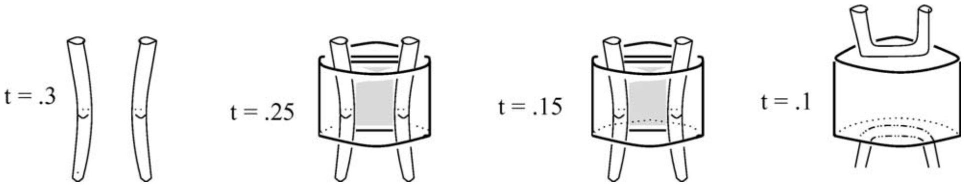

Fig. 6. Neighborhood of a 1-handle.

in $D^4 - \mathring{N}(F)$ (this is the previous computation). Fig. 6 displays a neighborhood of a 1-handle in F embedded in D^4. The cylinder around the band is a 2-handle in $D^4 - \mathring{N}(F)$. This illustrates the fact that 1-handles in F correspond to 2-handles in $D^4 - \mathring{N}(F)$. It also enables one to construct handle decompositions for $(S^3 - \mathring{N}(L)) \times I$. Dehn filling is accomplished by attaching a 2-handle and then attaching a 3-handle. This will complete a handle decomposition of $M^3 \times I$. The special cases when M is D^3, or $S^1 \times D^2$, or $S^2 \times D^1$ are instructive, when extending a handle decomposition of $M^3 \times I$ to a handle decomposition of $M^3 \times S^1$. In general, a $(k+1)$-handle is added for every k-handle of $M^3 \times I$.

We can apply these ideas to $M = S^3 - \mathring{N}(K)$. Let the knot K be expressed as the closure of a braid, β, in such a way that the black board framing of K is the zero framing. The result is the handle decomposition for $(S^3 - \mathring{N}(K)) \times S^1$ displayed in Fig. 7. Fig. 7 also has a handle decomposition of $(N_2, T) \# (S^3 \times S^1, K \times S^1)$ obtained from $(S^3 - \mathring{N}(K)) \times S^1$ by gluing on an $N_2 - \mathring{N}(T)$.

There are many different surgery descriptions of any given 3-manifold (see Fig. 8). Any of these descriptions will produce a handle decomposition of $M^3 \times S^1$. It is an interesting exercise to see how various 3-manifold moves translate into sequences of handle slides

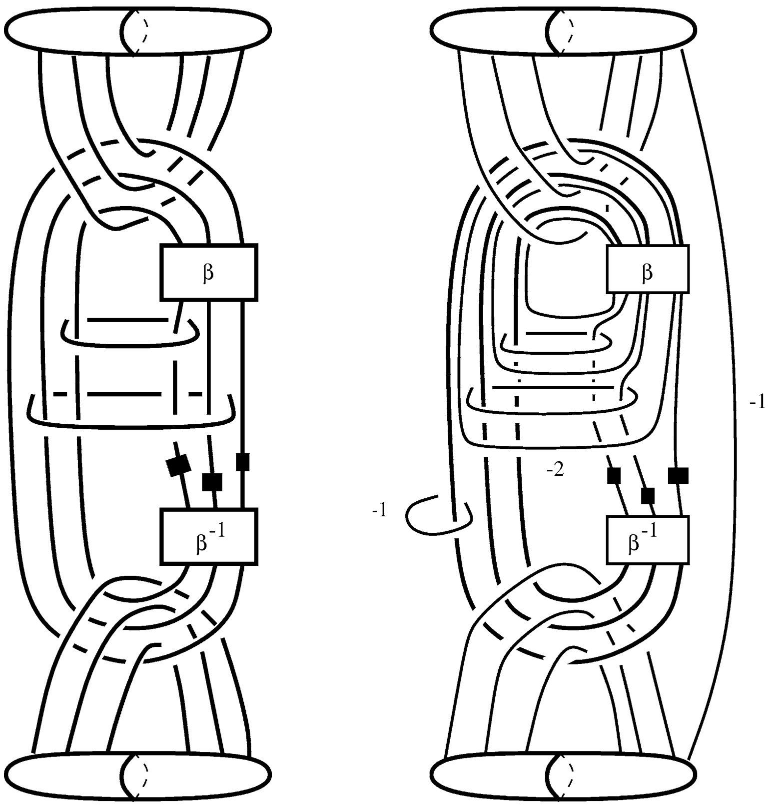

Fig. 7. $(S^3 - \mathring{N}(K)) \times S^1$ and $(N_2, T)\,\#\,(S^3 \times S^1, K \times S^1)$.

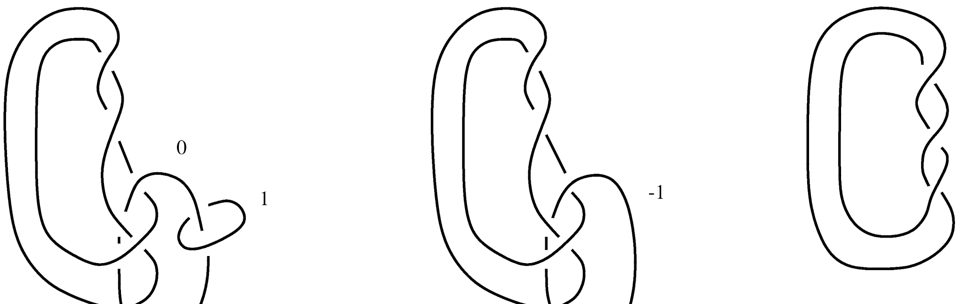

Fig. 8. Different descriptions of the same manifold.

and handle pair birth/deaths. In particular, it is interesting to see how Markov moves on the braid, handle slides, and Kirby moves effect the 4-dimensional handle decomposition.

Notice that any knot can be converted to the unknot by a sequence of ± 1 surgeries. This will enable us to understand the fiber sum with $S^3 \times S^1$ along a complicated knot crossed with the circle using one simple move. We will come back to this later in this paper.

2. Stabilization

For this paper, stabilizing a 4-manifold will simply refer to taking the connected sum with $S^2 \tilde{\times} S^2$. The manifold, $S^2 \tilde{\times} S^2$ is the nontrivial S^2 bundle over S^2. It may also be described as $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Stabilization is closely related to the surgery corresponding to the addition of a five-dimensional 2-handle. This surgery amounts to replacing an $S^2 \times D^3$ by a $D^2 \times S^2$ in the 4-manifold. If $S^1 \times \{0\}$ is homotopically trivial, we may assume that it is contained in a 4-disk. Since surgery on a trivial loop in the 4-disk either produces a punctured $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, it follows that surgery on a null homotopic loop is the same as taking the connected sum with either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ (see Fig. 9).

