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Global existence for the multi-dimensional compressible viscoelastic flows

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The global solutions in critical spaces to the multi-dimensional compressible viscoelastic flows are considered. The global existence of the Cauchy problem with initial data close to an equilibrium state is established in Besov spaces. Using uniform estimates for a hyperbolic–parabolic linear system with convection terms, we prove the global existence in the Besov space which is invariant with respect to the scaling of the associated equations. Several important estimates are achieved, including a smoothing effect on the velocity, and the L^1-decay of the density and deformation gradient.

1. Introduction

We consider the following equations of multi-dimensional compressible viscoelastic flows [9,11,15,21]:

$$\hat{\rho}_t + \operatorname{div}(\hat{\rho}\hat{\mathbf{u}}) = 0, \tag{1.1a}$$

$$(\hat{\rho}\hat{\mathbf{u}})_t + \operatorname{div}(\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) - \mu\Delta\hat{\mathbf{u}} - (\lambda+\mu)\nabla \operatorname{div}\hat{\mathbf{u}} + \nabla P(\hat{\rho}) = \alpha \operatorname{div}\big(\hat{\rho}\mathtt{F}\mathtt{F}^\top\big), \tag{1.1b}$$

$$\mathtt{F}_t + \hat{\mathbf{u}} \cdot \nabla\mathtt{F} = \nabla\hat{\mathbf{u}}\mathtt{F}, \tag{1.1c}$$

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where $\hat{\rho}$ stands for the density, $\hat{\mathbf{u}} \in \mathbb{R}^N$ $(N = 2, 3)$ the velocity, and $\mathtt{F} \in M^{N\times N}$ (the set of $N \times N$ matrices) the deformation gradient. The viscosity coefficients μ, λ are two constants satisfying $\mu > 0$, $2\mu + N\lambda > 0$, which ensures that the operator $-\mu\Delta\hat{\mathbf{u}} - (\lambda+\mu)\nabla \operatorname{div}\hat{\mathbf{u}}$ is a strongly elliptic operator. The pressure term $P(\hat{\rho})$ is an increasing and convex function of $\hat{\rho}$ for $\hat{\rho} > 0$. The symbol $\otimes$ denotes the Kronecker tensor product, $\mathtt{F}^\top$ means the transpose matrix of $\mathtt{F}$, and the notation $\hat{\mathbf{u}} \cdot \nabla\mathtt{F}$ is understood to be $(\hat{\mathbf{u}} \cdot \nabla)\mathtt{F}$. For system (1.1), the corresponding elastic energy is chosen to be the special form of the Hookean linear elasticity:

$$W(\mathtt{F}) = \frac{\alpha}{2}|\mathtt{F}|^2, \quad \alpha > 0,$$

which, however, does not reduce the essential difficulties for analysis. The methods and results of this paper can be applied to more general cases.

In this paper, we consider the Cauchy problem of system (1.1) subject to the initial condition:

$$(\hat{\rho}, \hat{\mathbf{u}}, \mathtt{F})|_{t=0} = \big(\hat{\rho}_0(x), \hat{\mathbf{u}}_0(x), \mathtt{F}_0(x)\big), \quad x \in \mathbb{R}^N, \tag{1.2}$$

and we are interested in the global existence and uniqueness of strong solution to the initial-value problem (1.1)–(1.2) near its equilibrium state in the multi-dimensional space $\mathbb{R}^N$. Here the equilibrium state of the system (1.1) is defined as: $\hat{\rho}$ is a positive constant (for simplicity, $\hat{\rho} = 1$), $\hat{\mathbf{u}} = 0$, and $\mathtt{F} = I$ (the identity matrix in $M^{3\times 3}$). We introduce a new unknown variable E by setting

$$\mathtt{F} = I + E.$$

Then, (1.1) becomes

$$\hat{\rho}_t + \operatorname{div}(\hat{\rho}\hat{\mathbf{u}}) = 0, \tag{1.3a}$$

$$(\hat{\rho}\hat{\mathbf{u}})_t + \operatorname{div}(\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) - \mu\Delta\hat{\mathbf{u}} - (\mu+\lambda)\nabla \operatorname{div}\hat{\mathbf{u}} + \nabla P(\hat{\rho}) = \alpha \operatorname{div}\big(\hat{\rho}(I+E)(I+E)^\top\big), \tag{1.3b}$$

$$E_t + \hat{\mathbf{u}} \cdot \nabla E = \nabla\hat{\mathbf{u}}E + \nabla\hat{\mathbf{u}}, \tag{1.3c}$$

with the initial data

$$(\hat{\rho}, \hat{\mathbf{u}}, E)|_{t=0} = \big(\hat{\rho}_0(x), \hat{\mathbf{u}}_0(x), E_0(x)\big), \quad x \in \mathbb{R}^N. \tag{1.4}$$

There have been some results about the local existence of strong solutions to the compressible viscoelastic flows, see [10,15] and the references therein. The global existence to (1.1) is a difficult problem due to the appearance of the deformation gradient. The challenge is to identify an appropriate functional space where the Cauchy problem (1.1)–(1.2) is well-posed globally in time. In this paper, to construct a global solution, we are going to use the scaling for the compressible viscoelastic flow to guess which space may be critical. We observe that system (1.1) is invariant under the transformation

$$\big(\hat{\rho}_0(x), \hat{\mathbf{u}}_0(x), \mathtt{F}_0(x)\big) \to \big(\hat{\rho}_0(lx), l\hat{\mathbf{u}}_0(lx), \mathtt{F}_0(lx)\big),$$
$$\big(\hat{\rho}(t,x), \hat{\mathbf{u}}(t,x), \mathtt{F}(t,x)\big) \to \big(\hat{\rho}\big(l^2t, lx\big), l\hat{\mathbf{u}}\big(l^2t, lx\big), \mathtt{F}\big(l^2t, lx\big)\big),$$

up to changes of the pressure law P into l^2P, and α into $l^2\alpha$. This suggests the following definition: *A functional space $\mathfrak{A} \subset \mathcal{S}'(\mathbb{R}^N) \times (\mathcal{S}'(\mathbb{R}^N))^N \times (\mathcal{S}'(\mathbb{R}^N))^{N\times N}$ is called a critical space if the associated norm is invariant under the transformation $(\rho, \mathbf{u}, \mathtt{F}) \to (\rho(l\cdot), l\mathbf{u}(l\cdot), \mathtt{F}(l\cdot))$ (up to a constant independent of l), where $\mathcal{S}'$ is the space of tempered distributions, i.e., the dual of the Schwartz space $\mathcal{S}$.* According to this definition, $B^{\frac{N}{2}} \times (B^{\frac{N}{2}-1})^N \times B^{\frac{N}{2}}$ (see Section 2 for the definition of $B^s := \dot{B}^s_{2,1}(\mathbb{R}^N)$) is a critical space. The motivations to use the homogeneous Besov space B^s with the derivative index $\frac{N}{2}$ include two points:

first, $B^{\frac{N}{2}}$ is an algebra embedded in L^∞, which allows us to control the density and the deformation gradient from below and from above without requiring more regularity on derivatives of $\hat{\rho}$ and $\mathtt{F}$; second, the product is continuous from $B^{\frac{N}{2}-\alpha} \times B^{\frac{N}{2}}$ to $B^{\frac{N}{2}-\alpha}$ for $0 \leqslant \alpha < N$.

For the global existence, the hardest part of the argument is to deal with the linear terms $\nabla\hat{\rho}$, $\operatorname{div} E$ and $\nabla\hat{\mathbf{u}}$, especially the first two terms. It turns out that finding some dissipation for $\operatorname{div} E$ is a crucial step. This step for the incompressible case has been fulfilled successfully in [14]. For the compressible system (1.1), we will reformulate the system, and use the divergence-free property of compressible viscoelastic flows for the "compressible" part of the velocity, while the property on curl is used to deal with the "incompressible" part of the velocity. Meanwhile, we decompose the deformation gradient into two parts: the symmetric part and the antisymmetric part. With this technique and decomposition, we will be able to obtain successfully the dissipation estimates on the density and the deformation gradient for an auxiliary system with convection terms. These estimates are crucial for the global existence. We remark that for the global existence of solutions to (1.1) near equilibrium, the intrinsic properties of the divergence and the curl are important and necessary. During the final stage of this paper, we noticed that some similar results are also obtained independently in [20], where the intrinsic properties of the divergence and curl of viscoelastic flows (see Appendix A) to system (1.1) are used to control the dissipation of the deformation gradient $\mathtt{F}$.

For the incompressible viscoelastic flows and related models, there are many papers in literature on classical solutions (cf. [7,8,12,13,17] and the references therein). On the other hand, the global existence of weak solutions to the incompressible viscoelastic flows with large initial data is still an outstanding open question, although there are some progress in that direction [16,18,19]. For the well-posedness of global solutions to the compressible Navier–Stokes equations, see [3,4] (for barotropic cases), and [5] (for barotropic cases with heat conduction). For the inviscid elastodynamics, see [22] and their references on the global existence of classical solutions.

The rest of this paper is organized as follows. In Section 2, we review the definitions of Besov spaces and show some good property of the Besov spaces. In Section 3, we reformulate the system (1.1) and state the main theorem. Section 4 is devoted to a priori estimates for an auxiliary linear system with convection terms. In Section 5, we give the proof of our main result, while in Appendix A, we prove two intrinsic properties of compressible viscoelastic flows.

2. Basic properties of Besov spaces

Throughout this paper, we use C for a generic constant, and denote $A \leqslant CB$ by $A \lesssim B$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. Also we use $(\alpha_q)_{q\in\mathbb{Z}}$ to denote a sequence such that $\sum_{q\in\mathbb{Z}} \alpha_q \leqslant 1$. $(f|g)$ denotes the inner product of two functions f, g in $L^2(\mathbb{R}^N)$. The standard summation notation over the repeated index is adopted in this paper.

The definition of homogeneous Besov spaces is built on a homogeneous Littlewood–Paley decomposition. First, we introduce a function $\psi \in C^\infty(\mathbb{R}^N)$, supported in the shell

$$\mathcal{C} = \left\{\xi \in \mathbb{R}^N\colon \frac{5}{6} \leqslant |\xi| \leqslant \frac{12}{5}\right\},$$

such that

$$\sum_{q\in\mathbb{Z}} \psi\big(2^{-q}\xi\big) = 1, \quad \text{if } \xi \neq 0.$$

Denoting by $h := \mathcal{F}^{-1}\psi$ the inverse Fourier transform of ψ, we define the dyadic blocks as follows:

$$\Delta_q f = \psi\big(2^{-q}D\big) f = 2^{qN} \int_{\mathbb{R}^N} h\big(2^q y\big) f(x-y)\,dy$$

and

$$S_q f = \sum_{p \leqslant q-1} \Delta_p f,$$

where D is the first-order differential operator. The formal decomposition

$$f = \sum_{q \in \mathbb{Z}} \Delta_q f \tag{2.1}$$

is called the homogeneous Littlewood–Paley decomposition. But unfortunately, the above identity is not always true in $\mathcal{S}'(\mathbb{R}^N)$ as pointed out in [4]. Nevertheless, (2.1) is true modulo polynomials (see [1,2,6]).

For $s \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^N)$, we denote

$$\|f\|_{B^s} := \sum_{q \in \mathbb{Z}} 2^{sq} \|\Delta_q f\|_{L^2}.$$

Notice that $\|\cdot\|_{B^s}$ is only a semi-norm on $\{f \in \mathcal{S}'(\mathbb{R}^N)\colon \|f\|_{B^s} < \infty\}$, because $\|f\|_{B^s}$ vanishes if and only if f is a polynomial. This leads us to introduce the following definition for homogeneous Besov spaces:

Definition 2.1. Let $s \in \mathbb{R}$ and $m = -[\frac{N}{2} + 1 - s]$. If $m < 0$, we set

$$B^s = \left\{ f \in \mathcal{S}'\big(\mathbb{R}^N\big)\colon \|f\|_{B^s} < \infty \text{ and } f = \sum_{q \in \mathbb{Z}} \Delta_q f \text{ in } \mathcal{S}'\big(\mathbb{R}^N\big) \right\}.$$

If $m \geqslant 0$, we denote by $\mathcal{P}_m$ the set of polynomials with N variables of degree $\leqslant m$ and define

$$B^s = \left\{ f \in \mathcal{S}'\big(\mathbb{R}^N\big)/\mathcal{P}_m\colon \|f\|_{B^s} < \infty \text{ and } f = \sum_{q \in \mathbb{Z}} \Delta_q f \text{ in } \mathcal{S}'\big(\mathbb{R}^N\big)/\mathcal{P}_m \right\}.$$

Functions in B^s have many good properties (see Proposition 2.5 in [4]):

Proposition 2.1. *The following properties hold*:

- *Density*: *the set* C_0^∞ *is dense in* B^s *if* $|s| \leqslant \frac{N}{2}$.
- *Derivation*: $\|f\|_{B^s} \approx \|\nabla f\|_{B^{s-1}}$.
- *Fractional derivation*: *let* $\Gamma = \sqrt{-\Delta}$ *and* $\sigma \in \mathbb{R}$; *then the operator* Γ^σ *is an isomorphism from* B^s *to* $B^{s-\sigma}$.
- *Algebraic properties*: *for* $s > 0$, $B^s \cap L^\infty$ *is an algebra.*
- *Interpolation*: $(B^{s_1}, B^{s_2})_{\theta,1} = B^{\theta s_1 + (1-\theta)s_2}$.

For the composition in B^s, we refer to [2] for the proof of the following estimates:

Lemma 2.1. *Given* $s > 0$ *and* $f \in L^\infty \cap B^s$.

- *Let* $\Psi \in W^{[s]+2}_{\mathrm{loc}}(\mathbb{R}^N)$ *such that* $\Psi(0) = 0$. *Then* $\Psi(f) \in B^s$. *Moreover, there exists a function* C *of one variable depending only on* s, N *and* Ψ, *and such that*

$$\big\|\Psi(f)\big\|_{B^s} \leqslant C\big(\|f\|_{L^\infty}\big)\|f\|_{B^s}.$$

- *Let $\Phi \in W^{[s]+2}_{loc}(\mathbb{R}^N)$ such that $\Phi'(0) = 0$. Suppose that f and g belong to $B^{\frac{N}{2}}$ and that $(f - g) \in B^s$ for some $s \in (-\frac{N}{2}, \frac{N}{2}]$. Then $\Phi(f) - \Phi(g)$ belongs to B^s and there exists a function of two variables C depending only on s, N and Φ, and such that*

$$\left\|\Phi(f) - \Phi(g)\right\|_{B^s} \leqslant C\left(\|f\|_{L^\infty}, \|g\|_{L^\infty}\right)\left(\|f\|_{B^{\frac{N}{2}}} + \|g\|_{B^{\frac{N}{2}}}\right)\|f - g\|_{B^s}.$$

But, different from the nonhomogeneous Besov space, the homogeneous Besov spaces fail to have nice inclusion properties. For example, owing to the low frequencies, the inclusion $B^s \hookrightarrow B^r$ does not hold for $s > r$. Still, the functions of B^s are locally more regular than those of B^r: for any $\varphi \in C_0^\infty$ and $f \in B^s$, the function φf is in B^r. This motivates the definition of *hybrid Besov spaces* where the growth conditions satisfied by the dyadic blocks are not the same for low and high frequencies. Let us recall that using hybrid Besov spaces has been crucial for proving global well-posedness for compressible gases in critical spaces (see [4,5]). The definition of the hybird Besov space is given as follows (see Definition 2.8 in [4] or [5]).

Definition 2.2. Let $s, t \in \mathbb{R}$. We set

$$\|f\|_{\tilde{B}^{s,t}} = \sum_{q\leqslant 0} 2^{qs}\|\Delta_q f\|_{L^2} + \sum_{q>0} 2^{qt}\|\Delta_q f\|_{L^2}.$$

Denoting $m = -[\frac{N}{2} + 1 - s]$, we define

$$\tilde{B}^{s,t} = \left\{f \in \mathcal{S}'\left(\mathbb{R}^N\right):\ \|f\|_{\tilde{B}^{s,t}} < \infty\right\} \quad \text{if } m < 0,$$
$$\tilde{B}^{s,t} = \left\{f \in \mathcal{S}'\left(\mathbb{R}^N\right)/\mathcal{P}_m:\ \|f\|_{\tilde{B}^{s,t}} < \infty\right\} \quad \text{if } m \geqslant 0.$$

Remark 2.1. Some remarks about the hybrid Besov spaces are in order:

- $\tilde{B}^{s,s} = B^s$.
- If $s \leqslant t$, then $\tilde{B}^{s,t} = B^s \cap B^t$. Otherwise, $\tilde{B}^{s,t} = B^s + B^t$. In particular, $\tilde{B}^{s,\frac{N}{2}} \hookrightarrow L^\infty$ as $s \leqslant \frac{N}{2}$.
- The space $\tilde{B}^{0,s}$ coincides with the usual nonhomogeneous Besov space

$$\left\{f \in \mathcal{S}'\left(\mathbb{R}^N\right):\ \left\|\chi(D)f\right\|_{L^2} + \sum_{q\geqslant 0} 2^{qs}\|\Delta_q f\|_{L^2} < \infty\right\}, \quad \text{where } \chi(\xi) = 1 - \sum_{q\geqslant 0}\phi\left(2^{-q}\xi\right).$$

- If $s_1 \leqslant s_2$ and $t_1 \geqslant t_2$, then $\tilde{B}^{s_1,t_1} \hookrightarrow \tilde{B}^{s_2,t_2}$.