Combining this with the observations that $(S^2 \tilde{\times} S^2) \# (S^2 \times S^2) \cong (S^2 \tilde{\times} S^2) \# (S^2 \tilde{\times} S^2)$, and that any five-dimensional h-cobordism may be constructed with just 2-handles and 3-handles proves that two homotopy equivalent, simply connected 4-manifolds become diffeomorphic after some number of stabilizations [10,20]. See also [4].

Computing the number of stabilizations required is an interesting open problem. For every known example, one stabilization is enough. The main argument used to prove that one stabilization is enough is a five-dimensional handle argument due to Mandelbaum [11–13]. In fact, many manifolds are known to become diffeomorphic to $(\mathbb{C}P^2)^{\# n} \# (\overline{\mathbb{C}P^2})^{\# m}$ after taking the connected sum with just $\mathbb{C}P^2$ [15]. Many related facts may be found in [7]. If S and T are tori in X and Y, the basic five-dimensional argument analyzes a natural cobordism between $X \amalg Y$ and $(X, S) \# (Y, T)$. Let S have a standard handle decomposition, $S = h^{(0)} \cup h_1^{(1)} \cup h_2^{(1)} \cup h^{(2)}$. The natural cobordism is then

$$\begin{aligned} W = {} & (I \times (X \amalg Y) \cup D^1 \times h^{(0)} \times D^2 \\ & \cup D^1 \times h_1^{(1)} \times D^2 \cup D^1 \times h_2^{(1)} \times D^2 \cup D^1 \times h^{(2)} \times D^2. \end{aligned}$$

The level of W after the 1-handle, $D^1 \times h^{(0)} \times D^2$, is $X \# Y$. The level after the 2-handles, $D^1 \times h_1^{(1)} \times D^2$ and $D^1 \times h_2^{(1)} \times D^2$, is $X \# Y \# (S^2 \times S^2) \times (S^2 \times S^2)$. The section of the cobordism from this level to the end is obtained by attaching a 3-handle. By turning this section upside down, we see that it is also obtained by attaching a five-dimensional 2-handle to $(X, S) \# (Y, T)$. The level is therefore $(X, S) \# (Y, T) \# (S^2 \times S^2)$.

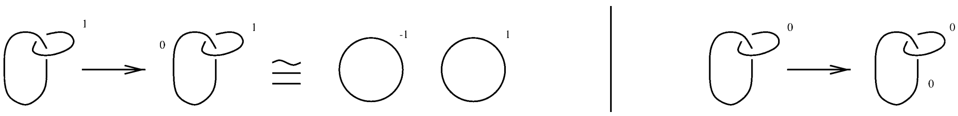

Fig. 9. Surgery and stabilization.

Thus $X \# Y \# (S^2 \times S^2) \# (S^2 \times S^2) \cong (X, S) \# (Y, T) \# (S^2 \times S^2)$. In the above argument, we assumed that X and Y were simply connected, and that the framings on all of the five-dimensional 2-handles are arranged so that factors of $S^2 \times S^2$ appear, not factors of $S^2 \widetilde{\times} S^2$.

Instead of checking the framings directly, we will use the five-dimensional argument as a guide for a four-dimensional handle sliding argument that $X_N \# (S^2 \widetilde{\times} S^2) \cong (\mathbb{C}P^2)^{\#4N} \# (\overline{\mathbb{C}P^2})^{20N}$.

The E_8 Milnor fiber is embedded in K3 disjoint from the nucleus [10]. It follows that E_8 is also embedded in X_N disjoint from all of the tori used in the fiber sum. The argument begins by showing that $E_8 \# (S^2 \widetilde{\times} S^2) \cong W_1 \# (\overline{\mathbb{C}P^2})^{\#7} \# (S^2 \times S^2)$ (see Fig. 10). Sliding the factor of $S^2 \times S^2$ into the $(N_2, T^2) \# (N_2, T^2)$ from Fig. 4, and performing the moves indicated in Fig. 11 produces Fig. 12. The handle slides in Fig. 11 correspond to the last section of the cobordism in the five-dimensional argument. Sliding the complicated zero framed 2-handle over the 2-handle dual to the complicated $0I$ handle will allow the complicated zero framed 2-handle to be pushed to the right of the figure as in Fig. 13.

The next step is to add two canceling 1-handle/2-handle pairs to produce the 1-handles in the right-side-up N_2. This is done in Fig. 14, resulting in the handle decomposition in Fig. 15. The handle slides in Fig. 16 will make the right side look exactly like a right-side-up nucleus. Now, introduce two canceling 2-handle/3-handle pairs. Slide one of the new 2-handles over the simple $0I$ component, then use the 2-handles dual to the $1I$ and complicated $0I$ components to arrange the new 2-handle as in Fig. 17. Repeat with the second new 2-handle.

Adding the 2-handles in the five-dimensional cobordism corresponds to the handle slides in Fig. 18. The handle slides in this figure show that $(N_2, T^2) \# (N_2, T^2) \# (S^2 \times S^2) \cong N_2 \# N_2 \# (S^2 \times S^2) \# (S^2 \times S^2)$. This argument may be repeated on each $(N_2, T^2) \# (N_2, T^2)$. This will show that

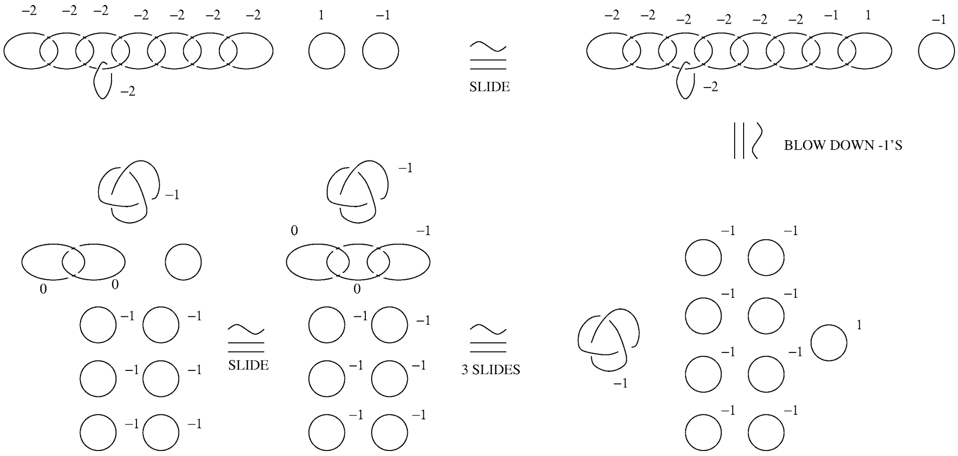

Fig. 10. $E_8 \# (S^2 \widetilde{\times} S^2) \cong W_1 (\overline{\mathbb{C}P^2})^{\#7} \# (S^2 \times S^2)$.