For products of functions in hybrid Besov spaces, we have (see Proposition 2.10 in [4]):

Proposition 2.2. *Given $s_1, s_2, t_1, t_2 \in \mathbb{R}$.*

- *For all $s_1, s_2 > 0$,*

$$\|fg\|_{\tilde{B}^{s_1,s_2}} \lesssim \|f\|_{L^\infty}\|g\|_{\tilde{B}^{s_1,s_2}} + \|g\|_{L^\infty}\|f\|_{\tilde{B}^{s_1,s_2}}.$$

- *For all $s_1, s_2 \leqslant \frac{N}{2}$ such that $\min\{s_1 + t_1, s_2 + t_2\} > 0$,*

$$\|fg\|_{\tilde{B}^{s_1+s_2-\frac{N}{2},t_1+t_2-\frac{N}{2}}} \lesssim \|f\|_{\tilde{B}^{s_1,s_2}}\|g\|_{\tilde{B}^{t_1,t_2}}.$$

In order to state our existence result, we introduce some functional spaces and explain the notations. Let $T>0$, $r\in[0,\infty]$ and X be a Banach space. We denote by $\mathcal{M}(0,T;X)$ the set of measurable functions on $(0,T)$ valued in X. For $f\in\mathcal{M}(0,T;X)$, we define

$$\|f\|_{L^r_T(X)}=\left(\int_0^T\|f(\tau)\|_X^r\,d\tau\right)^{\frac{1}{r}}\quad\text{if } r<\infty,$$

$$\|f\|_{L^\infty_T(X)}=\sup_{\tau\in(0,T)}\operatorname{ess}\|f(\tau)\|_X.$$

Denote

$$L^r(0,T;X)=\left\{f\in\mathcal{M}(0,T;X):\ \|f\|_{L^r_T(X)}<\infty\right\}.$$

If $T=\infty$, we denote by $L^r(\mathbb{R}^+;X)$ and $\|f\|_{L^r(X)}$ the corresponding spaces and norms. Also denote by $C([0,T],X)$ (or $C(\mathbb{R}^+,X)$) the set of continuous X-valued functions on $[0,T]$ (resp. $\mathbb{R}^+$). We shall further denote by $C_b(\mathbb{R}^+;X)$ the set of bounded continuous X-valued functions.

For $\alpha\in(0,1)$, $C^\alpha([0,T];X)$ (or $C^\alpha(\mathbb{R}^+;X)$) stands for the space of the Hölder continuous functions in time with order α, that is, for every t,s in $[0,T]$ (resp. $\mathbb{R}^+$), we have

$$\|f(t)-f(s)\|_X\lesssim|t-s|^\alpha.$$

In this paper, the following estimates for the convection terms arising in the localized system will be used several times (cf. Lemma 5.1 in [5] or Lemma 6.2 in [4]).

Lemma 2.2. *Let G be a homogeneous smooth function of degree m. Suppose $-\frac{N}{2}<s_i,t_i\leqslant 1+\frac{N}{2}$ for $i=1,2$. Then the following inequalities hold*:

$$\begin{aligned}&\left|\left(G(D)\Delta_q(\mathbf{u}\cdot\nabla f)|G(D)\Delta_q f\right)\right|\\&\quad\leqslant C\alpha_q 2^{-q(\phi^{s_1,s_2}(q)-m)}\|\mathbf{u}\|_{B^{1+\frac{N}{2}}}\|f\|_{\tilde{B}^{s_1,s_2}}\left\|G(D)\Delta_q f\right\|_{L^2}\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}&\left|\left(G(D)\Delta_q(\mathbf{u}\cdot\nabla f)|\Delta_q g\right)+\left(\Delta_q(\mathbf{u}\cdot\nabla g)|G(D)\Delta_q f\right)\right|\\&\quad\leqslant C\alpha_q\|\mathbf{u}\|_{B^{1+\frac{N}{2}}}\left(2^{-q(\phi^{t_1,t_2}(q)-m)}\left\|G(D)\Delta_q f\right\|_{L^2}\|g\|_{\tilde{B}^{t_1,t_2}}\right.\\&\qquad\left.+2^{-q(\phi^{s_1,s_2}-m)}\|f\|_{\tilde{B}^{s_1,s_2}}\|\Delta_q g\|_{L^2}\right),\end{aligned}\tag{2.3}$$

where

$$\phi^{s,t}(q):=\begin{cases}s, & \text{if } q\leqslant 0,\\ t, & \text{if } q\geqslant 1.\end{cases}$$

For the nonlinear term $\nabla\mathbf{u}E$, we have the following estimates.

Lemma 2.3. *If $-\frac{N}{2}<s_i\leqslant 1+\frac{N}{2}$ for $i=1,2$, then*

$$\|\nabla\mathbf{u}E\|_{\tilde{B}^{s_1,s_2}}\leqslant C\|\mathbf{u}\|_{B^{1+\frac{N}{2}}}\|E\|_{\tilde{B}^{s_1,s_2}}.\tag{2.4}$$

To proof the above lemma, we need to recall the paradifferential calculus which enables us to define a generalized product between distributions. The paraproduct between f and g is defined by

$$T_f g = \sum_{q\in\mathbb{Z}} S_{q-1} f \Delta_q g.$$

We also have the following formal decomposition (modulo a polynomial):

$$fg = T_f g + T_g f + R(f,g),$$

with

$$R(f,g) = \sum_{q\in\mathbb{Z}} \Delta_q f \tilde{\Delta}_q g,$$

where $\tilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}$.

Proof of Lemma 2.3. Denoting $T'_f g = T_f g + R(g,f)$, we get the following decomposition

$$\Delta_q(\nabla \mathbf{u} E) = \Delta_q T'_E \nabla \mathbf{u} + J_q, \tag{2.5}$$

with

$$J_q = \sum_{|q'-q|\leqslant 3} \big(\big[\Delta_q, S_{q'-1}(\nabla \mathbf{u})\big]\Delta_{q'} E + (S_{q'-1} - S_{q-1})(\nabla \mathbf{u})\Delta_q \Delta_{q'} E + S_{q-1}(\nabla \mathbf{u})\Delta_q E\big).$$

Applying Proposition 5.2 in [5] or Proposition 6.1 in [4], we see that the first term on the right-hand side of (2.5) satisfies (2.4) provided $-\frac{N}{2} < s_i \leqslant \frac{N}{2} + 1$ for $i = 1, 2$. Next we estimate J_q term by term. First, for the commutator $[\Delta_q, S_{q'-1}(\nabla \mathbf{u})]\Delta_{q'} E$, we have

$$\big[\Delta_q, S_{q'-1}(\nabla \mathbf{u})\big]\Delta_{q'} E(x) = 2^{-q} \int_{\mathbb{R}^N}\int_0^1 h(y)\big(y \cdot S_{q'-1}(\nabla\nabla \mathbf{u})\big(x - 2^{-q}\tau y\big)\big)\Delta_{q'} E\big(x - 2^{-q} y\big)\, d\tau\, dy.$$

The above identity, together with Young's inequality for convolution operator, yields

$$\big\|\big[\Delta_q, S_{q'-1}(\nabla \mathbf{u})\big]\Delta_{q'} E\big\|_{L^2} \leqslant C\|\nabla \mathbf{u}\|_{L^\infty}\|\Delta_{q'} E\|_{L^2},$$

since

$$\|S_{q'-1}\nabla\nabla \mathbf{u}\|_{L^\infty} \lesssim 2^q\|\nabla \mathbf{u}\|_{L^\infty} \lesssim 2^q\|\mathbf{u}\|_{B^{\frac{N}{2}+1}}$$

according to Bernstein's Lemma (cf. [1,2]). Hence, we easily obtain the following inequality:

$$\|J_q\|_{L^2} \leqslant C\alpha_q 2^{-q\phi^{s_1,s_2}(q)}\|\mathbf{u}\|_{B^{1+\frac{N}{2}}}\|E\|_{\tilde{B}^{s_1,s_2}}.$$

The proof is complete. □

3. Reformulation and main results

In this section, we state our global existence result. We first reformulate system (1.1). Assume that the pressure $P(\hat{\rho})$ is an increasing convex function with $P'(1) > 0$, and denote $\chi_0 = (P'(1))^{-\frac{1}{2}}$. For $\hat{\rho} > 0$, system (1.1) can be rewritten as

$$\hat{\rho}_t + \hat{\mathbf{u}} \cdot \nabla \hat{\rho} + \hat{\rho} \operatorname{div} \hat{\mathbf{u}} = 0, \tag{3.1a}$$

$$\partial_t \hat{\mathbf{u}}_i + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}_i - \frac{1}{\hat{\rho}}\big(\mu \Delta \hat{\mathbf{u}}_i - (\lambda + \mu)\partial_{x_i} \operatorname{div} \hat{\mathbf{u}}\big) + \frac{P'(\hat{\rho})}{\hat{\rho}} \partial_{x_i} \hat{\rho} = \alpha \mathtt{F}_{jk} \partial_{x_j} \mathtt{F}_{ik}, \tag{3.1b}$$

$$\mathtt{F}_t + \hat{\mathbf{u}} \cdot \nabla \mathtt{F} = \nabla \hat{\mathbf{u}} \mathtt{F}, \tag{3.1c}$$

where we used the condition $\operatorname{div}(\hat{\rho}\mathtt{F}^\top) = 0$ (see Lemma A.1) for all $t \geqslant 0$, which ensures that the i-th component of the vector $\operatorname{div}(\rho \mathtt{F}\mathtt{F}^\top)$ is

$$\begin{aligned} \partial_{x_j}(\hat{\rho}\mathtt{F}_{ik}\mathtt{F}_{jk}) &= \hat{\rho}\mathtt{F}_{jk}\partial_{x_j}\mathtt{F}_{ik} + \mathtt{F}_{ik}\partial_{x_j}(\hat{\rho}\mathtt{F}_{jk}) \\ &= \hat{\rho}\mathtt{F}_{jk}\partial_{x_j}\mathtt{F}_{ik}. \end{aligned}$$

Define

$$\rho(t,x) = \hat{\rho}\big(\chi_0^2 t, \chi_0 x\big) - 1, \qquad \mathbf{u}(t,x) = \chi_0 \hat{\mathbf{u}}\big(\chi_0^2 t, \chi_0 x\big), \qquad E(t,x) = \mathtt{F}\big(\chi_0^2 t, \chi_0 x\big) - I,$$

then

$$\rho_t + \mathbf{u} \cdot \nabla \rho + \operatorname{div} \mathbf{u} = -\rho \operatorname{div} \mathbf{u}, \tag{3.2a}$$

$$\partial_t \mathbf{u}_i + \mathbf{u} \cdot \nabla \mathbf{u}_i - \mathcal{A}\mathbf{u} + \nabla_{x_i}\rho - a\partial_{x_j} E_{ij} = aE_{jk}\partial_{x_j}E_{ik} - \frac{\rho}{1+\rho}\mathcal{A}\mathbf{u} - K(\rho)\partial_{x_i}\rho, \tag{3.2b}$$

$$E_t + \mathbf{u} \cdot \nabla E - \nabla \mathbf{u} = \nabla \mathbf{u} E, \tag{3.2c}$$

with

$$K(\rho) := \frac{P'(\rho+1)}{(1+\rho)P'(1)} - 1, \qquad \mathcal{A} := \mu\Delta + (\lambda+\mu)\nabla \operatorname{div}, \qquad a = \frac{\alpha}{P'(1)}.$$

We remark that for simplicity of the presentation, we will assume that $a = 1$ for the rest of this paper.

For $s \in \mathbb{R}$, we denote

$$\Lambda^s f := \mathcal{F}^{-1}\big(|\xi|^s \mathcal{F}(f)\big).$$

Let

$$d = \Lambda^{-1} \operatorname{div} \mathbf{u}$$

be the "compressible part" of the velocity, and

$$\omega = \Lambda^{-1} \operatorname{curl} \mathbf{u}, \quad \text{with } \big(\operatorname{curl}(\mathbf{u})\big)_i^j = \partial_{x_j}\mathbf{u}^i - \partial_{x_i}\mathbf{u}^j$$

be the "incompressible part". Setting $\nu = \lambda + 2\mu$, system (3.2) can be rewritten as

$$\rho_t + \Lambda d = -\rho \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla \rho, \tag{3.3a}$$

$$\partial_t d - \nu \Delta d - \Lambda \rho - \mathcal{T} E = \Lambda^{-1} \operatorname{div}\left(-\mathbf{u} \cdot \nabla \mathbf{u} + E_{jk} \partial_{x_j} E_{ik} - \frac{\rho}{1+\rho} \mathcal{A} \mathbf{u} - K(\rho) \partial_{x_i} \rho\right), \tag{3.3b}$$

$$\partial_t \omega - \mu \Delta \omega - \mathcal{R} E = \Lambda^{-1} \operatorname{curl}\left(-\mathbf{u} \cdot \nabla \mathbf{u} + E_{jk} \partial_{x_j} E_{ik} - \frac{\rho}{1+\rho} \mathcal{A} \mathbf{u} - K(\rho) \partial_{x_i} \rho\right), \tag{3.3c}$$

$$E_t + \mathbf{u} \cdot \nabla E - \nabla \mathbf{u} = \nabla \mathbf{u} E, \tag{3.3d}$$

$$\mathbf{u} = -\Lambda^{-1} \nabla d + \Lambda^{-1} \operatorname{curl} \omega, \tag{3.3e}$$

where

$$\mathcal{R} = \Lambda^{-1} \operatorname{curl} \operatorname{div}, \qquad \mathcal{T} = \Lambda^{-1} \operatorname{div} \operatorname{div}.$$

The operators Λ, $\mathcal{T}$ and $\mathcal{R}$ are differential operators of order one.

Notice that the condition $\operatorname{div}(\hat{\rho}\mathtt{F}) = 0$ for all $t \geqslant 0$ implies that $\frac{\partial^2(\hat{\rho}\mathtt{F}_{ij})}{\partial x_i \partial x_j} = 0$ for all $t \geqslant 0$ and smooth functions $\hat{\rho}$, $\mathtt{F}$. Hence, we have

$$\begin{aligned} \mathcal{T} E &= \Lambda^{-1}\left(\frac{\partial^2 E_{ij}}{\partial x_i \partial x_j}\right) \\ &= \underbrace{\Lambda^{-1}\left(\frac{\partial^2[(1+\rho)(\delta_{ij} + E_{ij})]}{\partial x_i \partial x_j}\right)}_{=0} - \Lambda^{-1} \operatorname{div} \operatorname{div}(\rho I + \rho E) \\ &= \Lambda \rho - \Lambda^{-1} \operatorname{div} \operatorname{div}(\rho E), \end{aligned} \tag{3.4}$$

where

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

On the other hand, according to Lemma A.2 (see Appendix A), we have

$$\begin{aligned} (\mathcal{R} E)_{ij} &= \Lambda^{-1}\left(\partial_{x_j}\left(\frac{\partial E_{ik}}{\partial x_k}\right) - \partial_{x_i}\left(\frac{\partial E_{jk}}{\partial x_k}\right)\right) \\ &= \Lambda^{-1}\left(\partial_{x_k}\left(\frac{\partial E_{ik}}{\partial x_j}\right) - \partial_{x_k}\left(\frac{\partial E_{jk}}{\partial x_i}\right)\right) \\ &= \Lambda^{-1}\left(\partial_{x_k}\left(\frac{\partial E_{ij}}{\partial x_k}\right) - \partial_{x_k}\left(\frac{\partial E_{ji}}{\partial x_k}\right)\right) + \Lambda^{-1} \partial_{x_k}(E_{lk} \nabla_l E_{ij} - E_{lj} \nabla_l E_{ik}) \\ &\quad - \Lambda^{-1} \partial_{x_k}(E_{lk} \nabla_l E_{ji} - E_{li} \nabla_l E_{jk}) \\ &= -\Lambda(E_{ij} - E_{ji}) + \Lambda^{-1} \partial_{x_k}(E_{lk} \nabla_l E_{ij} - E_{lj} \nabla_l E_{ik}) - \Lambda^{-1} \partial_{x_k}(E_{lk} \nabla_l E_{ji} - E_{li} \nabla_l E_{jk}). \end{aligned}$$

Thus, we finally obtain

$$\rho_t + \Lambda d = -\rho \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla \rho, \tag{3.5a}$$

$$\begin{aligned} &\partial_t d - \nu \Delta d - 2\Lambda \rho \\ &\quad = \Lambda^{-1} \operatorname{div}\left(-\mathbf{u} \cdot \nabla \mathbf{u} + E_{jk} \partial_{x_j} E_{ik} - \frac{\rho}{1+\rho} \mathcal{A}\mathbf{u} - K(\rho)\partial_{x_i}\rho - \operatorname{div}(\rho E)\right), \end{aligned} \tag{3.5b}$$

$$\begin{aligned} &\partial_t \omega - \mu \Delta \omega + \Lambda\big(E - E^\top\big) \\ &\quad = \Lambda^{-1} \operatorname{curl}\left(-\mathbf{u} \cdot \nabla \mathbf{u} + E_{jk} \partial_{x_j} E_{ik} - \frac{\rho}{1+\rho} \mathcal{A}\mathbf{u} - K(\rho)\partial_{x_i}\rho\right) + \mathcal{S}, \end{aligned} \tag{3.5c}$$

$$\big(E^\top - E\big)_t + \mathbf{u} \cdot \nabla\big(E^\top - E\big) + \Lambda \omega = (\nabla \mathbf{u} E)^\top - \nabla \mathbf{u} E, \tag{3.5d}$$

$$\begin{aligned} \mathcal{E}_t + 2\Lambda d = {} & -\Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}\big(\mathbf{u} \cdot \nabla(E_{ij} + E_{ji})\big) \\ & + \Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}\big((\nabla \mathbf{u} E)_{ij} + (\nabla \mathbf{u} E)_{ji}\big), \end{aligned} \tag{3.5e}$$

$$\mathbf{u} = -\Lambda^{-1}\nabla d + \Lambda^{-1} \operatorname{curl} \omega, \tag{3.5f}$$

where the antisymmetric matrix $\mathcal{S}$ is defined as

$$\mathcal{S}_{ij} = \Lambda^{-1}\partial_{x_k}(E_{lk}\nabla_l E_{ij} - E_{lj}\nabla_l E_{ik}) - \Lambda^{-1}\partial_{x_k}(E_{lk}\nabla_l E_{ji} - E_{li}\nabla_l E_{jk}),$$

and the scalar function $\mathcal{E}$ is defined as

$$\mathcal{E}_{ij} = \Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}(E_{ij} + E_{ji}).$$

Notice that from Proposition 2.1, we deduce that

$$\|\mathcal{E}\|_{B^s} \approx \big\|E + E^\top\big\|_{B^s}$$

and

$$\|\mathcal{E}\|_{B^s} + \big\|E - E^\top\big\|_{B^s} \approx \|E\|_{B^s}. \tag{3.6}$$

Also, according to (3.4) and the second equation of (3.5), we have

$$\begin{aligned} &\partial_t d - \nu \Delta d - 2\Lambda \mathcal{E} \\ &\quad = \Lambda^{-1} \operatorname{div}\left(-\mathbf{u} \cdot \nabla \mathbf{u} + E_{jk} \partial_{x_j} E_{ik} - \frac{\rho}{1+\rho} \mathcal{A}\mathbf{u} - K(\rho)\partial_{x_i}\rho + \operatorname{div}(\rho E)\right). \end{aligned} \tag{3.7}$$

The motivation to write the second equation of (3.5) as (3.7) is to obtain the estimate on the symmetric part $\mathcal{E}$ of the deformation gradient, as we will see in Section 4.