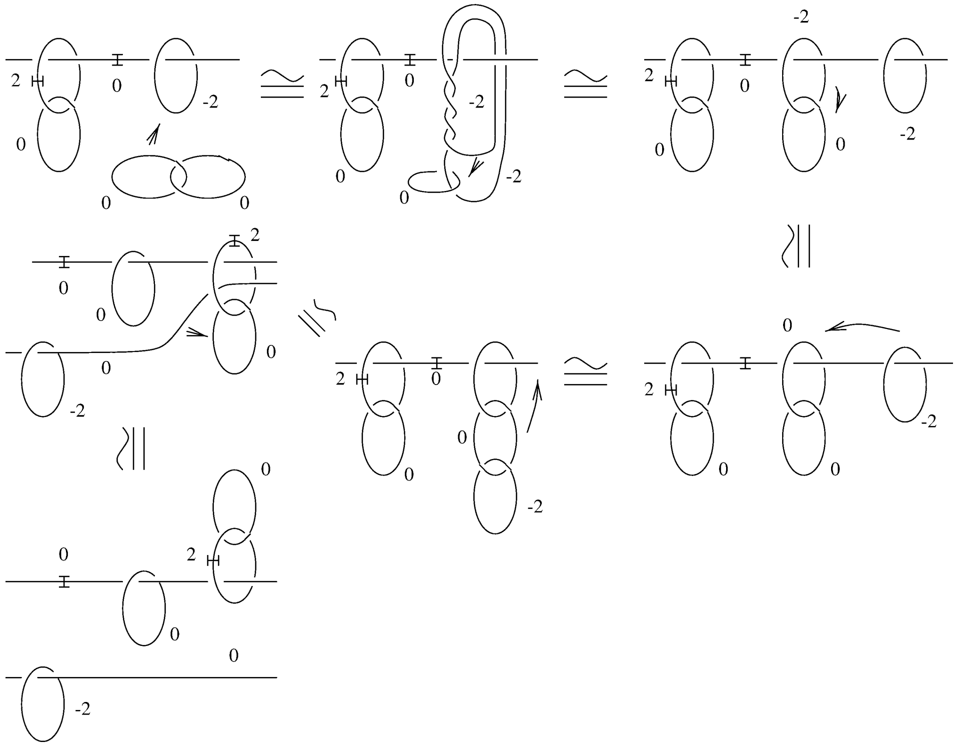

Fig. 11. Handle slides for the five dimensional 3-handle.

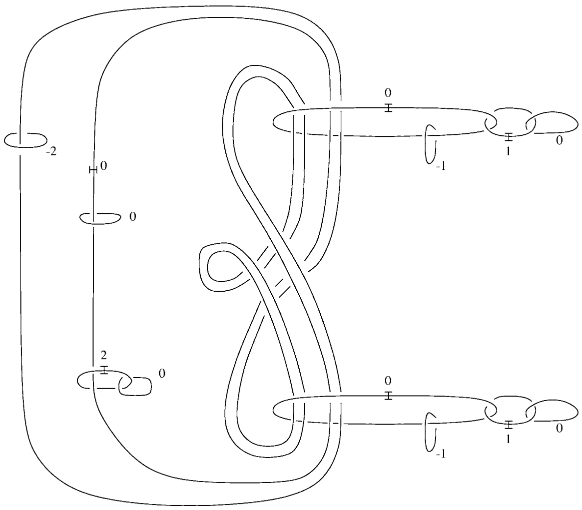

Fig. 12. $(N_2, T^2) \# (N_2, T^2) \# (S^2 \times S^2)$.

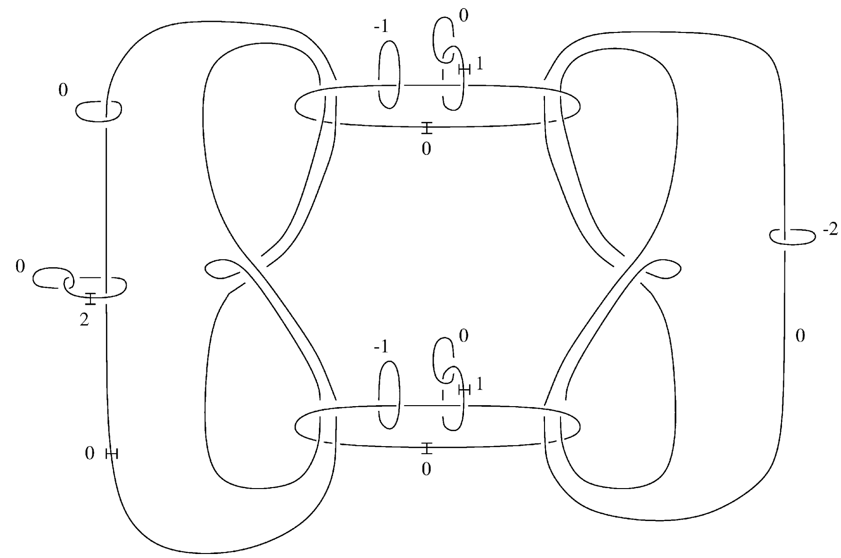

Fig. 13. $(N_2, T^2) \# (N_2, T^2) \# (S^2 \times S^2)$.

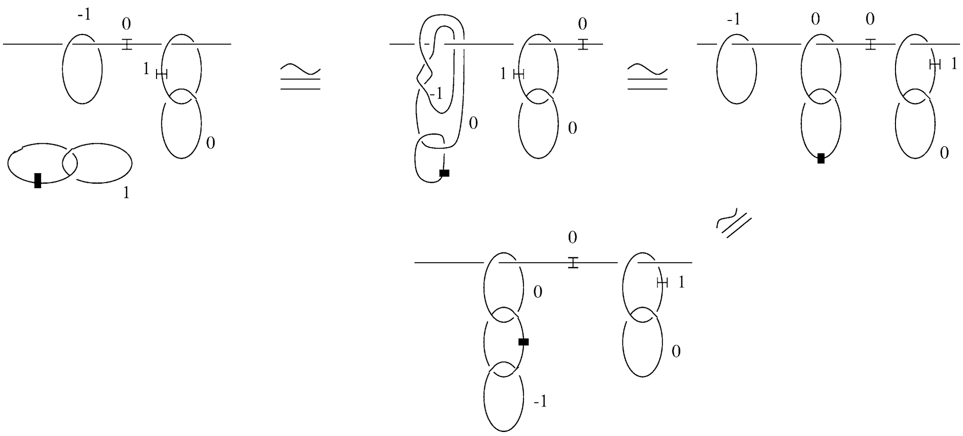

Fig. 14. Introducing 1-handles.

$$\begin{aligned}
X_N \# \big(S^2 \widetilde{\times} S^2\big) &\cong (K3)^{\#N-1} \# \big(S^2 \times S^2\big)^{\#N} \# \big(\overline{\mathbb{C}P^2}\big)^{\#7} \# W_1 \cup M \cup E_8 \\
&\cong (K3)^{\#N-1} \# \big(S^2 \times S^2\big)^{N-1} \# \big(S^2 \widetilde{\times} S^2\big) \# \big(\overline{\mathbb{C}P^2}\big)^{\#7} \# W_1 \cup M \cup E_8 \\
&\cong \big(\overline{\mathbb{C}P^2}\big)^{14N} \# \big(S^2 \times S^2\big)^{N-1} \# \big(S^2 \widetilde{\times} S^2\big) \# \big(W_1 \cup M \cup W_1\big)^{\#N} \\
&\cong \big(\mathbb{C}P^2\big)^{\#4N} \# \big(\overline{\mathbb{C}P^2}\big)^{\#20N}.
\end{aligned}$$

In the above argument, M is the complement of two E_8 manifolds in K3. Fig. 19 displays handle decompositions of M and $W_1 \# M \# W_1$.

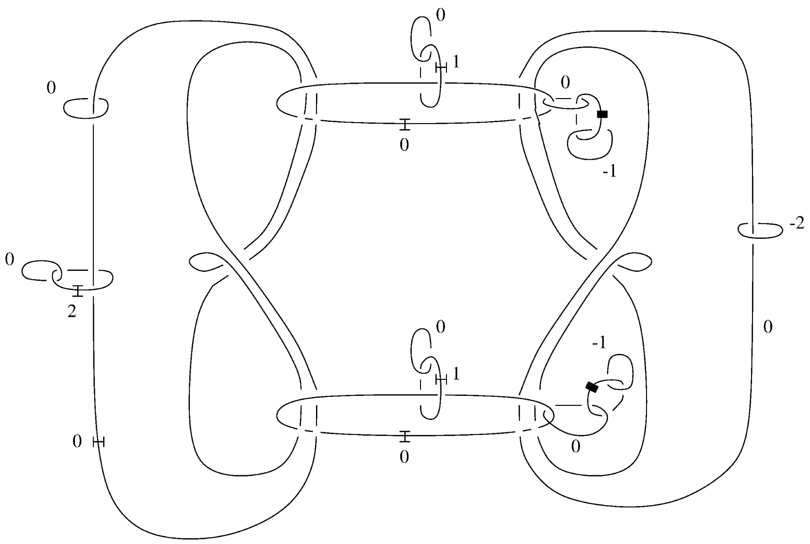

Fig. 15. $(N_2, T^2)\,\#\,(N_2, T^2)\,\#\,(S^2 \times S^2)$.

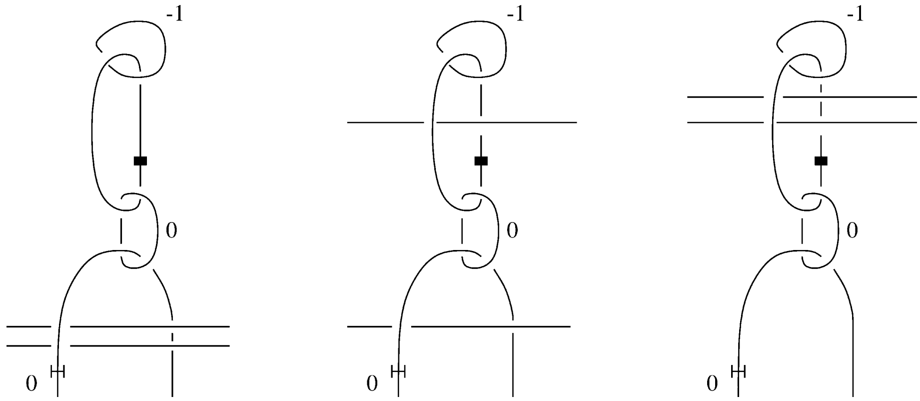

Fig. 16. Completing a nucleus.

We will now discuss the effect of a single stabilization on a manifold fiber summed with an $S^3 \times S^1$ along a knot cross a circle. Let K_1 and K_2 be two knots related by a single crossing change. By Markov moves, the relevant crossing may be assumed to be in the lower right corner of a braid representation of K_1. If $S^3 - \mathring{N}(K_1)$ is described with an extra non-interacting $+1$ Dehn surgery, then the manifold $(N_2, T)\,\#\,(S^3 \times S^1, K_1 \times S^1)$ will have the handle decomposition displayed in Fig. 20. All unlabeled 2-handles are zero framed.

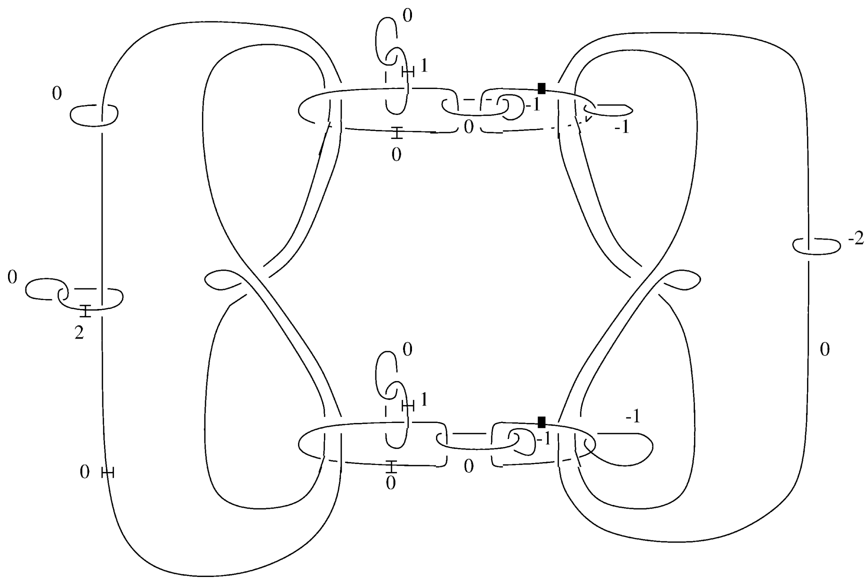

Fig. 17. $(N_2, T^2)\,\#\,(N_2, T^2)\,\#\,(S^2 \times S^2)$.