The fact that this new formulation ((3.5), supplemented with (3.7)) is equivalent to (3.1) requires some explanation: it is not immediately obvious that the second and the third equations in the above system are equivalent to the second equation in (3.1) under the condition that

$\operatorname{div}((1+\rho)(I+E)^\top)=0$. Notice however that the right-hand side (denote it by $\mathcal{O}$) of the third equation in (3.5) and the term $\Lambda(E^\top - E)$ are skew-symmetric matrices and satisfy the following Jacobi relation:

$$\partial_{x_i}\mathcal{O}_j^k + \partial_{x_j}\mathcal{O}_k^i + \partial_{x_k}\mathcal{O}_i^j = 0 \quad \text{for } 1 \leqslant i,j,k \leqslant N.$$

Since $\omega_0 := \Lambda^{-1}\operatorname{curl}\mathbf{u}_0$ is a skew-symmetric matrix satisfying the Jacobi relation, this is also the case for ω. We therefore have the equivalence

$$\mathbf{u} = -\Lambda^{-1}\nabla d + \Lambda^{-1}\operatorname{curl}\omega \quad \Leftrightarrow \quad \operatorname{div}\mathbf{u} = \Lambda d \quad \text{and} \quad \operatorname{curl}\mathbf{u} = \Lambda\omega,$$

which enables us to conclude that $\mathbf{u}$ indeed satisfies the second equation of (3.1) as soon as d and ω satisfy the second and the third equations in (3.5).

The existence of a solution to (1.1) is proved thanks to a classical (and tedious) iteration method: we define a sequence of approximate solutions of (1.1) which solve a linear systems to which Proposition 4.1 applies. For small enough initial data, we obtain uniform estimates so that we can use a compactness argument to show the convergence of such an approximate solution. Refer to Section 5 for more details of the complete proof.

Let us now introduce the functional space which appears in the global existence theorem.

Definition 3.1. For $T > 0$ and $s \in \mathbb{R}$, we denote

$$\begin{aligned}\mathfrak{B}_T^s = \Big\{(\rho,\mathbf{u},E) \in \big(L^1\big(0,T;\tilde{B}^{s+1,s}\big)\cap C\big([0,T];\tilde{B}^{s-1,s}\big)\big) \\ \times \big(L^1\big(0,T;B^{s+1}\big)\cap C\big([0,T];B^{s-1}\big)\big)^N \\ \times \big(L^1\big(0,T;\tilde{B}^{s+1,s}\big)\cap C\big([0,T];\tilde{B}^{s-1,s}\big)\big)^{N\times N}\Big\}\end{aligned}$$

and

$$\begin{aligned}\big\|(\rho,\mathbf{u},E)\big\|_{\mathfrak{B}_T^s} = \|\rho\|_{L_T^\infty(\tilde{B}^{s-1,s})} + \|\mathbf{u}\|_{L_T^\infty(B^{s-1,s})} + \|E\|_{L_T^\infty(\tilde{B}^{s-1,s})} \\ + \|\rho\|_{L_T^1(\tilde{B}^{s+1,s})} + \|\mathbf{u}\|_{L_T^1(B^{s+1})} + \|E\|_{L_T^1(\tilde{B}^{s+1,s})}.\end{aligned}$$

We use the notation $\mathfrak{B}^s$ if $T=+\infty$ by changing the interval $[0,T]$ into $[0,\infty)$ in the definition above.

Now it is ready to state our main result:

Theorem 3.1. *There exist two positive constants γ and Γ, such that, if $\hat{\rho}_0 - 1 \in \tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}$, $\hat{\mathbf{u}}_0 \in B^{\frac{N}{2}-1}$, $\mathtt{F}_0 - I \in \tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}$ satisfy*

- $\|\hat{\rho}_0 - 1\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\hat{\mathbf{u}}_0\|_{B^{\frac{N}{2}-1}} + \|\mathtt{F}_0 - I\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} \leqslant \gamma$;
- $\operatorname{div}(\hat{\rho}_0\mathtt{F}_0^\top) = 0$;
- $\mathtt{F}_{lk}(0)\nabla_{x_l}\mathtt{F}_{ij}(0) = \mathtt{F}_{lj}(0)\nabla_{x_l}\mathtt{F}_{ik}(0)$,

then system (1.1) *has a solution $(\hat{\rho},\hat{\mathbf{u}},\mathtt{F})$ with $(\hat{\rho}-1,\hat{\mathbf{u}},\mathtt{F}-I)$ in $\mathfrak{B}^{\frac{N}{2}}$ satisfying*

$$\big\|(\hat{\rho}-1,\hat{\mathbf{u}},\mathtt{F}-I)\big\|_{\mathfrak{B}^{\frac{N}{2}}} \leqslant \Gamma\big(\|\hat{\rho}_0 - 1\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\hat{\mathbf{u}}_0\|_{B^{\frac{N}{2}-1}} + \|\mathtt{F}_0 - I\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\big).$$

Remark 3.1. The solution in Theorem 3.1 is also unique, but we omit the proof of the uniqueness. The proof of uniqueness will be same as in [3] with a slightly modification due to the deformation gradient. See also [20] for a proof.

4. Estimates of a linear problem

In this section, we consider the following auxiliary linear system:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho + \Lambda d = \mathfrak{L}, \\ \partial_t d + \mathbf{u} \cdot \nabla d - \nu \Delta d - 2\Lambda \rho = \mathfrak{M}, \\ \partial_t \omega + \mathbf{u} \cdot \nabla \omega - \mu \Delta \omega + \Lambda \big(E - E^\top\big) = \mathfrak{N}, \\ \partial_t \big(E^\top - E\big) + \mathbf{u} \cdot \nabla \big(E^\top - E\big) + \Lambda \omega = \mathfrak{Q}, \\ \mathcal{E}_t + \mathbf{u} \cdot \nabla \mathcal{E} + 2\Lambda d = \mathfrak{K}, \end{cases} \tag{4.1}$$

where $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{Q}, \mathfrak{J}, \mathfrak{K}$, and $\mathbf{u}$ are given functions, and (3.6) gives a relation of E, $E - E^\top$, and $\mathcal{E}$. We remark that, as in (3.7), to obtain estimates on $\mathcal{E}$, we need to rewrite the second equation of (4.1) as

$$\partial_t d + \mathbf{u} \cdot \nabla d - \nu \Delta d - 2\Lambda \mathcal{E} = \mathfrak{J}, \tag{4.2}$$

under the constraint

$$2\Lambda \rho + \mathfrak{M} = 2\Lambda \mathcal{E} + \mathfrak{J}. \tag{4.3}$$

For this system, we have the following estimate:

Proposition 4.1. *Let* $(\rho, d, \omega, E - E^\top, \mathcal{E})$ *be a solution of* (4.1) *on* $[0, T)$, *and*

$$V(t) := \int_0^t \|\mathbf{u}(s)\|_{B^{\frac{N}{2}+1}}\, ds.$$

Under the condition (4.3), *the following estimate holds on* $[0, T)$:

$$\begin{aligned} &\|\rho(t)\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|E(t)\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|d(t)\|_{B^{\frac{N}{2}-1}} + \|\omega(t)\|_{B^{\frac{N}{2}-1}} \\ &\quad + \int_0^t \Big(\|\rho(s)\|_{\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}}} + \|d(s)\|_{B^{\frac{N}{2}+1}} + \|E(s)\|_{\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}}} + \|\omega(s)\|_{B^{\frac{N}{2}+1}}\Big)\, ds \\ &\leqslant Ce^{CV(t)} \Bigg\{ \|\rho_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|E_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|d_0\|_{B^{\frac{N}{2}-1}} + \|\omega_0\|_{B^{\frac{N}{2}-1}} \\ &\quad + \int_0^t e^{-CV(s)} \Big(\|\mathfrak{L}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\mathfrak{M}\|_{B^{\frac{N}{2}-1}} + \|\mathfrak{N}\|_{B^{\frac{N}{2}-1}} + \|\mathfrak{Q}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} \\ &\quad + \|\mathfrak{J}\|_{B^{\frac{N}{2}-1}} + \|\mathfrak{K}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\Big)\, ds \Bigg\}, \end{aligned}$$

where C depends only on N.

Remark 4.1. Notice that the constraint (4.3) is always satisfied by our system (3.5) in view of the divergence property of the deformation gradient E, and $\mathfrak{J}$ will be given as the right-hand side of (3.7). This implies that the estimates in Proposition 4.1 also hold for solutions to (3.5).

Proof of Proposition 4.1. To prove this proposition, we first localize (4.1) in low and high frequencies according to the Littlewood–Paley decomposition. We then use an energy method to estimate each dyadic block. To this end, we will divide our proof into four steps.

Let (ρ, d, ω, E) be a solution of (4.1) and $K > 0$. Define

$$\tilde{\rho} = e^{-KV(t)}\rho, \qquad \tilde{d} = e^{-KV(t)}d, \qquad \tilde{E}^\top - \tilde{E} = e^{-KV(t)}\big(E^\top - E\big),$$
$$\tilde{\omega} = e^{-KV(t)}\omega, \qquad \tilde{\mathcal{E}} = e^{-KV(t)}\mathcal{E},$$

and

$$\tilde{\mathfrak{L}} = e^{-KV(t)}\mathfrak{L}, \qquad \tilde{\mathfrak{M}} = e^{-KV(t)}\mathfrak{M}, \qquad \tilde{\mathfrak{N}} = e^{-KV(t)}\mathfrak{N},$$
$$\tilde{\mathfrak{J}} = e^{-KV(t)}\mathfrak{J}, \qquad \tilde{\mathfrak{K}} = e^{-KV(t)}\mathfrak{K}.$$

Applying the operator Δ_q to (4.1) and (4.2), we deduce that $(\Delta_q\tilde{\rho}, \Delta_q\tilde{d}, \Delta_q\tilde{\omega}, \Delta_q\tilde{E})$ satisfies

$$\begin{cases} \partial_t\Delta_q\tilde{\rho} + \Delta_q(\mathbf{u}\cdot\nabla\tilde{\rho}) + \Lambda\Delta_q\tilde{d} = \Delta_q\tilde{\mathfrak{L}} - KV'(t)\Delta_q\tilde{\rho}, \\ \partial_t\Delta_q\tilde{d} + \Delta_q(\mathbf{u}\cdot\nabla\tilde{d}) - \nu\Delta\Delta_q\tilde{d} - 2\Lambda\Delta_q\tilde{\rho} = \Delta_q\tilde{\mathfrak{M}} - KV'(t)\Delta_q\tilde{d}, \\ \partial_t\Delta_q\tilde{\omega} + \Delta_q(\mathbf{u}\cdot\nabla\tilde{\omega}) - \mu\Delta\Delta_q\tilde{\omega} + \Lambda\Delta_q\big(\tilde{E} - \tilde{E}^\top\big) = \Delta_q\tilde{\mathfrak{N}} - KV'(t)\Delta_q\tilde{\omega}, \\ \partial_t\Delta_q\big(\tilde{E}^\top - \tilde{E}\big) + \Delta_q\big(\mathbf{u}\cdot\nabla\big(\tilde{E}^\top - \tilde{E}\big)\big) + \Lambda\tilde{\omega} = \Delta_q\tilde{\mathfrak{Q}} - KV'(t)\Delta_q\big(\tilde{E}^\top - \tilde{E}\big), \\ \partial_t\Delta_q\tilde{d} + \Delta_q(\mathbf{u}\cdot\nabla\tilde{d}) - \nu\Delta\Delta_q\tilde{d} - 2\Lambda\Delta_q\tilde{\mathcal{E}} = \Delta_q\tilde{\mathfrak{J}}, \\ \partial_t\Delta_q\tilde{\mathcal{E}} + \Delta_q(\mathbf{u}\cdot\nabla\tilde{\mathcal{E}}) + 2\Lambda\Delta_q\tilde{d} = \Delta_q\tilde{\mathfrak{K}}. \end{cases} \tag{4.4}$$

Denote

$$g_q := 2^{q(\frac{N}{2}-1)}\bigg(2\|\Delta_q\tilde{\rho}\|_{L^2}^2 + 2\|\Delta_q\tilde{d}\|_{L^2}^2 + \big\|\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + \|\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2 + \|\Delta_q\tilde{\omega}\|_{L^2}^2$$
$$- \frac{\nu}{\eta}\big(\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big|\Delta_q\tilde{\omega}\big) - \frac{\nu}{\eta}(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d}) - \frac{\nu}{\eta}(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d})\bigg)^{\frac{1}{2}}$$

for $q \leqslant q_0$ with $\eta = \max\{\frac{4^{q_0}\nu^2+3}{2}, \nu, \frac{\nu}{\mu}, \frac{4^{q_0}\mu\nu}{2}\} + 1$;

$$g_q := 2^{q(\frac{N}{2}-1)}\big(\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2 + \big\|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + \|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2 + \|\Delta_q\tilde{d}\|_{L^2}^2 + \|\Delta_q\tilde{\omega}\|_{L^2}^2$$
$$- (\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d}) - \big(\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big|\Delta_q\tilde{\omega}\big) - (\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d})\big)^{\frac{1}{2}}$$

for $q > q_0$, where $\beta_1 = \frac{2}{\nu}$, $\beta_2 = \frac{2}{\mu}$, $\gamma = \max\{\frac{2}{\mu^2}, \frac{5}{\nu^2}\} + 1$, and q_0 is chosen to satisfy

$$\|\Lambda\Delta_q f\|_{L^2} \geqslant 2\gamma\|\Delta_q f\|_{L^2} \quad \text{for all } q \geqslant q_0. \tag{4.5}$$

Due to the fact $\operatorname{supp}\mathcal{F}(\Delta_q\rho) \subset 2^q\mathcal{C}$ and $\operatorname{supp}\mathcal{F}(\Delta_q E) \subset 2^q\mathcal{C}$, one deduces that

$$\left(\frac{g_q}{2^{q\phi^{\frac{N}{2}-1,\frac{N}{2}}(q)}(\|\Delta_q\tilde{\rho}\|_{L^2} + \|\Delta_q\tilde{E}\|_{L^2}) + 2^{q(\frac{N}{2}-1)}(\|\Delta_q\tilde{d}\|_{L^2} + \|\Delta_q\tilde{\omega}\|_{L^2})}\right)^{\pm 1} \leqslant C \tag{4.6}$$

for a universal constant C.

The first two steps of the proof are devoted to getting the following inequality:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}g_q^2+\kappa 2^{q\phi^{N,N-2}(q)}\big(\|\Delta_q\tilde{\rho}\|_{L^2}^2+\big\|\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big\|_{L^2}^2+\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2+\|\Delta_q\tilde{\omega}\|_{L^2}^2+\|\Delta_q\tilde{d}\|_{L^2}^2\big)\\&\quad\leqslant C\alpha_q g_q\big(\|\tilde{\mathfrak{L}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|\tilde{\mathfrak{Q}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|\tilde{\mathfrak{K}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|\tilde{\mathfrak{J}}\|_{B^{\frac{N}{2}-1}}+\|\tilde{\mathfrak{M}}\|_{B^{\frac{N}{2}-1}}+\|\tilde{\mathfrak{N}}\|_{B^{\frac{N}{2}-1}}\\&\qquad+V'(t)\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\big\|\tilde{E}^\top-\tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|\tilde{d}\|_{B^{\frac{N}{2}-1}}+\|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big)\big)\\&\qquad-KV'g_q^2,\end{aligned}\tag{4.7}$$

where κ is a universal constant.