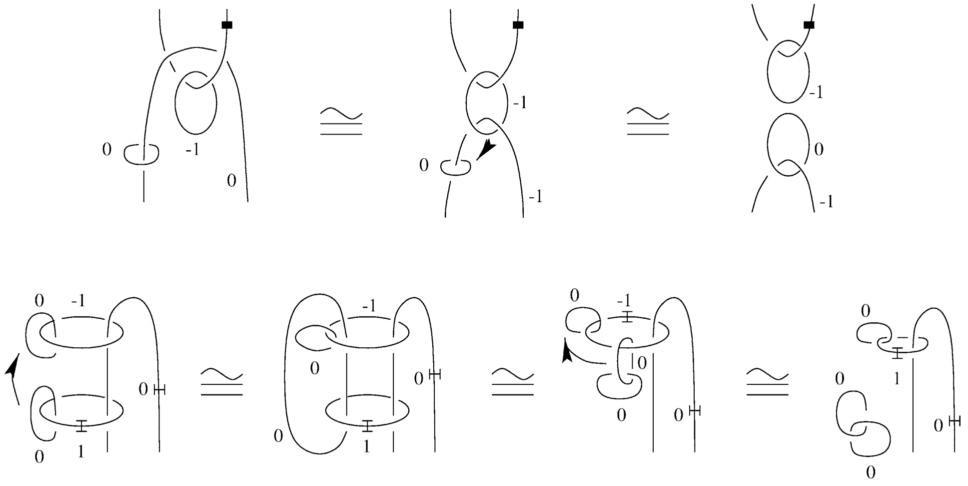

Fig. 18. $(N_2, T^2)\,\#\,(N_2, T^2)\,\#\,(S^2 \times S^2) \cong N_2^{\#2}\,\#\,(S^2 \times S^2)^{\#2}$.

To obtain Fig. 21, take the connected sum with $S^2 \widetilde{\times} S^2$ and slide handles. Now add two canceling 2-handle/3-handle pairs and one 1-handle/2-handle pair (Fig. 22). From here a long series of handle slides will demonstrate that

$$(N_2, T)\,\#\,\big(S^3 \times S^1, K_1 \times S^1\big)\,\#\,S^2 \widetilde{\times} S^2 \cong (N_2, T)\,\#\,\big(S^3 \times S^1, K_2 \times S^1\big)\,\#\,S^2 \widetilde{\times} S^2$$

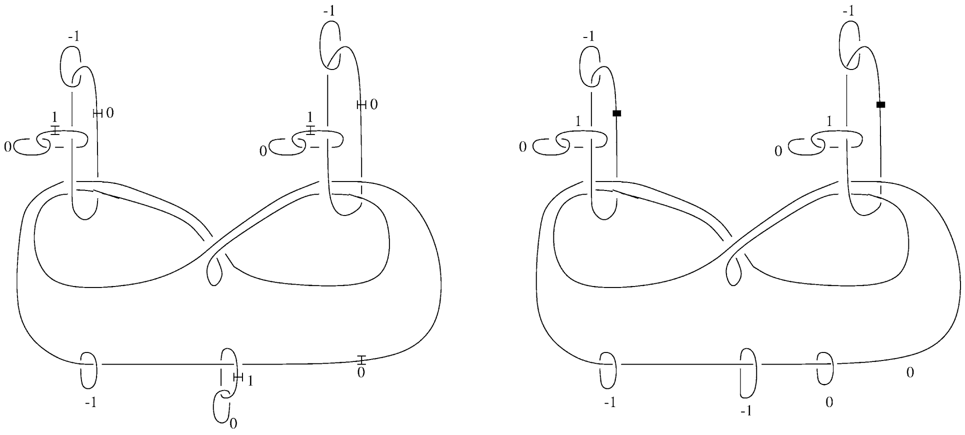

Fig. 19. M and $W_1 \cup M \cup W_1$.

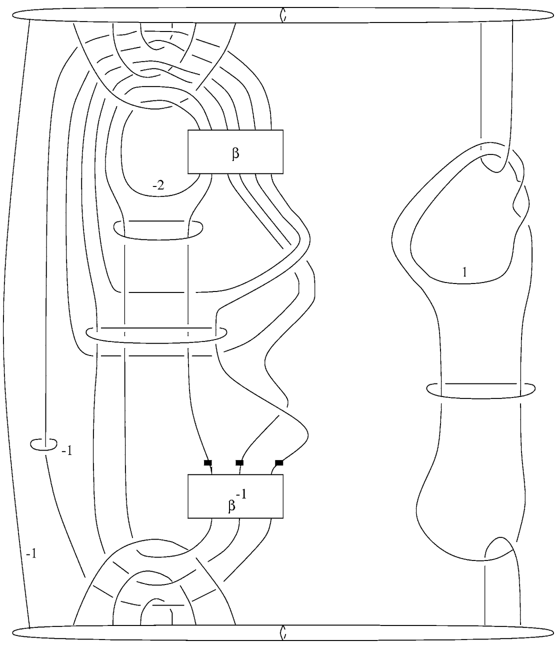

Fig. 20. $(N_2, T)\,\#\,(S^3 \times S^1, K_1 \times S^1)$.

(Figs. 23–26). The moves from Figs. 25, 26 are illustrated in Fig. 27. The 1-handle with feet is redrawn, represented by a circle with a dot. The rightmost strand may be pulled out from the braid by sliding it over some of the concentric 2-handles.

Finally notice that one can pass from any knot to the unknot by a series of crossing changes. Call the resulting sequence of knots $K_1, K_2, \ldots, K_n$, with K_n, the unknot. Then

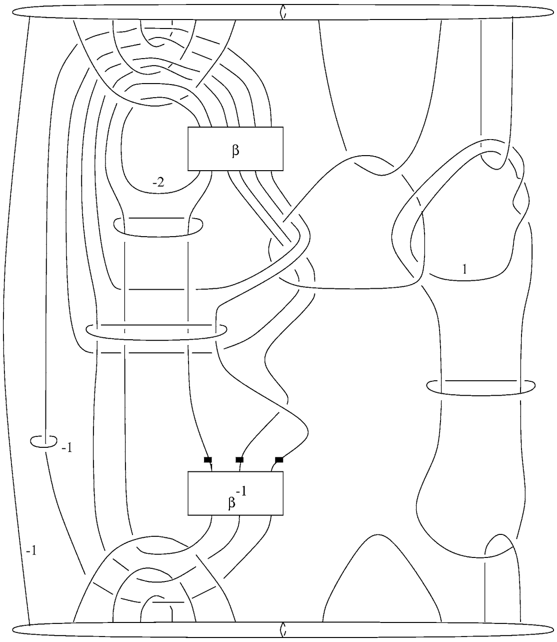

Fig. 21. $(N_2, T)\,\#\,(S^3 \times S^1, K_1 \times S^1)\,\#\,(S^2 \widetilde{\times} S^2)$.