First step: Low frequencies. Suppose $q\leqslant q_0$ and define

$$\begin{aligned}f_q^2&=2\|\Delta_q\tilde{\rho}\|_{L^2}^2+2\|\Delta_q\tilde{d}\|_{L^2}^2+\big\|\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big\|_{L^2}^2+\|\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2+\|\Delta_q\tilde{\omega}\|_{L^2}^2\\&\quad-\frac{\nu}{\eta}\big(\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big|\Delta_q\tilde{\omega}\big)-\frac{\nu}{\eta}(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d})-\frac{\nu}{\eta}(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d}).\end{aligned}$$

Taking the L^2-scalar product of the first equation of (4.4) with $\Delta_q\tilde{\rho}$, of the second equation with $\Delta_q\tilde{d}$, of the third equation with $\Delta_q\tilde{\omega}$, and of the fourth equation with $\Delta_q(\tilde{E}^\top-\tilde{E})$, we obtain the following four identities:

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q\tilde{\rho}\|_{L^2}^2+\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{\rho})|\Delta_q\tilde{\rho}\big)+(\Lambda\Delta_q\tilde{d}|\Delta_q\tilde{\rho})=(\Delta_q\tilde{\mathfrak{L}}|\Delta_q\tilde{\rho})-KV'\|\Delta_q\tilde{\rho}\|_{L^2}^2;\tag{4.8}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta_q\tilde{d}\|_{L^2}^2+\nu\|\Lambda\Delta_q\tilde{d}\|_{L^2}^2+\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})|\Delta_q\tilde{d}\big)-2(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d})\\&\quad=(\Delta_q\tilde{\mathfrak{M}}|\Delta_q\tilde{d})-KV'\|\Delta_q\tilde{d}\|_{L^2}^2;\end{aligned}\tag{4.9}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta_q\tilde{\omega}\|_{L^2}+\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{\omega})|\Delta_q\tilde{\omega}\big)+\mu\|\Lambda\Delta_q\tilde{\omega}\|_{L^2}^2+\big(\Lambda\Delta_q\big(\tilde{E}-\tilde{E}^\top\big)\big|\Delta_q\tilde{\omega}\big)\\&\quad=(\Delta_q\tilde{\mathfrak{N}}|\Delta_q\tilde{\omega})-KV'\|\Delta_q\tilde{\omega}\|_{L^2}^2;\end{aligned}\tag{4.10}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big\|_{L^2}^2+\big(\Delta_q\big(\mathbf{u}\cdot\nabla\big(\tilde{E}^\top-\tilde{E}\big)\big)\big|\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big)+\big(\Lambda\Delta_q\tilde{\omega}|\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big)\\&\quad=\big(\Delta_q\big(\tilde{\mathfrak{Q}}^\top-\tilde{\mathfrak{Q}}\big)\big|\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big)-KV'\big\|\big(\tilde{E}^\top-\tilde{E}\big)\big\|_{L^2}^2.\end{aligned}\tag{4.11}$$

And, we also have, from the fifth and sixth equations in (4.4)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta_q\tilde{d}\|_{L^2}^2+\nu\|\Lambda\Delta_q\tilde{d}\|_{L^2}^2+\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})|\Delta_q\tilde{d}\big)-2(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d})\\&\quad=(\Delta_q\tilde{\mathfrak{J}}|\Delta_q\tilde{d})-KV'\|\Delta_q\tilde{d}\|_{L^2}^2;\end{aligned}\tag{4.12}$$

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2+\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{\mathcal{E}})|\Delta_q\tilde{\mathcal{E}}\big)+2(\Lambda\Delta_q\tilde{d}|\Delta_q\tilde{\mathcal{E}})=(\Delta_q\tilde{\mathfrak{K}}|\Delta_q\tilde{\mathcal{E}})-KV'\|\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2.\tag{4.13}$$

For estimates of the term $(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d})$, we apply Λ to the first equation in (4.4) and take the L^2-scalar product with $\Delta_q\tilde{d}$, then take the scalar product of the second equation with $\Lambda\Delta_q\tilde{\rho}$ and sum both equalities to yield

$$\frac{d}{dt}(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d}) + \|\Lambda\Delta_q\tilde{d}\|_{L^2}^2 - 2\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2 + \big(\Lambda\Delta_q\tilde{\rho}|\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})\big)$$
$$+ \big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\rho})|\Delta_q\tilde{d}\big) + \nu\big(\Lambda^2\Delta_q\tilde{d}|\Lambda\Delta_q\tilde{\rho}\big)$$
$$= (\Lambda\Delta_q\tilde{\mathfrak{L}}|\Delta_q\tilde{d}) + (\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{\mathfrak{M}}) - 2KV'(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d}). \tag{4.14}$$

For estimates of the term $(\Lambda\Delta_q(\tilde{E}^\top - \tilde{E})|\Delta_q\tilde{\omega})$, we apply Λ to the fourth equation in (4.4) and take the L^2-scalar product with $\Delta_q\tilde{\omega}$, then take the scalar product of the third equation with $\Lambda\Delta_q(\tilde{E}^\top - \tilde{E})$ and sum both equalities to yield

$$\frac{d}{dt}\big(\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big|\Delta_q\tilde{\omega}\big) - \big\|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + \|\Lambda\Delta_q\tilde{\omega}\|_{L^2}^2 + \big(\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big|\Delta_q(\mathbf{u}\cdot\nabla\tilde{\omega})\big)$$
$$+ \big(\Lambda\Delta_q\big(\mathbf{u}\cdot\nabla\big(\tilde{E}^\top - \tilde{E}\big)\big)\big|\Delta_q\tilde{\omega}\big) + \mu\big(\Lambda^2\Delta_q\tilde{\omega}|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big)$$
$$= (\Lambda\Delta_q\tilde{\mathfrak{Q}}|\Delta_q\tilde{\omega}) + \big(\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big|\Delta_q\tilde{\mathfrak{N}}\big) - 2KV'\big(\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big|\Delta_q\tilde{\omega}\big). \tag{4.15}$$

For estimates of the term $(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d})$, we apply Λ to the last equation in (4.4) and take the L^2-scalar product with $\Delta_q\tilde{d}$, then take the scalar product of the fifth equation with $\Lambda\Delta_q\tilde{\mathcal{E}}$ and sum both equalities to yield

$$\frac{d}{dt}(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d}) + \|\Lambda\Delta_q\tilde{d}\|_{L^2}^2 - 2\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2 + \big(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})\big)$$
$$+ \big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\mathcal{E}})|\Delta_q\tilde{d}\big) + \nu\big(\Lambda^2\Delta_q\tilde{d}|\Lambda\Delta_q\tilde{\mathcal{E}}\big)$$
$$= (\Lambda\Delta_q\tilde{\mathfrak{K}}|\Delta_q\tilde{d}) + (\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{\mathfrak{J}}) - 2KV'(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d}). \tag{4.16}$$

Taking linear combination of (4.8)–(4.16), we obtain

$$\frac{1}{2}\frac{d}{dt}f_q^2 + \frac{\nu}{2\eta}\big(2\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2 + \big\|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + 2\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2 + (2\eta - 3)\|\Lambda\Delta_q\tilde{d}\|_{L^2}$$
$$- \big(\nu\Lambda^2\Delta_q\tilde{d}|\Lambda\Delta_q\tilde{\rho}\big) - \big(\mu\Lambda^2\Delta_q\tilde{\omega}|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big) - \nu\big(\Lambda^2\Delta_q\tilde{d}|\Lambda\Delta_q\tilde{\mathcal{E}}\big)\big)\big) + KV'f_q^2$$
$$+ \left(\mu - \frac{\nu}{2\eta}\right)\|\Lambda\Delta_q\tilde{\omega}\|_{L^2}^2$$
$$= \mathcal{X}, \tag{4.17}$$

where

$$\mathcal{X} := 2(\Delta_q\tilde{\mathfrak{L}}|\Delta_q\tilde{\rho}) + (\Delta_q\tilde{\mathfrak{M}}|\Delta_q\tilde{d}) + (\Delta_q\tilde{\mathfrak{N}}|\Delta_q\tilde{\omega}) + \big(\Delta_q\tilde{\mathfrak{Q}}|\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big) + (\Delta_q\tilde{\mathfrak{J}}|\Delta_q\tilde{d})$$
$$+ (\Delta_q\tilde{\mathfrak{K}}|\Delta_q\tilde{\mathcal{E}}) - 2\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{\rho})|\Delta_q\tilde{\rho}\big) - 2\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})|\Delta_q\tilde{d}\big)$$
$$- \big(\Delta_q\big(\mathbf{u}\cdot\nabla\big(\tilde{E}^\top - \tilde{E}\big)\big)\big|\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big) - \big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{\omega})|\Delta_q\tilde{\omega}\big) - \big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{\mathcal{E}})|\Delta_q\tilde{\mathcal{E}}\big)$$
$$+ \frac{\nu}{2\eta}\big\{\big(\Lambda\Delta_q\tilde{\rho}|\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})\big) + \big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\rho})|\Delta_q\tilde{d}\big) + \big(\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big|\Delta_q(\mathbf{u}\cdot\nabla\tilde{\omega})\big)$$
$$+ \big(\Lambda\Delta_q\big(\mathbf{u}\cdot\nabla\big(\tilde{E}^\top - \tilde{E}\big)\big)\big|\Delta_q\tilde{\omega}\big) - (\Lambda\Delta_q\tilde{\mathfrak{L}}|\Delta_q\tilde{d}) - \big(\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big|\Delta_q\tilde{\mathfrak{N}}\big)$$
$$- (\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{\mathfrak{M}}) - (\Lambda\Delta_q\tilde{\mathfrak{Q}}|\Delta_q\tilde{\omega}) + \big(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})\big) + \big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\mathcal{E}})|\Delta_q\tilde{d}\big)$$
$$- (\Lambda\Delta_q\tilde{K}|\Delta_q\tilde{d}) - (\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{J})\big\}.$$

As $q \leqslant q_0$, there exists a constant $c_0 \geqslant 1$, which depends on λ, μ, such that

$$\frac{1}{c_0} f_q^2 \leqslant \|\Delta_q \tilde{\rho}\|_{L^2}^2 + \|\Delta_q \tilde{d}\|_{L^2}^2 + \big\|\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + \|\Delta_q \tilde{\mathcal{E}}\|_{L^2}^2 + \|\Delta_q \tilde{\omega}\|_{L^2}^2 \leqslant c_0 f_q^2, \tag{4.18}$$

and this equivalence implies that there exists a universal positive constant κ depending on λ and μ, such that

$$\begin{aligned} &\frac{\nu}{2\eta}\big(2\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2 + \big\|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + 2\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2 + (2\eta-3)\|\Lambda\Delta_q\tilde{d}\|_{L^2} \\ &\quad - \big(\nu\Lambda^2\Delta_q\tilde{d}|\Lambda\Delta_q\tilde{\rho}\big) - \big(\mu\Lambda^2\Delta_q\tilde{\omega}|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big) - \nu\big(\Lambda^2\Delta_q\tilde{d}|\Lambda\Delta_q\tilde{\mathcal{E}}\big)\big)\big) \\ &\quad + \left(\mu - \frac{\nu}{2\eta}\right)\|\Lambda\Delta_q\tilde{\omega}\|_{L^2}^2 \\ &\quad \geqslant \kappa 2^{2q}\big(\|\Delta_q\tilde{\rho}\|_{L^2}^2 + \big\|\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + \|\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2 + \|\Delta_q\tilde{\omega}\|_{L^2}^2 + \|\Delta_q\tilde{d}\|_{L^2}^2\big). \end{aligned} \tag{4.19}$$

For terms on the right-hand side of (4.17), we use Lemma 2.2, (4.18), and the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} |\mathcal{X}| &\leqslant C f_q\big(\|\Delta_q\tilde{\mathfrak{L}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{M}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{N}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{Q}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{K}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{J}}\|_{L^2} \\ &\quad + 2^{-q(\frac{N}{2}-1)}\alpha_q V'\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \big\|\tilde{E}^\top - \tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} \\ &\quad + \|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big)\big). \end{aligned} \tag{4.20}$$

Hence, combining (4.17)–(4.20) together, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} f_q^2 + \kappa 2^{2q}\big(\|\Delta_q\tilde{\rho}\|_{L^2}^2 + \big\|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + \|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2 + \|\Delta_q\tilde{\omega}\|_{L^2}^2 + \|\Delta_q\tilde{d}\|_{L^2}^2\big) \\ &\quad \leqslant C f_q\big(\|\Delta_q\tilde{\mathfrak{L}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{M}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{N}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{Q}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{K}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{J}}\|_{L^2} \\ &\quad + 2^{-q(\frac{N}{2}-1)}\alpha_q V'\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \big\|\tilde{E}^\top - \tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big)\big) \\ &\quad - KV' f_q^2. \end{aligned} \tag{4.21}$$

Second step: High frequencies. In this step, we assume $q > q_0$. We apply the operator Λ to the first equation of (4.4), multiply by $\Lambda\Delta_q\tilde{\rho}$ and integrate over $\mathbb{R}^N$ to yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2 + \big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\rho})|\Lambda\Delta_q\tilde{\rho}\big) + \big(\Lambda^2\Delta_q\tilde{d}|\Lambda\Delta_q\tilde{\rho}\big) \\ &\quad = (\Lambda\Delta_q\tilde{\mathfrak{L}}|\Lambda\Delta_q\tilde{\rho}) - KV'\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2. \end{aligned} \tag{4.22}$$

Applying the operator Λ to the fourth equation of (4.4), multiplying by $\Lambda\Delta_q(\tilde{E}^\top - \tilde{E})$ and integrating over $\mathbb{R}^N$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big\|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2 + \big(\Lambda\Delta_q\big(\mathbf{u}\cdot\nabla\big(\tilde{E}^\top - \tilde{E}\big)\big)\big|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big) + \big(\Lambda^2\Delta_q\tilde{\omega}|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big) \\ &\quad = \big(\Lambda\Delta_q\tilde{\mathfrak{Q}}|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big) - KV'\big\|\Lambda\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}^2. \end{aligned} \tag{4.23}$$

Applying the operator Λ to the sixth equation of (4.4), multiplying by $\Lambda\Delta_q\tilde{\mathcal{E}}$ and integrating over $\mathbb{R}^N$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2+\big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\mathcal{E}})|\Lambda\Delta_q\tilde{\mathcal{E}}\big)+2\big(\Lambda^2\Delta_q\tilde{d}|\Lambda\Delta_q\tilde{\mathcal{E}}\big)$$
$$=(\Lambda\Delta_q\tilde{\mathfrak{K}}|\Lambda\Delta_q\tilde{\mathcal{E}})-KV'\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2. \tag{4.24}$$

Denote

$$f_q^2=\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2+\big\|\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big\|_{L^2}^2+\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2+2\gamma\|\Delta_q\tilde{d}\|_{L^2}^2+\gamma\|\Delta_q\tilde{\omega}\|_{L^2}^2$$
$$-\beta_1(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d})-\beta_2\big(\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big|\Delta_q\tilde{\omega}\big)-2\beta_1(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d}).$$

Combining (4.9), (4.10), (4.12), (4.14), (4.15), (4.22), and (4.24), we obtain

$$\frac{1}{2}\frac{d}{dt}f_q^2+\beta_1\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2+2\beta_1\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2+\left(2\gamma\nu-\frac{5\beta_1}{2}\right)\|\Lambda\Delta_q\tilde{d}\|_{L^2}^2+\left(\mu\gamma-\frac{\beta_2}{2}\right)\|\Lambda\Delta_q\tilde{\omega}\|_{L^2}^2$$
$$+\frac{\beta_2}{2}\big\|\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big\|_{L^2}^2-2\gamma(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d})-2\gamma(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d})$$
$$-\gamma\big(\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big|\Delta_q\tilde{\omega}\big)+KV'f_q^2$$
$$=\mathcal{Y}, \tag{4.25}$$

where

$$\mathcal{Y}=\gamma(\Delta_q\tilde{\mathfrak{M}}|\Delta_q\tilde{d})+\gamma(\Delta_q\tilde{\mathfrak{J}}|\Delta_q\tilde{d})+\gamma(\Delta_q\tilde{\mathfrak{N}}|\Delta_q\tilde{\omega})-2\gamma\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})|\Delta_q\tilde{d}\big)$$
$$-\gamma\big(\Delta_q(\mathbf{u}\cdot\nabla\tilde{\omega})|\Delta_q\tilde{\omega}\big)+\frac{\beta_1}{2}\big(\big(\Lambda\Delta_q\tilde{\rho}|\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})\big)+\big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\rho})|\Delta_q\tilde{d}\big)-(\Lambda\Delta_q\tilde{\mathfrak{L}}|\Delta_q\tilde{d})$$
$$-(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{\mathfrak{M}})+2\big(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q(\mathbf{u}\cdot\nabla\tilde{d})\big)+2\big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\mathcal{E}})|\Delta_q\tilde{d}\big)-2(\Lambda\Delta_q\tilde{\mathfrak{K}}|\Delta_q\tilde{d})$$
$$-2(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{\mathfrak{J}})\big)+\frac{\beta_2}{2}\big(\big(\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big|\Delta_q(\mathbf{u}\cdot\nabla\tilde{\omega})\big)+\big(\Lambda\Delta_q\big(\mathbf{u}\cdot\nabla\big(\tilde{E}^\top-\tilde{E}\big)\big)\big|\Delta_q\tilde{\omega}\big)$$
$$-(\Lambda\Delta_q\tilde{\mathfrak{Q}}|\Delta_q\tilde{\omega})-\big(\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big|\Delta_q\tilde{\mathfrak{N}}\big)\big)-\big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\rho})|\Lambda\Delta_q\tilde{\rho}\big)$$
$$+(\Lambda\Delta_q\tilde{\mathfrak{L}}|\Lambda\Delta_q\tilde{\rho})-\big(\Lambda\Delta_q\big(\mathbf{u}\cdot\nabla\big(\tilde{E}^\top-\tilde{E}\big)\big)\big|\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big)+\big(\Lambda\Delta_q\tilde{\mathfrak{Q}}|\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big)$$
$$-\big(\Lambda\Delta_q(\mathbf{u}\cdot\nabla\tilde{\mathcal{E}})|\Lambda\Delta_q\tilde{\mathcal{E}}\big)+(\Lambda\Delta_q\tilde{\mathfrak{K}}|\Lambda\Delta_q\tilde{\mathcal{E}}).$$