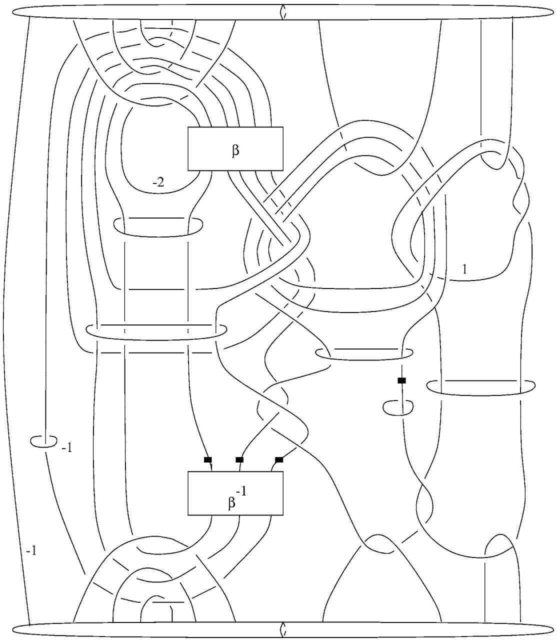

Fig. 22. $(N_2, T)\,\#\,(S^3 \times S^1, K_1 \times S^1)\,\#\,(S^2 \widetilde{\times} S^2)$.

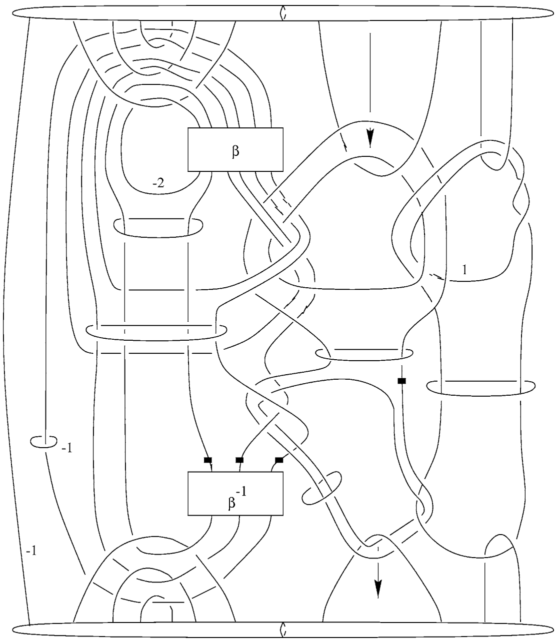

Fig. 23. $(N_2, T)\,\#\,(S^3 \times S^1, K_1 \times S^1)\,\#\,(S^2 \tilde{\times} S^2)$.

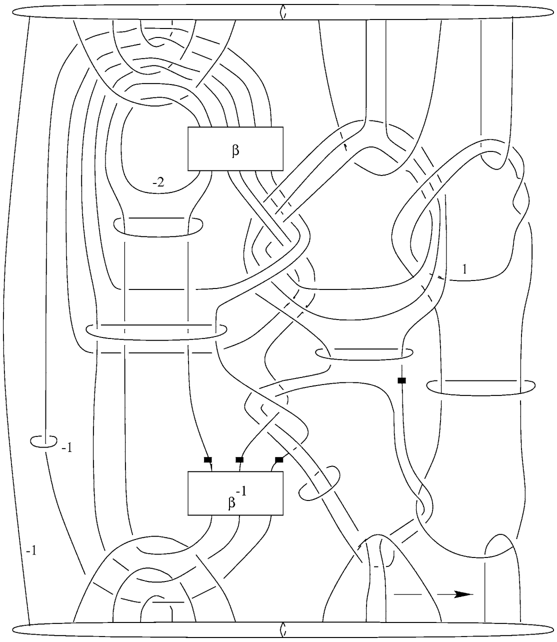

Fig. 24. $(N_2, T)\,\#\,(S^3 \times S^1, K_1 \times S^1)\,\#\,(S^2 \tilde{\times} S^2)$.

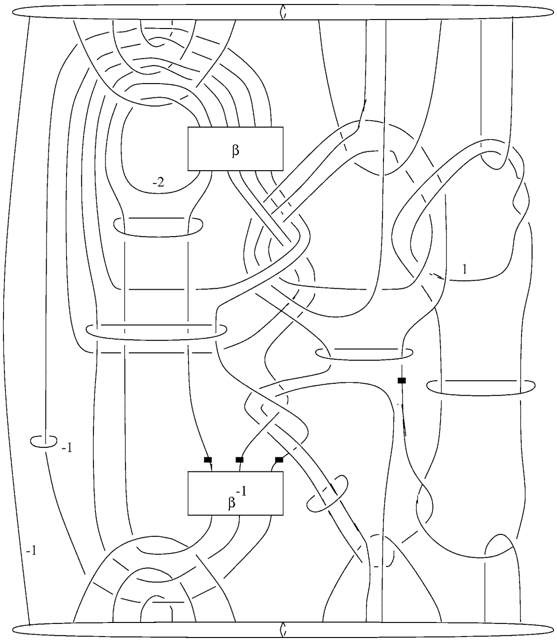

Fig. 25. $(N_2, T)\,\#\,(S^3 \times S^1, K_1 \times S^1)\,\#\,(S^2 \widetilde{\times} S^2)$.

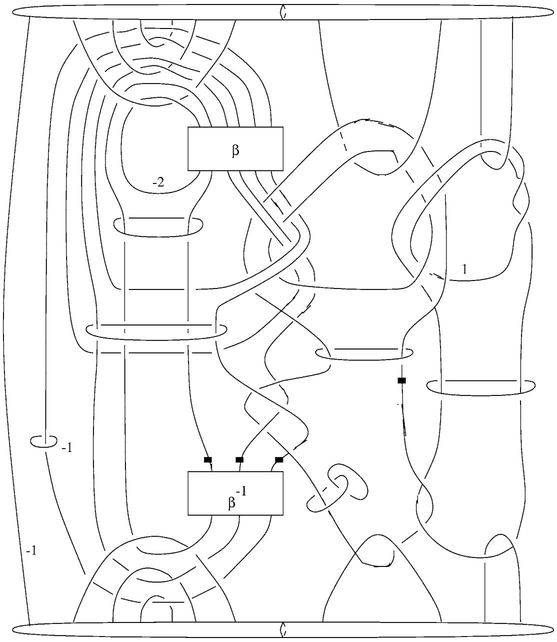

Fig. 26. $(N_2, T)\,\#\,(S^3 \times S^1, K_1 \times S^1)\,\#\,(S^2 \widetilde{\times} S^2)$.