Notice that, for $q\geqslant q_0$, we have

$$f_q^2\approx\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2+\|\Lambda\Delta_q\tilde{E}\|_{L^2}^2+\|\Delta_q\tilde{d}\|_{L^2}^2+\|\Delta_q\tilde{\omega}\|_{L^2}^2; \tag{4.26}$$

and

$$\beta_1\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2+2\beta_1\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2+\left(2\gamma\nu-\frac{5\beta_1}{2}\right)\|\Lambda\Delta_q\tilde{d}\|_{L^2}^2+\left(\mu\gamma-\frac{\beta_2}{2}\right)\|\Lambda\Delta_q\tilde{\omega}\|_{L^2}^2$$
$$+\frac{\beta_2}{2}\big\|\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big\|_{L^2}^2-2\gamma(\Lambda\Delta_q\tilde{\rho}|\Delta_q\tilde{d})-2\gamma(\Lambda\Delta_q\tilde{\mathcal{E}}|\Delta_q\tilde{d})-\gamma\big(\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big|\Delta_q\tilde{\omega}\big)$$
$$\approx\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2+\|\Delta_q\tilde{d}\|_{L^2}^2+\|\Delta_q\tilde{\omega}\|_{L^2}^2+\big\|\Lambda\Delta_q\big(\tilde{E}^\top-\tilde{E}\big)\big\|_{L^2}^2+\|\Lambda\Delta_q\tilde{\mathcal{E}}\|_{L^2}^2. \tag{4.27}$$

Next, we apply Lemma 2.2 to obtain, using (4.26)

$$|\mathcal{Y}| \leqslant C f_q\big(\|\Lambda\Delta_q\tilde{\mathfrak{L}}\|_{L^2} + \|\Lambda\Delta_q\tilde{\mathfrak{Q}}\|_{L^2} + \|\Lambda\Delta_q\tilde{\mathfrak{K}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{J}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{M}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{N}}\|_{L^2} + \alpha_q 2^{-q(\frac{N}{2}-1)} V'\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \big\|\tilde{E}^\top - \tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big)\big). \tag{4.28}$$

Therefore, there exists a universal positive constant κ, which depends on μ and ν, such that

$$\frac{1}{2}\frac{d}{dt} f_q^2 + \kappa\big(\|\Lambda\Delta_q\tilde{\rho}\|_{L^2}^2 + \|\Lambda\Delta_q\tilde{E}\|_{L^2}^2 + \|\Delta_q\tilde{d}\|_{L^2}^2 + \|\Delta_q\tilde{\omega}\|_{L^2}^2\big) \leqslant C f_q\big(\|\Lambda\Delta_q\tilde{\mathfrak{L}}\|_{L^2} + \|\Lambda\Delta_q\tilde{\mathfrak{Q}}\|_{L^2} + \|\Lambda\Delta_q\tilde{\mathfrak{K}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{J}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{M}}\|_{L^2} + \|\Delta_q\tilde{\mathfrak{N}}\|_{L^2} + \alpha_q 2^{-q(\frac{N}{2}-1)} V'\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \big\|\tilde{E}^\top - \tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big)\big) - KV' f_q^2. \tag{4.29}$$

Third step: Damping effect. We are now going to show that inequality (4.7) entails a decay for ρ, E, d and ω. We postpone the proof of smoothing properties for d and ω to the next step. Let $\delta > 0$ be a small parameter (which will tend to 0) and denote $h_q^2 = g_q^2 + \delta^2$. From (4.7), and dividing by h_q, we obtain

$$\frac{d}{dt} h_q + \kappa h_q \leqslant C\alpha_q\big(\|\tilde{\mathfrak{L}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathfrak{Q}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathfrak{K}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathfrak{J}}\|_{B^{\frac{N}{2}-1}} + \|\tilde{\mathfrak{M}}\|_{B^{\frac{N}{2}-1}} + \|\tilde{\mathfrak{N}}\|_{B^{\frac{N}{2}-1}}\big) + C\alpha_q V'\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \big\|\tilde{E}^\top - \tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big) - KV' h_q + \delta KV' + \delta\kappa.$$

Integrating over $[0, t]$ and having δ tend to 0, we obtain

$$g_q(t) + \kappa\int_0^t g_q(\tau)\,d\tau \leqslant g_q(0) + C\int_0^t \alpha_q(s)\big(\|\tilde{\mathfrak{L}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathfrak{K}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathfrak{J}}\|_{B^{\frac{N}{2}-1}} + \|\tilde{\mathfrak{M}}\|_{B^{\frac{N}{2}-1}} + \|\tilde{\mathfrak{N}}\|_{B^{\frac{N}{2}-1}} + \|\tilde{\mathfrak{Q}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\big)\,ds + \int_0^t V'(s)\big\{C\alpha_q(s)\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \big\|\tilde{E}^\top - \tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big) - Kg_q(s)\big\}\,ds. \tag{4.30}$$

Thanks to (4.6), we have

$$C\alpha_q(s)\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \big\|\tilde{E}^\top - \tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big) - Kg_q(s) \leqslant C\alpha_q(s)\big\|\tilde{\rho}(s)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} - KC^{-1}2^{\phi^{\frac{N}{2}-1,\frac{N}{2}}(q)}\|\Delta_q\tilde{\rho}\|_{L^2} + C\alpha_q(s)\big\|\tilde{E}^\top(s) - \tilde{E}(s)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} - KC^{-1}2^{\phi^{\frac{N}{2}-1,\frac{N}{2}}(q)}\big\|\Delta_q\big(\tilde{E}^\top - \tilde{E}\big)\big\|_{L^2}$$

$$+ C\alpha_q(s)\big\|\tilde{\mathcal{E}}(s)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} - KC^{-1}2^{\phi^{\frac{N}{2}-1,\frac{N}{2}}(q)}\|\Delta_q\tilde{\mathcal{E}}\|_{L^2}$$

$$+ C\alpha_q(s)\big\|\tilde{d}(s)\big\|_{B^{\frac{N}{2}-1}} - KC^{-1}2^{q(\frac{N}{2}-1)}\|\Delta_q\tilde{d}\|_{L^2}$$

$$+ C\alpha_q(s)\big\|\tilde{\omega}(s)\big\|_{B^{\frac{N}{2}-1}} - KC^{-1}2^{q(\frac{N}{2}-1)}\|\Delta_q\tilde{\omega}\|_{L^2}.$$

If we choose $K \geqslant C^2$, we have

$$\sum_{q\in\mathbb{Z}}\big\{C\alpha_q(s)\big(\|\tilde{\rho}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \big\|\tilde{E}^\top - \tilde{E}\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\mathcal{E}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big) - Kg_q(s)\big\} \leqslant 0.$$

According to the last inequality, and thanks to (4.6) and (4.30), we conclude, after summation on $\mathbb{Z}$, that

$$\begin{aligned}
&\big\|\tilde{\rho}(t)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \big\|\tilde{E}(t)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \big\|\tilde{\omega}(t)\big\|_{B^{\frac{N}{2}-1}} + \big\|\tilde{d}(t)\big\|_{B^{\frac{N}{2}-1}} \\
&\quad + \kappa\left(\int_0^t \big(\big\|\tilde{\rho}(\tau)\big\|_{\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}}} + \big\|\tilde{E}(\tau)\big\|_{\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}}} + \big\|\tilde{d}(\tau)\big\|_{\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}-1}} + \big\|\tilde{\omega}(\tau)\big\|_{\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}-1}}\big)\,d\tau\right) \\
&\leqslant C\Bigg\{\|\rho_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|E_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\omega_0\|_{B^{\frac{N}{2}-1}} + \|d_0\|_{B^{\frac{N}{2}-1}} + \int_0^t \big(\big\|\tilde{\mathfrak{L}}(s)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} \\
&\quad + \big\|\tilde{\mathfrak{M}}(s)\big\|_{B^{\frac{N}{2}-1}} + \big\|\tilde{\mathfrak{J}}(s)\big\|_{B^{\frac{N}{2}-1}} + \|\tilde{\mathfrak{K}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \big\|\tilde{\mathfrak{N}}(s)\big\|_{B^{\frac{N}{2}-1}} + \|\tilde{\mathfrak{Q}}\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\big)\,ds\Bigg\}.
\end{aligned}\tag{4.31}$$

Fourth step: Smoothing effect. Once showing the damping effect for ρ and E, we can further get the smoothing effect on d and ω for the system (4.1) by considering the second and third equations with terms $\Lambda\rho$ and ΛE being seen as given source terms. Indeed, thanks to (4.31), it suffices to state the proof for high frequencies only. We therefore assume in this step $q \geqslant q_0$. Define

$$I_q = 2^{q(\frac{N}{2}-1)}\|\Delta_q\tilde{d}\|_{L^2} + 2^{q(\frac{N}{2}-1)}\|\Delta_q\tilde{\omega}\|_{L^2}.$$

Then, from the energy estimates for the system

$$\begin{cases}\partial_t\Delta_q\tilde{d} + \Delta_q(\mathbf{u}\cdot\nabla\tilde{d}) - \nu\Delta\Delta_q\tilde{d} = 2\Lambda\Delta_q\tilde{\rho} + \Delta_q\tilde{\mathfrak{M}} - KV'(t)\Delta_q\tilde{d},\\ \partial_t\Delta_q\tilde{\omega} + \Delta_q(\mathbf{u}\cdot\nabla\tilde{\omega}) - \mu\Delta\Delta_q\tilde{\omega} = \mathcal{R}\Delta_q\tilde{E} + \Delta_q\tilde{\mathfrak{N}} - KV'(t)\Delta_q\tilde{\omega},\end{cases}$$

we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}I_q^2 + \kappa 2^{2q}I_q^2 &\leqslant I_q\big(2^{q\frac{N}{2}}\|\Delta_q\tilde{\rho}\|_{L^2} + 2^{q(\frac{N}{2}-1)}\|\Delta_q\tilde{\mathfrak{M}}\|_{L^2} + 2^{q(\frac{N}{2}-1)}\|\Delta_q\tilde{\mathfrak{N}}\|_{L^2} + 2^{q\frac{N}{2}}\|\Delta_q E\|_{L^2}\big) \\
&\quad + I_qV'(t)\big(C\alpha_q\big(\|\tilde{d}\|_{B^{\frac{N}{2}-1}} + \|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big) - KI_q\big),
\end{aligned}$$

for a universal positive constant κ. Using $J_q^2 = I_q^2 + \delta^2$, integrating over $[0,t]$ and then taking the limit as $\delta \to 0$, we deduce

$$
\begin{aligned}
& I_q(t)+\kappa 2^{2q}\int_0^t I_q(s)\,ds \\
& \quad\leqslant I_q(0)+\int_0^t 2^{q(\frac{N}{2}-1)}\big(\big\|\Delta_q\tilde{\mathfrak{M}}(s)\big\|_{L^2}+\big\|\Delta_q\tilde{\mathfrak{N}}(s)\big\|_{L^2}\big)\,ds+\int_0^t 2^{q\frac{N}{2}}\big(\big\|\Delta_q\tilde{\rho}(s)\big\|_{L^2}+\big\|\Delta_q\tilde{E}(s)\big\|_{L^2}\big)\,ds \\
& \qquad + C\int_0^t V'(s)\alpha_q(s)\big(\big\|\tilde{d}(s)\big\|_{B^{\frac{N}{2}-1}}+\big\|\tilde{\omega}(s)\big\|_{B^{\frac{N}{2}-1}}\big)\,ds.
\end{aligned}
\tag{4.32}
$$

We therefore get

$$
\begin{aligned}
& \sum_{q\geqslant q_0}2^{q(\frac{N}{2}-1)}\big(\big\|\Delta_q\tilde{d}(t)\big\|_{L^2}+\big\|\Delta_q\tilde{\omega}(t)\big\|_{L^2}\big)+\kappa\int_0^t\sum_{q\geqslant q_0}2^{q(\frac{N}{2}+1)}\big(\big\|\Delta_q\tilde{d}(s)\big\|_{L^2}+\big\|\Delta_q\tilde{\omega}(t)\big\|_{L^2}\big)\,ds \\
& \quad\leqslant \|d_0\|_{B^{\frac{N}{2}-1}}+\|\omega_0\|_{B^{\frac{N}{2}-1}}+\int_0^t\big(\big\|\tilde{\mathfrak{M}}(s)\big\|_{B^{\frac{N}{2}-1}}+\big\|\tilde{\mathfrak{N}}(s)\big\|_{B^{\frac{N}{2}-1}}\big)\,ds \\
& \qquad+\int_0^t\sum_{q\geqslant q_0}2^{q\frac{N}{2}}\big(\big\|\Delta_q\tilde{\rho}(s)\big\|_{L^2}+\big\|\Delta_q\tilde{E}(s)\big\|_{L^2}\big)\,ds+CV(t)\sup_{s\in[0,t]}\big(\|\tilde{d}\|_{B^{\frac{N}{2}-1}}+\|\tilde{\omega}\|_{B^{\frac{N}{2}-1}}\big).
\end{aligned}
$$

Using (4.31), we eventually conclude that

$$
\begin{aligned}
& \kappa\int_0^t\sum_{q\geqslant q_0}2^{q(\frac{N}{2}+1)}\big(\big\|\Delta_q\tilde{d}(s)\big\|_{L^2}+\big\|\Delta_q\tilde{d}(s)\big\|_{L^2}\big)\,ds \\
& \quad\leqslant C\big(1+V(t)\big)\bigg\{\|\rho_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|E_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|\omega_0\|_{B^{\frac{N}{2}-1}}+\|d_0\|_{B^{\frac{N}{2}-1}} \\
& \qquad+\int_0^t\big(\big\|\tilde{\mathfrak{L}}(s)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\big\|\tilde{\mathfrak{M}}(s)\big\|_{B^{\frac{N}{2}-1}}+\big\|\tilde{\mathfrak{N}}(s)\big\|_{B^{\frac{N}{2}-1}}+\big\|\tilde{\mathfrak{K}}(s)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} \\
& \qquad+\big\|\tilde{\mathfrak{Q}}(s)\big\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\big\|\tilde{\mathfrak{J}}(s)\big\|_{B^{\frac{N}{2}-1}}\big)\,ds\bigg\}.
\end{aligned}
$$

Combining the last inequality with (4.31), we finish the proof of Proposition 4.1. □

In the rest of this section, we will sketch the proof of the existence of a unique global solution to (4.1). To this end, we only need to use some properties of transport and heat equations in nonhomogeneous Sobolev spaces H^s with high regularity order, see [1,2]. Indeed, we set $(\rho^0, d^0, \omega^0, E^0) = (\rho_0, d_0, \omega_0, E_0)$ and

$$\begin{cases} \partial_t \rho^{n+1} + \mathbf{u} \cdot \nabla \rho^{n+1} = -\Lambda d^n + \mathfrak{L}, \\ \partial_t d^{n+1} - \nu \Delta d^{n+1} = \mathfrak{M} - \mathbf{u} \cdot \nabla d^n + 2\Lambda \rho^n, \\ \partial_t \omega^{n+1} - \mu \Delta \omega^{n+1} = \mathfrak{N} + \mathbf{u} \cdot \nabla \omega^n + \Lambda\big(\big(E^n\big)^\top - E^n\big), \\ \partial_t\big(\big(E^{n+1}\big)^\top - E^{n+1}\big) + \mathbf{u} \cdot \nabla\big(\big(E^{n+1}\big)^\top - E^{n+1}\big) = -\Lambda \omega^n + \mathfrak{Q}, \\ \partial_t \mathcal{E}^{n+1} + \mathbf{u} \cdot \nabla \mathcal{E}^{n+1} = -2\Lambda d^n + \mathfrak{K}, \end{cases} \tag{4.33}$$

where

$$\big(\rho^{n+1}, d^{n+1}, \omega^{n+1}, E^{n+1}\big)\big|_{t=0} = (\rho_0, d_0, \omega_0, E_0).$$

Let $T > 0$ and s (large enough) be fixed, and let K be a suitably large positive constant (depending on s, T and $\mathbf{u}$). We set

$$\tilde{\rho}^n = e^{-Kt}\rho^n, \qquad \tilde{d}^n = e^{-Kt} d^n, \qquad \tilde{\omega}^n = e^{-Kt}\omega^n, \qquad \tilde{E}^n = e^{-Kt} E^n.$$

The straightforward computations show that $\{(\tilde{\rho}^n, \tilde{d}^n, \tilde{\omega}^n, \tilde{E}^n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in

$$C\big([0,T]; H^s\big) \times \big(C\big([0,T]; H^{s-1} \times H^{s-1}\big) \cap L^1\big([0,T]; H^{s+1} \times H^{s+1}\big)\big)^{2N} \times C\big([0,T]; H^s\big)^{N\times N}.$$

Denoting by $(\tilde{\rho}, \tilde{d}, \tilde{\omega}, \tilde{E})$ the limit, then it is easy to show that $(e^{Kt}\tilde{\rho}, e^{Kt}\tilde{d}, e^{Kt}\tilde{\omega}, e^{Kt}\tilde{E})$ solves (4.1).

5. Global existence

The goal of this section is to prove the global existence of solutions to (1.3) by building approximating solutions $(\rho^n, \mathbf{u}^n, E^n)$ using an iterative method. Those approximate solutions are solutions of auxiliary systems of (4.1) to which Proposition 4.1 applies.

We set the first term $(\rho^0, \mathbf{u}^0, E^0)$ to $(0,0,0)$. We then define $\{(\rho^n, \mathbf{u}^n, E^n)\}_{n\in\mathbb{N}}$ by induction. In fact, we choose $(\rho^{n+1}, \mathbf{u}^{n+1}, E^{n+1})$ as the solution of the following system:

$$\begin{cases} \partial_t \rho^{n+1} + \mathbf{u}^n \cdot \nabla \rho^{n+1} + \Lambda d^{n+1} = \mathfrak{L}^n, \\ \partial_t d^{n+1} + \mathbf{u}^n \cdot \nabla d^{n+1} - \nu \Delta d^{n+1} - 2\Lambda \rho^{n+1} = \mathfrak{M}^n, \\ \partial_t \omega^{n+1} + \mathbf{u}^n \cdot \nabla \omega^{n+1} - \mu \Delta \omega^{n+1} - \Lambda\big(E^{n+1} - (E^{n+1})^\top\big) = \mathfrak{N}^n, \\ \partial_t\big(\big(E^{n+1}\big)^\top - E^{n+1}\big) + \mathbf{u} \cdot \nabla\big(\big(E^{n+1}\big)^\top - E^{n+1}\big) + \Lambda \omega^{n+1} = \mathfrak{Q}^n, \\ \partial_t \mathcal{E}^{n+1} + \mathbf{u} \cdot \nabla \mathcal{E}^{n+1} + 2\Lambda d^{n+1} = \mathfrak{K}^n, \\ \mathbf{u}^{n+1} = -\Lambda^{-1}\nabla d^{n+1} - \Lambda^{-1} \operatorname{curl} \omega^{n+1}, \\ \big(\rho^{n+1}, d^{n+1}, \omega^{n+1}, E^{n+1}\big)\big|_{t=0} = \big(\rho_n, \Lambda^{-1} \operatorname{div} \mathbf{u}_n, \Lambda^{-1} \operatorname{curl} \mathbf{u}_n, E_n\big), \end{cases} \tag{5.1}$$

where

$$\rho_n = \sum_{|q|\leqslant n} \Delta_q \rho_0, \qquad \mathbf{u}_n = \sum_{|q|\leqslant n} \Delta_q \mathbf{u}_0, \qquad E_n = \sum_{|q|\leqslant n} \Delta_q E_0,$$

$$\mathfrak{L}^n = -\rho^n \operatorname{div} \mathbf{u}^n,$$

$$\mathfrak{M}^n = \mathbf{u}^n \cdot \nabla d^n - \Lambda^{-1} \operatorname{div}\bigg(\mathbf{u}^n \cdot \nabla \mathbf{u}^n + K\big(\rho^n\big)\nabla \rho^n + \frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n - E^n_{jk}\partial_{x_j} E^n_{ik} + \operatorname{div}\big(\rho^n E^n\big)\bigg),$$

$$\mathfrak{N}^n = \mathbf{u}^n \cdot \nabla\omega^n - \Lambda^{-1}\operatorname{curl}\left(\mathbf{u}^n \cdot \nabla\mathbf{u}^n + K(\rho^n)\nabla\rho^n + \frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n - E^n_{jk}\partial_{x_j}E^n_{ik}\right),$$
$$\mathfrak{Q}^n = \left(\nabla\mathbf{u}^n E^n\right)^\top - \nabla\mathbf{u}^n E^n,$$

and

$$\begin{aligned}\mathfrak{K}^n = {} & \mathbf{u}^n \cdot \nabla\mathcal{E}^n - \Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}\left(\mathbf{u}\cdot\nabla\left(E^n_{ij}+E^n_{ji}\right)\right)\\ & + \Lambda^{-1}\partial_{x_i}\Lambda^{-1}\partial_{x_j}\left(\left(\nabla\mathbf{u}^n E^n\right)_{ij} + \left(\nabla\mathbf{u}^n E^n\right)_{ji}\right).\end{aligned}$$

As in (3.7), due to the divergence property of the deformation gradient, we can rewrite the second equation in (5.1) as

$$\partial_t d^{n+1} + \mathbf{u}\cdot\nabla d^{n+1} - \nu\Delta d^{n+1} - 2\Lambda\mathcal{E}^{n+1} = \mathfrak{J}^n, \tag{5.2}$$

where $\mathfrak{J}^n$ is given by

$$\mathfrak{J}^n = \mathbf{u}^n \cdot \nabla d^n - \Lambda^{-1}\operatorname{div}\left(\mathbf{u}^n \cdot \nabla\mathbf{u}^n + K(\rho^n)\nabla\rho^n + \frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n - E^n_{jk}\partial_{x_j}E^n_{ik} - \operatorname{div}\left(\rho^n E^n\right)\right).$$

The argument in the previous section guarantees that the system (5.1) is solvable and Remark 4.1 tells us that, in view of (5.2), the solution to (5.1) satisfies the estimates in Proposition 4.1.

5.1. Uniform estimates in the critical regularity case

In this subsection, we establish uniform estimates in $\mathfrak{B}^{\frac{N}{2}}$ for $(\rho^n, \mathbf{u}^n, E^n)$. Denote

$$\gamma = \|\rho_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\mathbf{u}_0\|_{B^{\frac{N}{2}-1}} + \|E_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}.$$

We are going to show that there exists a positive constant Γ such that, if γ is small enough, the following estimate holds for all $n \in \mathbb{N}$:

$$\left\|\left(\rho^n, \mathbf{u}^n, E^n\right)\right\|_{\mathfrak{B}^{\frac{N}{2}}} \leqslant \Gamma\gamma. \tag{$\mathfrak{P}_n$}$$

We will prove $(\mathfrak{P}_n)$ by the mathematical induction. Suppose that $(\mathfrak{P}_n)$ is satisfied and let us prove that $(\mathfrak{P}_{n+1})$ also holds.

According to Proposition 4.1 and the definition of $(\rho_n, \mathbf{u}_n, E_n)$, the following inequality holds

$$\begin{aligned}\left\|\left(\rho^{n+1}, \mathbf{u}^{n+1}, E^{n+1}\right)\right\|_{\mathfrak{B}^{\frac{N}{2}}} \leqslant {} & Ce^{CV^n}\Big(\|\rho_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\mathbf{u}_0\|_{B^{\frac{N}{2}-1}} + \|E_0\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \left\|\mathfrak{L}^n\right\|_{L^1(\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\\ & + \left\|\mathfrak{M}^n\right\|_{L^1(B^{\frac{N}{2}-1})} + \left\|\mathfrak{N}^n\right\|_{L^1(B^{\frac{N}{2}-1})} + \left\|\mathfrak{Q}^n\right\|_{L^1(\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\\ & + \left\|\mathfrak{K}^n\right\|_{L^1(\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}})} + \left\|\mathfrak{J}^n\right\|_{L^1(B^{\frac{N}{2}-1})}\Big)\end{aligned} \tag{5.3}$$

where

$$V^n = \int_0^\infty \left\|\mathbf{u}^n(s)\right\|_{B^{\frac{N}{2}+1}} ds.$$

Hence, it remains to obtain the estimates for $\mathfrak{L}^n$, $\mathfrak{M}^n$, $\mathfrak{N}^n$, $\mathfrak{K}^n$, $\mathfrak{J}^n$ and $\mathfrak{Q}^n$ by using $(\mathfrak{P}_n)$.

Estimate of $\mathfrak{L}^n$: The estimate of $\mathfrak{L}^n$ is straightforward; thanks to Proposition 2.2, we have

$$\begin{aligned}\|\mathfrak{L}^n\|_{L^1(\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}})} &\leqslant C\|\rho^n\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\operatorname{div}\mathbf{u}^n\|_{L^1(B^{\frac{N}{2}})}\\ &\leqslant C\Gamma^2\gamma^2.\end{aligned} \tag{5.4}$$

Estimates of $\mathfrak{M}^n$, $\mathfrak{N}^n$, and $\mathfrak{J}^n$: To estimate $\mathfrak{M}^n$, $\mathfrak{N}^n$, and $\mathfrak{J}^n$, we assume that γ satisfies

$$\gamma \leqslant \frac{1}{2\Gamma\mathfrak{C}},$$

where $\mathfrak{C}$ is the continuity modulus of $B^{\frac{N}{2}-1,\frac{N}{2}} \hookrightarrow L^\infty$. If $(\mathfrak{P}_n)$ holds, then

$$\|\rho^n\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^N)} \leqslant \mathfrak{C}\|\rho^n\|_{B^{\frac{N}{2}-1,\frac{N}{2}}} \leqslant \frac{1}{2}$$

and

$$\|E^n\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^N)} \leqslant \mathfrak{C}\|E^n\|_{B^{\frac{N}{2}-1,\frac{N}{2}}} \leqslant \frac{1}{2}.$$

We will estimate $\mathfrak{M}^n$ term by term. First we have, according to Proposition 2.2 and Lemma 2.1,

$$\begin{aligned}\left\|\frac{\rho^n}{1+\rho^n}\nabla^2\mathbf{u}^n\right\|_{L^1(B^{\frac{N}{2}-1})} &\leqslant C\|\nabla^2\mathbf{u}^n\|_{L^1(B^{\frac{N}{2}-1})}\left\|\frac{\rho^n}{1+\rho^n}\right\|_{L^\infty(B^{\frac{N}{2}})}\\ &\leqslant C\|\mathbf{u}^n\|_{L^1(B^{\frac{N}{2}+1})}\|\rho^n\|_{L^\infty(B^{\frac{N}{2}})} \leqslant C\Gamma^2\gamma^2.\end{aligned} \tag{5.5}$$

Since $K(0)=0$ and Lemma 2.1, one has

$$\|K(\rho^n)\nabla\rho^n\|_{L^1(B^{\frac{N}{2}-1})} \leqslant C\|K(\rho^n)\|_{L^2(B^{\frac{N}{2}})}\|\nabla\rho^n\|_{L^2(B^{\frac{N}{2}-1})} \leqslant C\|\rho^n\|^2_{L^2(B^{\frac{N}{2}})}.$$

On the other hand, we have, by Hölder's inequality,

$$\begin{aligned}\|\rho^n\|^2_{L^2(B^{\frac{N}{2}})} &= \int_0^\infty\left(\sum_{q\in\mathbb{Z}}\left(2^{q\phi^{\frac{N}{2}-1,\frac{N}{2}}(q)}\|\Delta_q\rho^n(t)\|_{L^2}\right)^{\frac{1}{2}}\left(2^{q\phi^{\frac{N}{2}+1,\frac{N}{2}}(q)}\|\Delta_q\rho^n(t)\|_{L^2}\right)^{\frac{1}{2}}\right)^2 dt\\ &\leqslant \int_0^\infty \|\rho^n(t)\|_{\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\|\rho^n(t)\|_{\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}}}\,dt\\ &\leqslant \|\rho^n\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\rho^n\|_{L^1(\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}})} \leqslant C\Gamma^2\gamma^2.\end{aligned}$$

Thus, the above two inequalities imply that

$$\|K(\rho^n)\nabla\rho^n\|_{L^1(B^{\frac{N}{2}-1})} \leqslant C\Gamma^2\gamma^2. \tag{5.6}$$

From Proposition 2.2, we obtain the following estimates:

$$\begin{aligned}
&\left\|\mathbf{u}^n\cdot\nabla d^n\right\|_{L^1(B^{\frac{N}{2}-1})}+\left\|\mathbf{u}^n\cdot\nabla \mathbf{u}^n\right\|_{L^1(B^{\frac{N}{2}-1})}+\left\|\mathbf{u}^n\cdot\nabla \omega^n\right\|_{L^1(B^{\frac{N}{2}-1})}\\
&\quad\leqslant C\int_0^\infty \left\|\mathbf{u}^n(t)\right\|_{B^{\frac{N}{2}-1}}\left(\left\|d^n(t)\right\|_{B^{\frac{N}{2}+1}}+\left\|\omega^n(t)\right\|_{B^{\frac{N}{2}+1}}\right)dt\\
&\quad\leqslant C\left\|\mathbf{u}^n\right\|_{L^\infty(B^{\frac{N}{2}-1})}\left(\left\|d^n\right\|_{L^1(B^{\frac{N}{2}+1})}+\left\|\omega^n\right\|_{L^1(B^{\frac{N}{2}+1})}\right)\\
&\quad\leqslant C\Gamma^2\gamma^2, \qquad (5.7)
\end{aligned}$$

$$\begin{aligned}
\left\|E^n_{jk}\partial_{x_j}E^n_{ik}\right\|_{L^1(B^{\frac{N}{2}-1})}&\leqslant C\int_0^\infty \left\|E^n(t)\right\|_{\tilde B^{\frac{N}{2}-1,\frac{N}{2}}}\left\|E^n(t)\right\|_{\tilde B^{\frac{N}{2}+1,\frac{N}{2}}}dt\\
&\leqslant C\left\|E^n\right\|_{L^\infty(\tilde B^{\frac{N}{2}-1,\frac{N}{2}})}\left\|E^n\right\|_{L^1(\tilde B^{\frac{N}{2}+1,\frac{N}{2}})}\\
&\leqslant C\Gamma^2\gamma^2, \qquad (5.8)
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\operatorname{div}\left(\rho^n E^n\right)\right\|_{L^1(B^{\frac{N}{2}-1})}\\
&\quad\leqslant C\left(\int_0^\infty \left\|\rho^n(t)\right\|_{\tilde B^{\frac{N}{2}-1,\frac{N}{2}}}\left\|E^n(t)\right\|_{\tilde B^{\frac{N}{2}+1,\frac{N}{2}}}dt+\int_0^\infty \left\|E^n(t)\right\|_{\tilde B^{\frac{N}{2}-1,\frac{N}{2}}}\left\|\rho^n(t)\right\|_{\tilde B^{\frac{N}{2}+1,\frac{N}{2}}}dt\right)\\
&\quad\leqslant C\left(\left\|\rho^n\right\|_{L^\infty(\tilde B^{\frac{N}{2}-1,\frac{N}{2}})}\left\|E^n\right\|_{L^1(\tilde B^{\frac{N}{2}+1,\frac{N}{2}})}+\left\|E^n\right\|_{L^\infty(\tilde B^{\frac{N}{2}-1,\frac{N}{2}})}\left\|\rho^n\right\|_{L^1(\tilde B^{\frac{N}{2}+1,\frac{N}{2}})}\right)\\
&\quad\leqslant C\Gamma^2\gamma^2. \qquad (5.9)
\end{aligned}$$

Summarizing (5.5)–(5.9), we finally get

$$\left\|\mathfrak{M}^n\right\|_{L^1(B^{\frac{N}{2}-1})}+\left\|\mathfrak{N}^n\right\|_{L^1(B^{\frac{N}{2}-1})}+\left\|\mathfrak{I}^n\right\|_{L^1(B^{\frac{N}{2}-1})}\leqslant C\Gamma^2\gamma^2. \qquad (5.10)$$

Estimates of $\mathfrak{Q}^n$ and $\mathfrak{K}^n$: It is easy to obtain, using Proposition 2.2,

$$\begin{aligned}
\left\|\nabla\mathbf{u}^n E^n\right\|_{L^1(\tilde B^{\frac{N}{2}-1,\frac{N}{2}})}&\leqslant C\int_0^\infty \left\|\mathbf{u}^n(t)\right\|_{B^{\frac{N}{2}+1}}\left\|E^n(t)\right\|_{\tilde B^{\frac{N}{2}-1,\frac{N}{2}}}dt\\
&\leqslant C\left\|\mathbf{u}^n\right\|_{L^1(B^{\frac{N}{2}+1})}\left\|E^n\right\|_{L^\infty(\tilde B^{\frac{N}{2}-1,\frac{N}{2}})}\\
&\leqslant C\Gamma^2\gamma^2. \qquad (5.11)
\end{aligned}$$

Hence

$$\left\|\mathfrak{Q}^n\right\|_{\tilde B^{\frac{N}{2}-1,\frac{N}{2}}}+\left\|\mathfrak{K}^n\right\|_{\tilde B^{\frac{N}{2}-1,\frac{N}{2}}}\leqslant C\Gamma\gamma.$$

From (5.4), (5.10) and (5.11), we finally have

$$\left\|\left(\rho^{n+1},\mathbf{u}^{n+1},E^{n+1}\right)\right\|_{\mathfrak{B}^{\frac{N}{2}}}\leqslant Ce^{C\Gamma\gamma}\left(\Gamma^2\gamma^2+\gamma\right). \qquad (5.12)$$

Now we choose $\Gamma = 4C$ and choose γ such that

$$\Gamma^2\gamma \leqslant 1, \qquad e^{C\Gamma\gamma} \leqslant 2 \quad \text{and} \quad \Gamma\gamma \leqslant \frac{1}{2\mathfrak{C}}, \tag{$\mathfrak{H}$}$$

then $(\mathfrak{P}_n)$ holds for all $n \in \mathbb{N}$.

5.2. Existence of a solution

In this subsection, we show that, up to a subsequence, the sequence $(\rho^n, \mathbf{u}^n, E^n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$ to a solution $(\rho, \mathbf{u}, E)$ of (1.3). We will use some compactness arguments. The starting point is to show that the first-order time derivative of $(\rho^n, \mathbf{u}^n, E^n)$ is uniformly bounded in appropriate spaces. This enables us to apply the Ascoli–Arzela theorem and get the existence of a limit $(\rho, \mathbf{u}, E)$ for a subsequence. Then, the uniform bounds of the previous subsection provide us with some additional regularity and convergence properties so that we may pass to the limit in the system (5.1).

To begin with, we have to prove uniform bounds in the suitable functional spaces and the convergence of $\{(\rho^n, \mathbf{u}^n, E^n)\}_{n\in\mathbb{N}}$. The uniform bounds are summarized in the following lemma.

Lemma 5.1. $\{(\rho^n, \mathbf{u}^n, E^n)\}_{n\in\mathbb{N}}$ *is uniformly bounded in*

$$C^{\frac{1}{2}}_{\mathrm{loc}}\big(\mathbb{R}^+; B^{\frac{N}{2}-1}\big) \times \big(C^{\frac{1}{4}}_{\mathrm{loc}}\big(\mathbb{R}^+; B^{\frac{N}{2}-\frac{3}{2}}\big)\big)^N \times \big(C^{\frac{1}{2}}_{\mathrm{loc}}\big(\mathbb{R}^+; B^{\frac{N}{2}-1}\big)\big)^{N\times N}$$

(*and also in* $C^{\frac{1}{2}}_{\mathrm{loc}}(B^{\frac{N}{2}-1} \times (B^{\frac{N}{2}-2})^N \times (B^{\frac{N}{2}-1})^{N\times N})$ *if* $N \geqslant 3$).