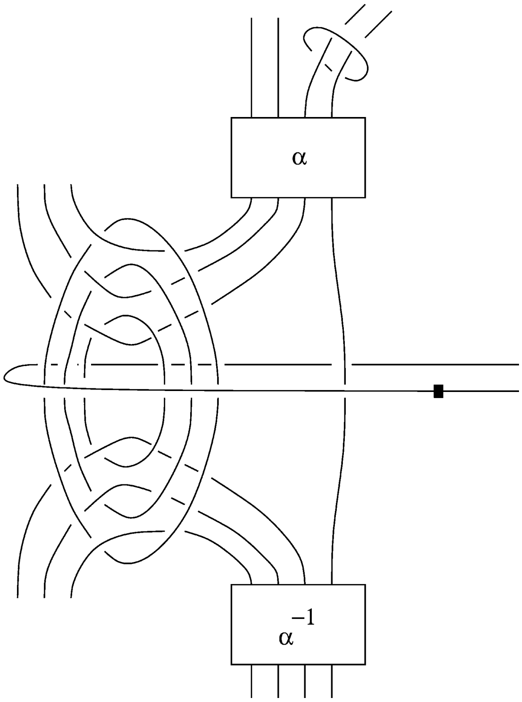

Fig. 27. Pulling a strand away from a braid.

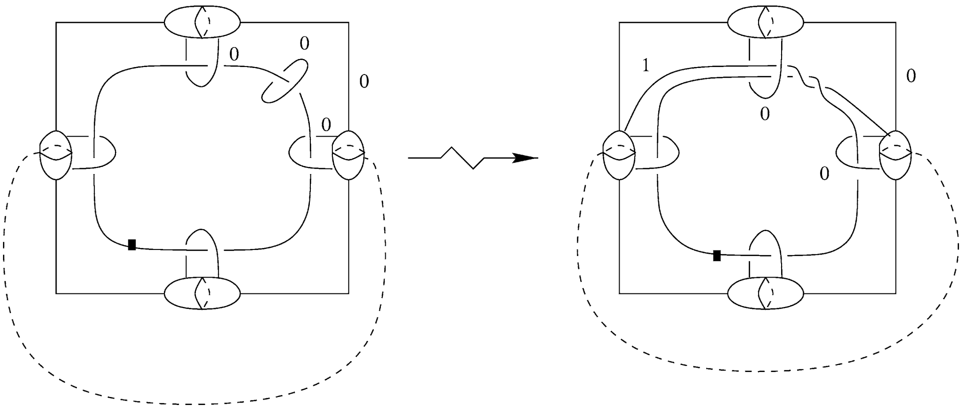

Fig. 28. Geometrically null $+1$ log transform.

$$
\begin{aligned}
&(N_2, t) \# \left(S^3 \times S^1, K_1 \times S^1\right) \# \left(S^2 \widetilde{\times} S^2\right) \\
&\quad \cong (N_2, t) \# \left(S^3 \times S^1, K_2 \times S^1\right) \# \left(S^2 \widetilde{\times} S^2\right) \cdots \\
&\quad \cong (N_2, t) \# \left(S^3 \times S^1, K_n \times S^1\right) \# \left(S^2 \widetilde{\times} S^2\right) \cong N_2 \# \left(S^2 \widetilde{\times} S^2\right).
\end{aligned}
$$

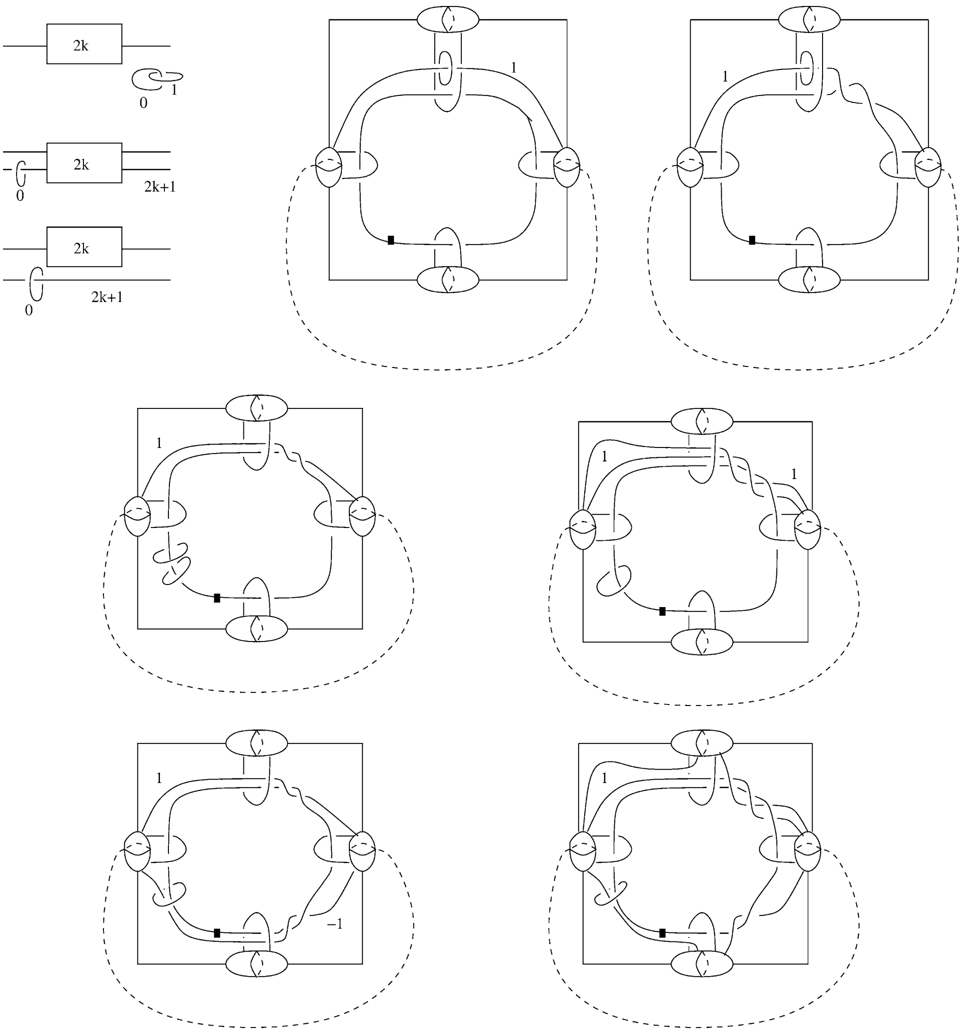

Fig. 29. Stabilizing a log transform.

The previous argument may be distilled to prove that any two manifolds related by a sequence of special moves become diffeomorphic after one stabilization. This special move is given in Fig. 28 which displays two different ways to attach a $T^2 \times S^2$ to an $I \times T^3$. If the dotted line bounds an evenly framed disk in some four-manifold, we will call the process of cutting out a $T^2 \times D^2$ and regluing it a geometrically null $+1$ log transform. This is just the product of $+1$ surgery with a circle. The Kirby calculus in Figs. 29, 30 demonstrates the following theorem.

Theorem. *Two manifolds related by a geometrically null $+1$ log transform become diffeomorphic after one stabilization.*

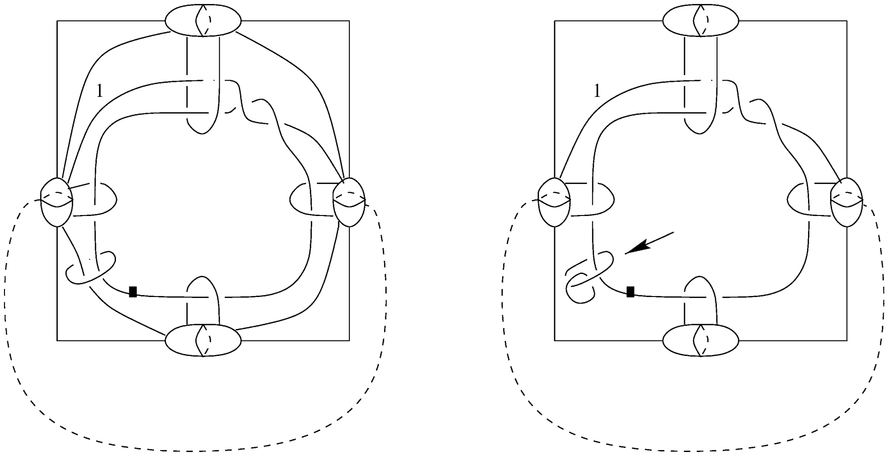

Fig. 30. Stabilizing a log transform. Finish by sliding the labeled handle over the $+1$ framed handle and reversing the moves from the beginning.

3. Seiberg–Witten invariants

Recall that the Seiberg–Witten series of a smooth 4-manifold with homology orientation is

$$SW_X = a_0 + \sum aj\big(\exp(K_j) + (-1)^{(\chi(x)+\alpha(x))/4}\exp(-k_j)\big)$$

where the set of basic classes is $\{\pm K_1, \pm K_2, \ldots, \pm K_n\} \subseteq H^2(X;\mathbb{Z})$, $a_0 = SW_X(0)$, and $a_j = SW_X(K_j)$. If $b_2^+(X) > 0$, then $SW_{X\,\#\,(S^2\widetilde{\times}S^2)} = 0$. Thus the Seiberg–Witten invariant cannot distinguish the two manifolds, $X\,\#\,(S^2\widetilde{\times}S^2)$ and $Y\,\#\,(S^2\widetilde{\times}S^2)$. One might hope that a diffeomorphism between $X\,\#\,(S^2\widetilde{\times}S^2)$ and $Y\,\#\,(S^2\widetilde{\times}S^2)$ would imply some restriction on the relationship between the Seiberg–Witten series, SW_X and SW_Y. We will compute the Seiberg–Witten series of all of the manifolds considered in the previous section. The number of basic classes, the rank of the space spanned by the basic classes, and the coefficients of the Seiberg–Witten series will vary arbitrarily in each family, F_N, of manifolds.

To compute the Seiberg–Witten series, we will use several gluing formula worked out by Morgan, Mrowka, and Szabo, and utilized by Fintushel and Stern [17,16,5].

Fact 1: $SW_{K3} = 1$.
Fact 2: $SW_{(X,T)\,\#\,(Y,S)} = SW_X \cdot SW_Y \cdot (\exp(T) - \exp(-T))^2$.
Fact 3: If $\pi_1(X) = 1$, $\pi_1(X - T) = 1$, $[T] \neq 0$ in $H_2(X)$ and $[T]^2 = 0$, then $SW_{(X,T)\,\#\,(S^3\times S^1, K\times S^1)} = SW_X \cdot \Delta_K(\exp(2T))$. Here, Δ_K is the Alexander polynomial of K.

The first fact is due to Witten, and is by now well known [21,19]. The second fact has not yet appeared in the literature, but it is similar to the results in [17,16]. We have not included the technical hypothesis for the second fact. The third fact is proved in [5].

Refine our original notation, to denote the tori in X_N by $T_{\alpha,i}$, with $\alpha = 1, 2 \ldots, N$ and $i = 1, 2, 3$ so that $T_{\alpha,3} = T_{\alpha+1,1}$ for $\alpha = 1, \ldots, N-1$. Using this notation, the Seiberg–Witten series of X_N is

$$SW_{X_N} = \prod_{\alpha=1}^{N-1} \bigl(\exp(T_{\alpha,3}) - \exp(-T_{\alpha,3})\bigr).$$

Finally, let

$$\begin{aligned}
&Y_0 = X_N, \qquad Y_{\alpha+1} = (Y_\alpha, T_{\alpha,2}) \# \bigl(S^3 \times S^1, K_{\alpha,2} \times S^1\bigr),\\
&Y' = (Y_N, T_{1,1}) \# \bigl(S^3 \times S^1, K_{1,1} \times S^1\bigr) \quad \text{and}\\
&Y = \bigl(Y', T_{N,3}\bigr) \# \bigl(S^3 \times S^1, K_{N,3}\bigr).
\end{aligned}$$

Then the Seiberg–Witten series of Y is

$$\begin{aligned}
SW_Y \;=\; &\prod_{\alpha=1}^{N-1} \bigl[\bigl(\exp(T_{\alpha,3}) - \exp(-T_{\alpha,3})\bigr) \cdot \Delta_{K_{\alpha,2}}\bigl(\exp(2T_{\alpha,2})\bigr)\bigr]\\
&\times \Delta_{K_{1,1}}\bigl(\exp(2T_{1,1})\bigr) \cdot \Delta_{K_{N,2}}\bigl(\exp(2TN, 2)\bigr) \cdot \Delta_{K_{N,3}}\bigl(\exp(2T_{N,3})\bigr).
\end{aligned}$$

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