Proof. We will finish the proof via five steps.

Step 1: Uniform bound of $\partial_t\rho^n$ in $L^2(B^{\frac{N}{2}-1})$. In fact, notice that

$$\partial_t\rho^{n+1} = -\rho^n \operatorname{div}\mathbf{u}^n - \mathbf{u}^n \cdot \nabla\rho^{n+1} - \Lambda d^n.$$

According to the estimates in the previous subsection and the interpolation result in Proposition 2.1, we see that $\{\mathbf{u}^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(B^{\frac{N}{2}})$, and $\{(\rho^n, E^n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $(L^\infty(B^{\frac{N}{2}}))^{N^2+1}$. Thus, $-\rho^n \operatorname{div}\mathbf{u}^n - \mathbf{u}^n \cdot \nabla\rho^{n+1} - \Lambda d^n$ is uniformly bounded in $L^2(B^{\frac{N}{2}-1})$, which implies that $\partial_t\rho^n$ is uniformly bounded in $L^2(B^{\frac{N}{2}-1})$, and furthermore $\{\rho^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}^+; B^{\frac{N}{2}-1})$.

Step 2: Uniform bound of $\partial_t E^n$ in $L^2(B^{\frac{N}{2}-1})$. In fact, notice that

$$\partial_t\big(\big(E^{n+1}\big)^\top - E^{n+1}\big) = -\Lambda\omega^n + \nabla\mathbf{u}^n E^n - \mathbf{u}^n \cdot \nabla\big(\big(E^{n+1}\big)^\top - E^{n+1}\big).$$

According to the estimates in the previous subsection and the interpolation result in Proposition 2.1, we conclude that $\{\mathbf{u}^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(B^{\frac{N}{2}})$, and $\{(\rho^n, E^n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $(L^\infty(B^{\frac{N}{2}}))^{N^2+1}$. Thus, $-\Lambda\omega^n + \nabla\mathbf{u}^n E^n - \mathbf{u}^n \cdot \nabla((E^{n+1})^\top - E^{n+1})$ is uniformly bounded in $L^2(B^{\frac{N}{2}-1})$. Therefore, $\partial_t E^n$ is uniformly bounded in $L^2(B^{\frac{N}{2}-1})$, and furthermore, $\{(E^n)^\top - E^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $(C^{\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}^+; B^{\frac{N}{2}-1}))^{N\times N}$. Similarly, we can show that $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}^+; B^{\frac{N}{2}-1})$. Hence, $\{E^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}^+; B^{\frac{N}{2}-1})$.

Step 3: Uniform bound of $\partial_t d^n$ in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})+L^4(B^{\frac{N}{2}-\frac{3}{2}})$. To this end, we recall that

$$\partial_t d^{n+1} = -\mathbf{u}^n\cdot\nabla d^{n+1} + \nu\Delta d^{n+1} + 2\Lambda\rho^{n+1} + \mathbf{u}^n\cdot\nabla d^n \\ - \Lambda^{-1}\operatorname{div}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n + K(\rho^n)\nabla\rho^n + \frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n - E^n_{jk}\partial_{x_j}E^n_{ik} + \operatorname{div}(\rho^n E^n)\right).$$

The estimates in the previous subsection and the interpolation result in Proposition 2.1 yield that $\{\mathbf{u}^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(B^{\frac{N}{2}-1})\cap L^{\frac{4}{3}}(B^{\frac{N+1}{2}})$. This uniform bound, combining together with the uniform bound of $\{\rho^n\}_{n\in\mathbb{N}}$ in $L^\infty(B^{\frac{N}{2}})$, gives a uniform bound of

$$-\mathbf{u}^n\cdot\nabla d^{n+1} + \nu\Delta d^{n+1} + \mathbf{u}^n\cdot\nabla d^n - \Lambda^{-1}\operatorname{div}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n + \frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n\right)$$

in $L^{\frac{4}{3}}(B^{\frac{N-3}{2}})$. Using the uniform bound of $\{\rho^n\}_{n\in\mathbb{N}}$ in $L^\infty(B^{\frac{N}{2}})\cap L^2(B^{\frac{N}{2}})$ obtained from the uniform bounds of $\{\rho^n\}_{n\in\mathbb{N}}$ in $L^\infty(\tilde{B}^{\frac{N}{2}-1,\frac{N}{2}})\cap L^1(\tilde{B}^{\frac{N}{2}+1,\frac{N}{2}})$ and the interpolation result in Proposition 2.1, we deduce that $\{\rho^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^4(B^{\frac{N-1}{2}})$, and hence $\{\Lambda\rho^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^4(B^{\frac{N-3}{2}})$. Finally, both $E^n_{jk}\partial_{x_j}E^n_{ik}$ and $\operatorname{div}(\rho^n E^n)$ are also uniformly bounded in $L^4(B^{\frac{N-3}{2}})$, since $\{\rho^n\}_{n\in\mathbb{N}}$ and $\{E^n\}_{n\in\mathbb{N}}$ are uniformly bounded in $L^\infty(B^{\frac{N}{2}})\cap L^4(B^{\frac{N-1}{2}})$ and $\operatorname{div}(\rho E)$ can be rewritten as the sum of $\rho\operatorname{div}E$ and $\nabla\rho E$. Therefore, $\{\partial_t d^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})+L^4(B^{\frac{N}{2}-\frac{3}{2}})$.

Step 4: Uniform bound of $\partial_t\omega^n$ in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})+L^4(B^{\frac{N-3}{2}})$. We recall that

$$\partial_t\omega^{n+1} = -\mathbf{u}^n\cdot\nabla\omega^{n+1} + \bar{\mu}\Delta\omega^{n+1} + \Lambda\left((E^{n+1})^\top - E^{n+1}\right) + \mathbf{u}^n\cdot\nabla\omega^n \\ - \Lambda^{-1}\operatorname{curl}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n + K(\rho^n)\nabla\rho^n + \frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n - E^n_{jk}\partial_{x_j}E^n_{ik}\right).$$

Similar to the argument in Step 3, we conclude that

$$-\mathbf{u}^n\cdot\nabla d^{n+1} + \nu\Delta d^{n+1} + \mathbf{u}^n\cdot\nabla d^n - \Lambda^{-1}\operatorname{curl}\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n + \frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n\right)$$

is uniformly bounded in $L^{\frac{4}{3}}(B^{\frac{N-3}{2}})$, using the uniform bound of $\{\rho^n\}_{n\in\mathbb{N}}$ in $L^\infty(B^{\frac{N}{2}})\cap L^2(B^{\frac{N}{2}})$. Also, similarly to Step 3, $\{\Lambda E^{n+1}\}_{n\in\mathbb{N}}$ and $\{E^n_{jk}\partial_{x_j}E^n_{ik}\}_{n\in\mathbb{N}}$ are uniformly bounded in $L^4(B^{\frac{N-3}{2}})$. Hence, $\{\partial_t\omega^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$ $+L^4(B^{\frac{N}{2}-\frac{3}{2}})$.

Step 5: Uniform bound as $N\geqslant 3$. Indeed, in this case, the only difference is the uniform bound on $\mathbf{u}^n$. Actually, from the uniform bounds on $\{\mathbf{u}^n\}_{n\in\mathbb{N}}$ in $L^\infty(B^{\frac{N}{2}-1})\cap L^2(B^{\frac{N}{2}})$, we deduce that $\{\mathbf{u}^n\cdot\nabla\mathbf{u}^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(B^{\frac{N}{2}-2})$. Then, following the same argument in Step 3 and Step 4, we can deduce that $\{\partial_t d^n\}_{n\in\mathbb{N}}$ and $\{\partial_t\omega^n\}_{n\in\mathbb{N}}$ are uniformly bounded in $L^2(B^{\frac{N}{2}-2})+L^\infty(B^{\frac{N}{2}-2})$, because $\{\Lambda\rho^n\}_{n\in\mathbb{N}}$, $\{\Lambda E^{n+1}\}_{n\in\mathbb{N}}$ and $\{E^n_{jk}\partial_{x_j}E^n_{ik}\}_{n\in\mathbb{N}}$ are uniformly bounded in $L^\infty(B^{\frac{N}{2}-2})$. This further implies that $\{\partial_t\mathbf{u}^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(B^{\frac{N}{2}-2})+L^\infty(B^{\frac{N}{2}-2})$, and hence is uniformly bounded in $C^{\frac{1}{2}}_{loc}(\mathbb{R}^+;B^{\frac{N}{2}-2})$. □

We can now turn to the proof of the existence in Theorem 3.1, using a compactness argument. To this end, we denote $\{\chi_p\}_{p\in\mathbb{N}}$ be a sequence of $C_0^\infty(\mathbb{R}^N)$ cut-off functions supported in the ball $B(0,p+1)$ in $\mathbb{R}^N$ and equal to 1 in a neighborhood of the ball $B(0,p)$. For any $p\in\mathbb{N}$, Lemma 5.1

tells us that $\{(\chi_p\rho^n, \chi_p\mathbf{u}^n, \chi_p E^n)\}_{n\in\mathbb{N}}$ is uniformly equicontinuous in $C(\mathbb{R}^+; B^{\frac{N}{2}-1} \times (B^{\frac{N-3}{2}})^N \times (B^{\frac{N}{2}-1})^{N\times N})$. Notice that the operator $f \mapsto \chi_p f$ is compact from $B^{\frac{N}{2}-1} \cap B^{\frac{N}{2}}$ into $\dot H^{\frac{N}{2}-1}$, and from $B^{\frac{N}{2}-1} \cap B^{\frac{N-3}{2}}$ into $\dot H^{\frac{N-3}{2}}$. This can be proved easily by noticing that $f \mapsto \chi_p f$ is compact from $\dot H^s \cap \dot H^{s'}$ into $\dot H^s$ for $s < s'$ and $B^s \hookrightarrow \dot H^s$. We now apply the Ascoli–Arzela theorem to the sequence $\{(\chi_p\rho^n, \chi_p\mathbf{u}^n, \chi_p E^n)\}_{n\in\mathbb{N}}$ on the time interval $[0, p]$. We then use Cantor's diagonal process. This finally provides us with a distribution $(\rho, \mathbf{u}, E)$ belonging to $C(\mathbb{R}^+; \dot H^{\frac{N}{2}-1} \times (\dot H^{\frac{N-3}{2}})^N \times (\dot H^{\frac{N}{2}-1})^{N\times N})$ and a subsequence (which we still denote by $\{(\rho^n, \mathbf{u}^n, E^n)\}_{n\in\mathbb{N}}$), such that, for all $p \in \mathbb{N}$, we have

$$\big(\chi_p\rho^n, \chi_p\mathbf{u}^n, \chi_p E^n\big) \to (\chi_p\rho, \chi_p\mathbf{u}, \chi_p E) \tag{5.13}$$

in $C([0, p]; \dot H^{\frac{N}{2}-1} \times (\dot H^{\frac{N-3}{2}})^N \times (\dot H^{\frac{N}{2}-1})^{N\times N})$. In particular, this implies that $(\rho^n, \mathbf{u}^n, E^n)$ tends to $(\rho, \mathbf{u}, E)$ in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$ as $n \to \infty$. Furthermore, according to the uniform bounds in the previous subsection, we deduce that $(\rho, \mathbf{u}, E)$ belongs to

$$\begin{aligned} &L^\infty\big(\mathbb{R}^+; \tilde B^{\frac{N}{2}-1,\frac{N}{2}} \times \big(B^{\frac{N}{2}-1}\big)^N \times \big(\tilde B^{\frac{N}{2}-1,\frac{N}{2}}\big)^{N\times N}\big) \\ &\quad \cap L^1\big(\mathbb{R}^+; \tilde B^{\frac{N}{2}+1,\frac{N}{2}} \times \big(B^{\frac{N}{2}+1}\big)^N \times \big(\tilde B^{\frac{N}{2}+1,\frac{N}{2}}\big)^{N\times N}\big), \end{aligned}$$

and belongs to $C^{\frac{1}{2}}(\mathbb{R}^+; B^{\frac{N}{2}-1}) \times (C^{\frac{1}{4}}(\mathbb{R}^+; B^{\frac{N-3}{2}}))^N \times (C^{\frac{1}{2}}(\mathbb{R}^+; B^{\frac{N}{2}-1}))^{N\times N}$ (and also belongs to $C^{\frac{1}{2}}(\mathbb{R}^+; B^{\frac{N}{2}-1} \times (B^{\frac{N}{2}-2})^N \times (B^{\frac{N}{2}-1})^{N\times N})$ if $N \geqslant 3$). And, obviously, we have the bounds provided by $(\mathfrak{P}_\mathbf{n})$ for this solution.

Next, we need to prove that $(\rho, \mathbf{u}, E)$ obtained above solves (1.1). To this end, we first observe that, according to (3.4),

$$\begin{cases} \partial_t\rho^{n+1} + \mathbf{u}^n \cdot \nabla\rho^{n+1} + \operatorname{div}\mathbf{u}^{n+1} = -\rho^n \operatorname{div}\mathbf{u}^n, \\ \partial_t\mathbf{u}^{n+1} + \mathbf{u}^n \cdot \nabla\mathbf{u}^{n+1} - \mathcal{A}\mathbf{u}^{n+1} + \nabla\rho^{n+1} - \operatorname{div} E^{n+1} \\ \quad = K\big(\rho^n\big)\nabla\rho^n + \dfrac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n + \operatorname{div}\big(\big(E^n\big)\big(E^n\big)^\top\big), \\ \partial_t E^{n+1} + \mathbf{u}^n \cdot \nabla E^{n+1} + \nabla\mathbf{u}^n = \nabla\mathbf{u}^n E^{n+1}. \end{cases} \tag{5.14}$$

Hence, the only problem now is to pass to the limit in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$ for each term in (5.14), especially for those nonlinear terms. Let $\theta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ and $p \in \mathbb{N}$ be such that $\operatorname{supp}\theta \subset [0, p] \times B(0, p)$. For the convergence of $\frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n$, we write

$$\begin{aligned} &\theta\bigg(\frac{\rho^n}{1+\rho^n}\mathcal{A}\mathbf{u}^n - \frac{\rho}{1+\rho}\mathcal{A}\mathbf{u}\bigg) \\ &\quad = \theta\frac{\rho^n}{1+\rho^n}\chi_p\mathcal{A}\big(\chi_p\mathbf{u}^n - \chi_p\mathbf{u}\big) + \theta\bigg(\frac{\chi_p\rho^n}{1+\chi_p\rho^n} - \frac{\chi_p\rho}{1+\chi_p\rho}\bigg)\chi_p\mathcal{A}\mathbf{u}. \end{aligned}$$

Since $\theta\frac{\rho^n}{1+\rho^n}$ is uniformly bounded in $L^\infty(B^{\frac{N}{2}}) \subset L^\infty(\dot H^{\frac{N}{2}})$ and $\chi_p\mathbf{u}^n$ tends to $\chi_p\mathbf{u}$ in $C([0, p]; \dot H^{\frac{N-3}{2}})$, the first term in the above identity tends to 0 in $C([0, p]; \dot H^{\frac{N-3}{2}})$, while $\frac{\chi_p\rho^n}{1+\chi_p\rho^n}$ tends to $\frac{\chi_p\rho}{1+\chi_p\rho}$ in $C([0, p]; \dot H^{\frac{N}{2}-1})$ by Lemma 2.1, which implies that the third term also tends to 0 in $C([0, p]; \dot H^{\frac{N-3}{2}})$. The convergence of other nonlinear terms can be treated similarly.

Finally, we now prove the continuity of ρ and E in time.

Lemma 5.2. *Let $(\rho, \mathbf{u}, E)$ be a solution to (1.1). Then ρ and E are continuous in $\tilde B^{\frac{N}{2}-1,\frac{N}{2}}$, and $\mathbf{u}$ belongs to $C(\mathbb{R}^+; B^{\frac{N}{2}-1})$.*

Proof. To prove this, we follow the argument in [4]. Indeed, the continuity of $\mathbf{u}$ is straightforward, because $\mathbf{u}$ satisfies

$$\partial_t \mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} + \mathcal{A}\mathbf{u} - \nabla\rho - \left(\frac{\rho}{1+\rho}\right)\mathcal{A}\mathbf{u} - K(\rho)\nabla\rho + E_{jk}\nabla_{x_j}E_{ik},$$

and the right-hand side belongs to $L^1(B^{\frac{N}{2}-1}) + L^2(B^{\frac{N}{2}-1})$ due to the facts that $\rho \in L^2(B^{\frac{N}{2}-1})$ and $E \in L^2(B^{\frac{N}{2}-1})$.

To prove $\rho, E \in C(\mathbb{R}^+; B^{\frac{N}{2}-1})$, we notice that $\rho_0, E_0 \in B^{\frac{N}{2}-1}$, $\rho, E \in L^\infty(\mathbb{R}^+; B^{\frac{N}{2}-1})$ and $\partial_t\rho, \partial_t E \in L^2(\mathbb{R}^+; B^{\frac{N}{2}-1})$. Thus, it remains to prove the continuity in time of ρ, E in $B^{\frac{N}{2}}$. To this end, we apply the operator Δ_q to the first equation of (1.1) to yield

$$\partial_t \Delta_q\rho = -\Delta_q(\mathbf{u}\cdot\nabla\rho) - \Lambda\Delta_q d - \Delta_q(\rho\,\mathrm{div}\,\mathbf{u}). \tag{5.15}$$

Obviously, for fixed q, the right-hand side belongs to $L^1(\mathbb{R}^+; L^2)$. Hence, each $\Delta_q\rho$ is continuous in time with values in L^2 (thus in $B^{\frac{N}{2}}$). Now, applying an energy method to (5.15), thanks to Lemma 2.2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q\rho\|_{L^2}^2 \leqslant C\|\Delta_q\rho\|_{L^2}\big(\alpha_q 2^{-q\frac{N}{2}}\|\rho\|_{B^{\frac{N}{2}}}\|\mathbf{u}\|_{B^{\frac{N}{2}+1}} + \|\Lambda\Delta_q d\|_{L^2} + \big\|\Delta_q(\rho\,\mathrm{div}\,\mathbf{u})\big\|_{L^2}\big).$$

Integrating the above inequality with respect to time on the interval $[t_1, t_2]$, we obtain

$$\begin{aligned}2^{q\frac{N}{2}}\big\|\Delta_q\rho(t_2)\big\|_{L^2} \leqslant{}& 2^{q\frac{N}{2}}\big\|\Delta_q\rho(t_1)\big\|_{L^2} + C\int_{t_1}^{t_2}\big(\alpha_q(\tau)\big\|\rho(\tau)\big\|_{B^{\frac{N}{2}}}\big\|\mathbf{u}(\tau)\big\|_{B^{\frac{N}{2}+1}} \\ &+ 2^{q(\frac{N}{2}+1)}\big\|\Delta_q d(\tau)\big\|_{L^2} + 2^{q\frac{N}{2}}\big\|\Delta_q(\rho\,\mathrm{div}\,\mathbf{u})(\tau)\big\|_{L^2}\big)\,d\tau.\end{aligned}$$

Since $\rho \in L^\infty(B^{\frac{N}{2}})$, $\mathbf{u} \in L^1(B^{\frac{N}{2}+1})$, and $\rho\,\mathrm{div}\,\mathbf{u} \in L^1(B^{\frac{N}{2}})$, we eventually obtain

$$\big\|\rho(t_2)\big\|_{B^{\frac{N}{2}}} \lesssim \big\|\rho(t_1)\big\|_{B^{\frac{N}{2}}} + \big(1 + \|\rho\|_{L^\infty(B^{\frac{N}{2}})}\big)\int_{t_1}^{t_2}\big\|\mathbf{u}(\tau)\big\|_{B^{\frac{N}{2}+1}}\,d\tau + \int_{t_1}^{t_2}\big\|\rho\,\mathrm{div}\,\mathbf{u}(\tau)\big\|_{B^{\frac{N}{2}}}\,d\tau,$$

which implies that ρ belongs to $C(\mathbb{R}^+; B^{\frac{N}{2}})$.

Similarly, we can prove that E also belongs to $C(\mathbb{R}^+; B^{\frac{N}{2}})$.

This finishes our proof. □

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Appendix A

For the completeness of the presentation, we give in this appendix the proof of two fundamental lemmas concerning the divergence and curl of the deformation gradient in the viscoelasticity system (1.1).

The first one we are going to state now is the following lemma (cf. Proposition 3.1 in [15]).

Lemma A.1. *Assume that* $\operatorname{div}(\rho_0 \mathtt{F}_0^\top) = 0$ *and* $(\rho, \mathbf{u}, \mathtt{F})$ *is the solution of the system* (1.1). *Then the following identity*

$$\operatorname{div}(\rho \mathtt{F}^\top) = 0 \tag{A.1}$$

holds for all time $t > 0$.

Proof. First, we transpose the third equation in (1.1) and apply the divergence operator to the resulting equation to yield

$$\partial_t(\partial_{x_j}\mathtt{F}_{ji}) + \mathbf{u}\cdot\nabla(\partial_{x_j}\mathtt{F}_{ji}) = \left(\frac{\partial^2 \mathbf{u}_j}{\partial x_k \partial x_j}\right)\mathtt{F}_{ki}. \tag{A.2}$$

Multiply the first equation in (1.1) by $\partial_{x_j}\mathtt{F}_{ji}$, multiply (A.2) by ρ, and summing them together, we obtain

$$\partial_t(\rho\partial_{x_j}\mathtt{F}_{ji}) + \mathbf{u}\cdot\nabla(\rho\partial_{x_j}\mathtt{F}_{ji}) = \rho\left(\frac{\partial^2 \mathbf{u}_j}{\partial x_k \partial x_j}\right)\mathtt{F}_{ki} - \rho\partial_{x_k}\mathbf{u}_k\partial_{x_j}\mathtt{F}_{ji}. \tag{A.3}$$

On the other hand, we differentiate the first equation in (1.1) with respect to x_j to yield

$$\partial_t(\partial_{x_j}\rho) + \mathbf{u}\cdot\nabla(\partial_{x_j}\rho) + \partial_{x_j}\mathbf{u}_k\partial_{x_k}\rho + \frac{\partial^2 \mathbf{u}_k}{\partial x_k \partial x_j}\rho + \partial_{x_j}\rho\partial_{x_k}\mathbf{u}_k = 0. \tag{A.4}$$

Multiplying (A.4) by $\mathtt{F}_{ji}$, multiplying the third equation in (1.1) by $\partial_{x_j}\rho$, and summing them together, we obtain

$$\partial_t(\partial_{x_j}\rho\mathtt{F}_{ji}) + \mathbf{u}\cdot\nabla(\partial_{x_j}\rho\mathtt{F}_{ji}) = -\rho\left(\frac{\partial^2 \mathbf{u}_k}{\partial x_k \partial x_j}\right)\mathtt{F}_{ji} - \partial_{x_j}\rho\partial_{x_k}\mathbf{u}_k\mathtt{F}_{ji}. \tag{A.5}$$

Adding (A.3) and (A.5) together yields

$$\partial_t\big(\operatorname{div}(\rho\mathtt{F}^\top)\big) + \operatorname{div}\big(\mathbf{u}\otimes\operatorname{div}(\rho\mathtt{F}^\top)\big) = 0. \tag{A.6}$$

If $(\rho, \mathbf{u}, \mathtt{F})$ is sufficiently smooth, we multiply (A.6) by $\operatorname{div}(\rho\mathtt{F}^\top)$, to get

$$\partial_t\big(\big|\operatorname{div}(\rho\mathtt{F}^\top)\big|^2\big) + \operatorname{div}\big(\mathbf{u}\big|\operatorname{div}(\rho\mathtt{F}^\top)\big|^2\big) = -\frac{1}{2}\operatorname{div}\mathbf{u}\big|\operatorname{div}(\rho\mathtt{F}^\top)\big|^2.$$

Integrating the above identity with respect to x over $\mathbb{R}^N$, we obtain

$$\begin{aligned}\frac{d}{dt}\big\|\operatorname{div}(\rho\mathtt{F}^\top)\big\|_{L^2}^2 &= -\frac{1}{2}\int_{\mathbb{R}^N}\operatorname{div}\mathbf{u}\big|\operatorname{div}(\rho\mathtt{F}^\top)\big|^2\,dx\\ &\leqslant \frac{1}{2}\|\nabla\mathbf{u}\|_{L^\infty}\big\|\operatorname{div}(\rho\mathtt{F}^\top)\big\|_{L^2}^2,\end{aligned}$$

which, by Gronwall's inequality, implies that, for all $t \geqslant 0$

$$\left\|\operatorname{div}\left(\rho \mathrm{F}^{\top}\right)(t)\right\|_{L^2}^2 \leqslant \left\|\operatorname{div}\left(\rho_0 \mathrm{F}_0^{\top}\right)\right\|_{L^2}^2 e^{\frac{1}{2}\int_0^t \|\nabla \mathbf{u}(\tau)\|_{L^\infty} d\tau}.$$

Hence, if $\operatorname{div}(\rho_0 \mathrm{F}_0^{\top}) = 0$, the above inequality will give $\|\operatorname{div}(\rho \mathrm{F}^{\top})\|_{L^2} = 0$ for all $t > 0$, which implies that $\operatorname{div}(\rho \mathrm{F}^{\top}) = 0$ for all $t > 0$.

This finishes the proof. □

Another hidden, but important, property of the viscoelastic fluids system (1.1) is concerned with the curl of the deformation gradient (for the incompressible case, see [13,16]). Actually, the following lemma says that the curl of the deformation gradient is of higher order.

Lemma A.2. *Assume that* (1.1c) *is satisfied and* $(\mathbf{u}, \mathrm{F})$ *is the solution of the system* (1.1)*. Then the following identity*

$$\mathrm{F}_{lk} \nabla_l \mathrm{F}_{ij} = \mathrm{F}_{lj} \nabla_l \mathrm{F}_{ik} \tag{A.7}$$

holds for all time $t > 0$ *if it initially satisfies* (A.7)*.*

Proof. First, we establish the evolution equation for the equality $\mathrm{F}_{lk} \nabla_l \mathrm{F}_{ij} - \mathrm{F}_{lj} \nabla_l \mathrm{F}_{ik}$. Indeed, by Eq. (1.1c), we can get

$$\partial_t \nabla_l \mathrm{F}_{ij} + \mathbf{u} \cdot \nabla \nabla_l \mathrm{F}_{ij} + \nabla_l \mathbf{u} \cdot \nabla \mathrm{F}_{ij} = \nabla_m \mathbf{u}_i \nabla_l \mathrm{F}_{mj} + \nabla_l \nabla_m \mathbf{u}_i \mathrm{F}_{mj}.$$

Thus,

$$\mathrm{F}_{lk}(\partial_t \nabla_l \mathrm{F}_{ij} + \mathbf{u} \cdot \nabla \nabla_l \mathrm{F}_{ij}) + \mathrm{F}_{lk} \nabla_l \mathbf{u} \cdot \nabla \mathrm{F}_{ij} = \mathrm{F}_{lk} \nabla_m \mathbf{u}_i \nabla_l \mathrm{F}_{mj} + \mathrm{F}_{lk} \nabla_l \nabla_m \mathbf{u}_i \mathrm{F}_{mj}. \tag{A.8}$$

Also, from (1.1c), we obtain

$$\nabla_l \mathrm{F}_{ij}(\partial_t \mathrm{F}_{lk} + \mathbf{u} \cdot \nabla \mathrm{F}_{lk}) = \nabla_l \mathrm{F}_{ij} \nabla_m \mathbf{u}_l \mathrm{F}_{mk}. \tag{A.9}$$

Now, adding (A.8) and (A.9), we deduce that

$$\begin{aligned}
&\partial_t(\mathrm{F}_{lk} \nabla_l \mathrm{F}_{ij}) + \mathbf{u} \cdot \nabla(\mathrm{F}_{lk} \nabla_l \mathrm{F}_{ij}) \\
&\quad = -\mathrm{F}_{lk} \nabla_l \mathbf{u} \cdot \nabla \mathrm{F}_{ij} + \mathrm{F}_{lk} \nabla_m \mathbf{u}_i \nabla_l \mathrm{F}_{mj} + \mathrm{F}_{lk} \nabla_l \nabla_m \mathbf{u}_i \mathrm{F}_{mj} + \nabla_l \mathrm{F}_{ij} \nabla_m \mathbf{u}_l \mathrm{F}_{mk} \\
&\quad = \mathrm{F}_{lk} \nabla_m \mathbf{u}_i \nabla_l \mathrm{F}_{mj} + \mathrm{F}_{lk} \nabla_l \nabla_m \mathbf{u}_i \mathrm{F}_{mj}.
\end{aligned} \tag{A.10}$$

Here, we used the identity which is derived by interchanging the roles of indices l and m:

$$\mathrm{F}_{lk} \nabla_l \mathbf{u} \cdot \nabla \mathrm{F}_{ij} = \mathrm{F}_{lk} \nabla_l \mathbf{u}_m \nabla_m \mathrm{F}_{ij} = \nabla_l \mathrm{F}_{ij} \nabla_m \mathbf{u}_l \mathrm{F}_{mk}.$$

Similarly, one has

$$\partial_t(\mathrm{F}_{lj} \nabla_l \mathrm{F}_{ik}) + \mathbf{u} \cdot \nabla(\mathrm{F}_{lj} \nabla_l \mathrm{F}_{ik}) = \mathrm{F}_{lj} \nabla_m \mathbf{u}_i \nabla_l \mathrm{F}_{mk} + \mathrm{F}_{lj} \nabla_l \nabla_m \mathbf{u}_i \mathrm{F}_{mk}. \tag{A.11}$$

Subtracting (A.11) from (A.10) yields

$$\begin{aligned}
&\partial_t(\mathrm{F}_{lk} \nabla_l \mathrm{F}_{ij} - \mathrm{F}_{lj} \nabla_l \mathrm{F}_{ik}) + \mathbf{u} \cdot \nabla(\mathrm{F}_{lk} \nabla_l \mathrm{F}_{ij} - \mathrm{F}_{lj} \nabla_l \mathrm{F}_{ik}) \\
&\quad = \nabla_m \mathbf{u}_i(\mathrm{F}_{lk} \nabla_l \mathrm{F}_{mj} - \mathrm{F}_{lj} \nabla_l \mathrm{F}_{mk}) + \nabla_l \nabla_m \mathbf{u}_i(\mathrm{F}_{mj} \mathrm{F}_{lk} - \mathrm{F}_{mk} \mathrm{F}_{lj}).
\end{aligned} \tag{A.12}$$

Due to the fact

$$\nabla_l \nabla_m \mathbf{u}_i = \nabla_m \nabla_l \mathbf{u}_i$$

in the sense of distributions, we have, again by interchanging the roles of indices l and m,

$$\begin{aligned}
\nabla_l \nabla_m \mathbf{u}_i (\mathtt{F}_{mj}\mathtt{F}_{lk} - \mathtt{F}_{mk}\mathtt{F}_{lj}) &= \nabla_l \nabla_m \mathbf{u}_i \mathtt{F}_{mj}\mathtt{F}_{lk} - \nabla_l \nabla_m \mathbf{u}_i \mathtt{F}_{mk}\mathtt{F}_{lj} \\
&= \nabla_l \nabla_m \mathbf{u}_i \mathtt{F}_{mj}\mathtt{F}_{lk} - \nabla_m \nabla_l \mathbf{u}_i \mathtt{F}_{lk}\mathtt{F}_{mj} \\
&= (\nabla_l \nabla_m \mathbf{u}_i - \nabla_m \nabla_l \mathbf{u}_i)\mathtt{F}_{lk}\mathtt{F}_{mj} = 0.
\end{aligned}$$

From this identity, Eq. (A.12) can be simplified as

$$\begin{aligned}
&\partial_t (\mathtt{F}_{lk}\nabla_l \mathtt{F}_{ij} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{ik}) + \mathbf{u}\cdot\nabla(\mathtt{F}_{lk}\nabla_l \mathtt{F}_{ij} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{ik}) \\
&\quad = \nabla_m \mathbf{u}_i (\mathtt{F}_{lk}\nabla_l \mathtt{F}_{mj} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{mk}).
\end{aligned} \tag{A.13}$$

Multiplying (A.13) by $\mathtt{F}_{lk}\nabla_l \mathtt{F}_{ij} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{ik}$, we get

$$\begin{aligned}
&\partial_t |\mathtt{F}_{lk}\nabla_l \mathtt{F}_{ij} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{ik}|^2 + \mathbf{u}\cdot\nabla|\mathtt{F}_{lk}\nabla_l \mathtt{F}_{ij} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{ik}|^2 \\
&\quad = 2(\mathtt{F}_{lk}\nabla_l \mathtt{F}_{ij} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{ik})\nabla_m \mathbf{u}_i (\mathtt{F}_{lk}\nabla_l \mathtt{F}_{mj} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{mk}) \\
&\quad \leqslant 2\|\nabla \mathbf{u}\|_{L^\infty(\mathbb{R}^3)} \mathcal{M}^2,
\end{aligned} \tag{A.14}$$

where $\mathcal{M}$ is defined as

$$\mathcal{M} = \max_{i,j,k} \{ |\mathtt{F}_{lk}\nabla_l \mathtt{F}_{ij} - \mathtt{F}_{lj}\nabla_l \mathtt{F}_{ik}|^2 \}.$$

Hence, (A.14) implies

$$\partial_t \mathcal{M} + \mathbf{u}\cdot\nabla\mathcal{M} \leqslant 2\|\nabla \mathbf{u}\|_{L^\infty(\mathbb{R}^3)} \mathcal{M}. \tag{A.15}$$

On the other hand, the characteristics of $\partial_t f + \mathbf{u}\cdot\nabla f = 0$ is given by

$$\frac{d}{ds} X(s) = \mathbf{u}\big(s, X(s)\big), \qquad X(t) = x.$$

Hence, (A.14) can be rewritten as

$$\frac{\partial U}{\partial t} \leqslant B(t, y) U, \qquad U(0, y) = \mathcal{M}_0(y), \tag{A.16}$$

where

$$U(t, y) = \mathcal{M}\big(t, X(t, x)\big), \qquad B(t, y) = 2\|\nabla \mathbf{u}\|_{L^\infty(\mathbb{R}^3)}\big(t, X(t, y)\big).$$

The differential inequality (A.16) implies that

$$U(t, y) \leqslant U(0) \exp\left(\int_0^t B(s, y)\, ds \right).$$

Hence,

$$\mathcal{M}(t,x) \leqslant \mathcal{M}(0)\exp\left(\int_0^t 2\|\nabla \mathbf{u}\|_{L^\infty(\mathbb{R}^3)}(s)\,ds\right).$$

Hence, if $\mathcal{M}(0) = 0$, then $\mathcal{M}(t) = 0$ for all $t > 0$, and the proof of the lemma is complete. □

Using $\mathtt{F} = I + E$, (A.7) means

$$\nabla_k E_{ij} + E_{lk}\nabla_l E_{ij} = \nabla_j E_{ik} + E_{lj}\nabla_l E_{ik}. \tag{A.17}$$

According to (A.17), it is natural to assume that the initial condition of E in the viscoelastic fluids system (1.3) should satisfy the compatibility condition

$$\nabla_k E(0)_{ij} + E(0)_{lk}\nabla_l E(0)_{ij} = \nabla_j E(0)_{ik} + E(0)_{lj}\nabla_l E(0)_{ik}. \tag{A.18}$$

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